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BOUNDARY INTEGRAL OPERATORS ON LIPSCHITZ DOMAINS: ELEMENTARY RESULTS*

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Abstract. The simple and double layer potentials for second order linear strongly elliptic differential operators on Lipschitz domains are studied and it is shown that in a certain range of Sobolev spaces, results on continuity and regularity can be obtained without using either Calderón's theorem on the L_2-continuity of the Cauchy integral on Lipschitz curves [J. L. Journé, "Calderón-Zygmuno operators, pseudo-differential operators and the Cauchy integral of Calderón," in *Lecture Notes in Math.* 994, Springer-Verlag, Berlin, 1983] or Dahlberg's estimates of harmonic measures ["On the Poisson integral for Lipschitz and C^1 domains," *Studia Math.*, 66 (1979), pp. 7–24]. The operator of the simple layer potential and of the normal derivative of the double layer potential are shown to be strongly elliptic in the sense that they satisfy Gårding inequalities in the respective energy norms. As an application, error estimates for Galerkin approximation schemes for integral equations of the first kind are derived.

Key words. Lipschitz domains, layer potentials, trace lemma, jump relations, Green's formula, Galerkin approximation, Gårding's inequality

AMS(MOS) subject classifications. 45E99, 35J25, 45L10, 58G15

1. Introduction. Boundary value problems on Lipschitz domains and the method of layer potentials for their solution have attracted some attention in recent years both in the theoretical and the applied mathematical literature.

On one hand, the final proof by Coifman, McIntosh, and Meyer [5] of Calderón's Theorem on the L_2-continuity of the Cauchy integral on Lipschitz curves and Dahlberg's estimates [10] of the Poisson kernel paved the way for investigations of the Dirichlet and Neumann problems for the Laplace equation by means of boundary integral equations [12], [21], [27]. This method was also applied to some boundary value problems for the equations of linear elasticity theory [20].

On the other hand, in the applied sciences, the so-called boundary element methods are frequently used for domains with corners and edges without mathematical analysis being available. As long as there exists no elementary proof of Calderón's theorem and its consequences, it seems justified to study the range of possible results obtainable without this deep and, for the nonspecialist, not easily accessible result.

We use throughout the weak (distributional) definition of boundary values and show that the operators of the simple layer, the double layer, the normal derivative of the simple layer, and the normal derivative of the double layer define bounded operators in those Sobolev spaces on the boundary that correspond to the "energy norm," i.e., to the variational formulation of the boundary value problem. The simple layer and the normal derivative of the double layer define strongly elliptic operators. This implies stability of corresponding Galerkin approximation schemes.

In order to show continuity of the operators in a certain range of Sobolev spaces, we prove a generalization of Gagliardo's Trace Lemma (Lemma 3.6) and use regularity results for the Dirichlet and Neumann problems by Nečas [23]. Nečas obtained these results by elementary means, applying an identity of Rellich that had been used for similar purposes by Payne and Weinberger [24] and recently by Jerison and Kenig [16], [17] and Verchota [21], [27]. The same tools yield regularity results for the solutions of the integral equations and also invertibility under some hypotheses on the

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differential operator and its fundamental solution satisfied for instance in the case of the Laplace operator unless the domain is a subset of $\mathbb{R}^2$ with analytic capacity equal to one.

This work is part of the author's habilitation thesis [7]. Further results concerning strong ellipticity of boundary integral operators for higher order differential equations on smooth domains have been published [9]. That paper also contains an extensive list of references on the Calderón-Seeley-Hörmander method of boundary integral equations for elliptic boundary value problems and on the history of strong ellipticity for boundary integral operators. Let us mention here only two references from each of these two fields: The books by Chazarain and Piriou [4] and by Dieudonné [11] describe the method of the Calderón projector of elliptic equations of any order on smooth domains. The lecture notes by Nedelec [22] and the paper by Hsiao and Wendland [15] contain, for the example of the Laplace operator on smooth domains, the idea of transforming the strong ellipticity of the differential operator via Green's formula into the strong ellipticity of certain operators on the boundary (see the proof of Theorem 2 below).

2. Main results. In this section we state the main results of this paper. Proofs are given in § 4.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. This means that its boundary Γ is locally the graph of a Lipschitz function. For properties of Lipschitz domains we refer to Nečas [23] and Grisvard [14]. Because of the invariance of the Sobolev spaces $H^s = W^{s,2}$ under Lipschitz coordinate transformations for $|s| \leqq 1$, we can define the spaces $H^s(\Gamma)(|s| \leqq 1)$ in the usual way using local coordinate representations of the Lipschitz manifold Γ. The same reason implies Gagliardo's Trace Lemma:

$$\text{(2.1)}\quad \begin{aligned} &\gamma_0 : u \mapsto \gamma_0 u := u|_\Gamma : H^s_{\mathrm{loc}}(\mathbb{R}^n) \to H^{s-1/2}(\Gamma) \text{ is continuous} \\ &\text{and has a continuous right inverse} \\ &\gamma_0^- : H^{s-1/2}(\Gamma) \to H^s_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{for all } s \in (\tfrac{1}{2}, 1]. \end{aligned}$$

Here traces are understood in the distributional sense, i.e., the mapping γ_0 is well defined for smooth (say continuous) functions, and for arbitrary $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$ it is defined by approximating u by smooth functions.

Let

$$\text{(2.2)}\quad P = -\sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \sum_{j=1}^n b_j \partial_j + c$$

be a differential operator with $C^\infty(\mathbb{R}^n; \mathbb{C})$-coefficients a_{jk}, b_j and c. Here $\partial_j = \partial/\partial x_j$.

We emphasize that all results will also be valid in the case of systems, i.e., for matrix valued coefficients a_{jk}, b_j and c. It is only for notational convenience that we stick to the scalar case.

We assume that P is strongly elliptic which implies that for the bilinear form

$$\text{(2.3)}\quad \Phi_\Omega(u, v) := \int_\Omega \left(\sum_{j,k=1}^n a_{jk} \partial_k u \overline{\partial_j v} + \sum_{j=1}^n b_j \partial_j u \bar{v} + cu\bar{v} \right) dx$$

there holds a Gårding inequality on all of $H^1(\Omega)$

$$\text{(2.4)}\quad \operatorname{Re} \Phi_\Omega(u, u) \geqq \lambda \|u\|^2_{H^1(\Omega)} - C\|u\|^2_{L_2(\Omega)} \quad \text{for all } u \in H^1(\Omega)$$

with some $\lambda > 0$. (In the case of systems we have to require (2.4) explicitly. It holds, for example, for the equations of linear elasticity theory by virtue of Korn's inequality [23, p. 194].)

Furthermore we assume that P has a fundamental solution G that is a two-sided inverse of P on the space of compactly supported distributions on $\mathbb{R}^n$. Then G has a weakly singular kernel that we also denote by G, and the function $(x, y) \mapsto G(x, y)$ is C^∞ outside the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$.

For a locally integrable function v on Γ we then can define the simple layer potential

$$\text{(2.5)}\quad K_0 v(x) := \int_\Gamma G(x, y) v(y)\, ds(y) \qquad (x \in \mathbb{R}^n \setminus \Gamma)$$

where ds is the surface measure on Γ, and the double layer potential

$$\text{(2.6)}\quad K_1 v(x) := \int_\Gamma \partial_{\nu(y)} G(x, y) v(y)\, ds(y).$$

Here ∂_ν is the conormal derivative

$$\text{(2.7)}\quad \partial_\nu := \sum_{j,k=1}^n n_j a_{jk} \partial_k,$$

where n_j are the components of the almost everywhere defined outward pointing normal vector.

The boundary integral operators are defined by taking the boundary data of K_0 and K_1 (in the distributional sense; see (2.1) and Lemma 3.2 below)

$$\text{(2.8)}\quad \begin{aligned} Av &:= \gamma_0 K_0 v, & Bv &:= \gamma_1 (K_0 v|_\Omega), \\ Cv &:= \gamma_0 (K_1 v|_\Omega), & Dv &:= -\tilde{\gamma}_1 (K_1 v|_\Omega). \end{aligned}$$

Here $\gamma_1 u := \partial_\nu u|_\Gamma$ and $\tilde{\gamma}_1 u := \partial_\nu u|_\Gamma - \sum_{j=1}^n n_j b_j u|_\Gamma$.

Under these assumptions, we have the following continuity result.

THEOREM 1. *For all $\sigma \in (-\frac{1}{2}, \frac{1}{2})$ the following operators are continuous*:

(i) $K_0 : H^{-1/2+\sigma}(\Gamma) \to H^{1+\sigma}_{\mathrm{loc}}(\mathbb{R}^n)$;
(ii) $K_1 : H^{1/2+\sigma}(\Gamma) \to H^{1+\sigma}(\Omega)$;
(iii) $A : H^{-1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma)$;
(iv) $B : H^{-1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma)$;
(v) $C : H^{1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma)$;
(vi) $D : H^{1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma)$.

Remark. As shown by Verchota [27] and Jerison and Kenig [16], [17], the Calderón and Dahlberg theorems give the above results for the endpoint $\sigma = \frac{1}{2}$. An argument using duality and interpolation then allows to cover the whole range $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, which is optimal in the sense that, for Lipschitz boundaries, Sobolev spaces $H^s(\Gamma)$ with $|\sigma| > 1$ cannot be defined in a unique invariant way.

The operators A and D are strongly elliptic.

THEOREM 2. *There exist compact operators*

$$T_A : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad T_D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

and constants $\lambda_A, \lambda_D > 0$ such that

$$\text{(2.9)}\quad \operatorname{Re}\langle (A + T_A)v, \bar{v} \rangle \geqq \lambda_A \|v\|^2_{H^{-1/2}(\Gamma)} \quad \textit{for all } v \in H^{-1/2}(\Gamma),$$

$$\text{(2.10)}\quad \operatorname{Re}\langle (D + T_D)v, \bar{v} \rangle \geqq \lambda_D \|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H^{1/2}(\Gamma).$$

Here the brackets $\langle \cdot, \cdot \rangle$ denote the natural duality between a Sobolev space $H^s(\Gamma)$ and its dual $H^{-s}(\Gamma)$.

The following regularity result holds.

THEOREM 3. *Let $\sigma \in [0, \frac{1}{2}]$ and let $\psi \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$ satisfy*

$$A\psi \in H^{1/2+\sigma}(\Gamma) \quad \textit{or} \quad B\psi \in H^{-1/2+\sigma}(\Gamma)$$

and

$$Cv \in H^{1/2+\sigma}(\Gamma) \quad or \quad Dv \in H^{-1/2+\sigma}(\Gamma).$$

Then $\psi \in H^{-1/2+\sigma}(\Gamma)$ *and* $v \in H^{1/2+\sigma}(\Gamma)$, *and there hold the a priori estimates*

(2.11) $$\|\psi\|_{H^{-1/2+\sigma}(\Gamma)} \leqq C(\|A\psi\|_{H^{1/2+\sigma}(\Gamma)} + \|\psi\|_{H^{-1/2}(\Gamma)}),$$

(2.12) $$\|\psi\|_{H^{-1/2+\sigma}(\Gamma)} \leqq C(\|B\psi\|_{H^{-1/2+\sigma}(\Gamma)} + \|\psi\|_{H^{-1/2}(\Gamma)}),$$

(2.13) $$\|v\|_{H^{1/2+\sigma}(\Gamma)} \leqq C(\|Cv\|_{H^{1/2+\sigma}(\Gamma)} + \|v\|_{H^{1/2}(\Gamma)}),$$

(2.14) $$\|v\|_{H^{1/2+\sigma}(\Gamma)} \leqq C(\|Dv\|_{H^{-1/2+\sigma}(\Gamma)} + \|v\|_{H^{1/2}(\Gamma)}).$$

Now let $(S^h)_{h>0}$ be a family of subspaces of $H^{-1/2}(\Gamma)$ with the property that the orthogonal projection operators onto S^h tend strongly to the identity in $H^{-1/2}(\Gamma)$ for $h \to 0$.

For the equation

(2.15) $$Av = g \quad \text{with } g \in H^{1/2}(\Gamma)$$

we consider the Galerkin scheme

(2.16) Find $v_h \in S^h$ such that $\langle Av_h, w\rangle = \langle g, w\rangle$ for all $w \in S^h$.

From Theorem 2 then follows stability and convergence in $H^{-1/2}(\Gamma)$. Note that Theorem 2 implies that the operators A and D are Fredholm operators of index zero.

THEOREM 4. *If the operator* $A: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is injective then for any* $g \in H^{1/2}(\Gamma)$ *there is a* $h_0 > 0$ *such that for all* $0 < h < h_0$ *the Galerkin scheme* (2.16) *has a unique solution* $v_h \in S^h$. *For* $h \to 0$, v_h *converges to the unique solution* $v \in H^{-1/2}(\Gamma)$ *of* (2.15) *quasioptimally, i.e., there exists a constant C such that for all* $0 < h < h_0$

(2.17) $$\|v - v_h\|_{H^{-1/2}(\Gamma)} \leqq C \inf_{w \in S^h} \|v - w\|_{H^{-1/2}(\Gamma)}.$$

Of course, a corresponding result holds for the operator D.

From Theorem 3 we can deduce asymptotic error estimates using (2.17). We assume for instance that S^h are regular finite element spaces, in the simplest case, e.g., consisting of functions piecewise constant on Γ that are constant on the faces of a triangulation of Γ quasiuniform with respect to h where h is the meshsize. Then there holds the following theorem.

THEOREM 5. *Let A be injective as above and* $g \in H^1(\Gamma)$. *Then there is a constant C such that for all* $0 < h < h_0$

(2.18) $$\|v - v_h\|_{H^{-1/2}(\Gamma)} \leqq Ch^{1/2}\|g\|_{H^1(\Gamma)}.$$

3. The tools. In this section we collect some results, some new but most of them known, and adapt them to the present situation.

We need some further notation.

$H^s_{\text{comp}}(\mathbb{R}^n)$ is the space of distributions in H^s with compact support. It is thus in a natural way the dual space of $H^{-s}_{\text{loc}}(\mathbb{R}^n)$.

$$H^s_P(\Omega) := \{u \in H^s(\Omega) \mid Pu \in L_2(\Omega)\}, \qquad \|u\|^2_{H^s_P(\Omega)} := \|u\|^2_{H^s} + \|Pu\|^2_{L_2},$$

$$P' := -\sum_{j,k=1}^n \partial_j a_{jk} \partial_k - \sum_{j=1}^n \partial_j b_j + c \quad \text{is the formal transpose of } P.$$

We may assume $a_{jk} = a_{kj}$ without restriction.

By P^0 we denote any operator with the same principal part as P and positive on $H^1(\Omega)$. We may take, e.g.,

$$-\sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \lambda \quad \text{with } \lambda > 0.$$

Thus there holds with some $\lambda > 0$

(3.1) $$\operatorname{Re}(P^0 u, u)_{L_2(\Omega)} \geqq \lambda \|u\|^2_{H^1(\Omega)} \quad \text{for all } u \in C^2(\bar\Omega).$$

It follows with the trace lemma (2.1) that for P^0 the Dirichlet problem is uniquely solvable in the weak sense.

LEMMA 3.1. *The Dirichlet problem*

$$P^0 u = 0 \quad in\ \Omega, \qquad \gamma_0 u = v$$

has for $v \in H^{1/2}(\Gamma)$ *a unique solution* $u := Tv \in H^1(\Omega)$. *The solution operator* $T: H^{1/2}(\Gamma) \to H^1_P(\Omega)$ *is continuous.*

From the partial integration formula

$$\int_\Omega (\partial_j u v + u\, \partial_j v)\, dx = \int_\Gamma u v n_j\, ds \quad \text{for } u, v \in H^1(\Omega)$$

follow [23] the *first Green formula*

(3.2) $$\int_\Omega \bar v P u\, dx = \Phi_\Omega(u, v) - \int_\Gamma \partial_\nu u \bar v\, ds \quad \text{for } v \in H^1(\Omega), \quad u \in H^2(\Omega)$$

and the *second Green formula*

(3.3) $$\int_\Omega (uP'v - vPu)\, dx = \int_\Gamma (v\, \partial_\nu u - u\, \partial_{\tilde\nu} v)\, ds \quad \text{for } u, v \in H^2(\Omega)$$

where we defined

$$\partial_{\tilde\nu} u := \partial_\nu u - \sum_{j=1}^n n_j b_j u.$$

Now let u be a function defined on $\mathbb{R}^n$ such that

$$u_1 := u|_\Omega \in C^\infty(\bar\Omega) \quad \text{and} \quad u_2 := u|_{\Omega^c} \in C^\infty_{\text{comp}}(\bar\Omega^c),$$

where $\Omega^c := \mathbb{R}^n \setminus \bar\Omega$ is the exterior domain. Let $f := Pu|_{\mathbb{R}^n \setminus \Gamma}$ and let

$$[u] := \gamma_0 u_2 - \gamma_0 u_1 \quad \text{denote the jump of } u \text{ across } \Gamma.$$

Then there holds the *representation formula* (for $x \in \mathbb{R}^n \setminus \Gamma$)

(3.4) $$u(x) = Gf(x) + \int_\Gamma (\partial_{\nu(y)} G(x, y)[u(y)] - G(x, y)[\partial_{\tilde\nu(y)} u(y)])\, ds(y).$$

We shall need equations (3.2)–(3.4) for more general functions. To this purpose, we first define the conormal derivative in the weak sense by using the first Green formula (3.2). Recall γ_0^- from (2.1).

LEMMA 3.2. *Let* $u \in H^1_P(\Omega)$. *Then the mapping*

$$\varphi \mapsto \langle \gamma_1 u, \varphi\rangle := \Phi_\Omega(u, \gamma_0^- \varphi) - \int_\Omega Pu \cdot \gamma_0^- \varphi\, dx$$

is a continuous linear functional $\gamma_1 u$ on $H^{1/2}(\Gamma)$ that coincides for $u \in H^2(\Omega)$ with the functional defined by $\partial_\nu u|_\Gamma \in L_2(\Gamma) \subset H^{-1/2}(\Gamma)$.

(3.5) *The mapping $\gamma_1: H^1_P(\Omega) \mapsto H^{-1/2}(\Gamma)$ is continuous.*

The following lemma was shown by Grisvard [14] for the case $P = -\Delta$. The proof works verbatim for the present case.

LEMMA 3.3. *$C^\infty(\bar{\Omega})$ is dense in the Hilbert space $H^1_P(\Omega)$.*

Thus we can extend (3.2)-(3.4) by continuity.

LEMMA 3.4. (i) *The first Green formula in the form*

$$\int_\Omega \bar{v} Pu \, dx = \Phi_\Omega(u, v) - \langle \gamma_1 u, \overline{\gamma_0 v} \rangle \tag{3.6}$$

holds for all $u \in H^1_P(\Omega)$, $v \in H^1(\Omega)$.

(ii) *The second Green formula in the form*

$$\int_\Omega (uP'v - vPu) \, dx = \langle \widetilde{\gamma_1} u, \gamma_0 v \rangle - \langle \gamma_1 v, \gamma_0 u \rangle \tag{3.7}$$

holds for all u, $v \in H^1_P(\Omega)$. Here we define, corresponding to the definition of $\partial_{\tilde{\nu}}$:

$$\widetilde{\gamma_1} u := \gamma_1 u - \sum_{j=1}^n n_j b_j \gamma_0 u.$$

(iii) *The representation formula in the form*

$$u(x) = Gf(x) + \langle \gamma_1 G(x, \cdot), [\gamma_0 u] \rangle - \langle [\widetilde{\gamma_1} u], G(x, \cdot) \rangle \qquad (x \in \mathbb{R}^n \setminus \Gamma) \tag{3.8}$$

holds for all $u \in L_2(\mathbb{R}^n)$ with $u|_\Omega \in H^1(\Omega)$, $u|_{\Omega^c} \in H^1_{\text{comp}}(\Omega^c)$, and $f = Pu|_{\mathbb{R}^n \setminus \Gamma} \in L_2(\mathbb{R}^n)$.

The proof is immediate if we keep in mind that $H^1_{P'} = H^1_P(\Omega)$ and that γ_1 remains the same, whether defined from P or from P'. For (3.8) we need only (3.7) and the representation formula (3.4) for a smooth domain, let us say a small ball enclosing the point x.

The following result will be needed in the proof of the jump relations (Lemma 4.1).

LEMMA 3.5. *The trace map*

$$(\gamma_0, \gamma_1): \varphi \mapsto (\gamma_0 \varphi, \gamma_1 \varphi)$$

maps $C^\infty_{\text{comp}}(\mathbb{R}^n)$ onto a dense subspace of $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

Proof. Assume that for some $(\chi, \psi) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ there holds

$$\langle \chi, \gamma_1 \varphi \rangle = \langle \psi, \gamma_0 \varphi \rangle \quad \text{for all } \varphi \in C^\infty_{\text{comp}}(\mathbb{R}^n). \tag{3.9}$$

We have to show that $\chi = \psi = 0$.

Let $T\chi \in H^1_P(\Omega)$ be the solution of the Dirichlet problem (see Lemma 3.1)

$$P^0 T\chi = 0 \quad \text{in } \Omega, \qquad \gamma_0 T\chi = \chi.$$

For arbitrary $f \in L_2(\Omega)$ let $Sf \in H^1_P(\Omega)$ be the unique weak solution of the Dirichlet problem

$$P^0 Sf = f \quad \text{in } \Omega, \qquad \gamma_0 Sf = 0.$$

The second Green formula (3.7) for the operator $P^0 = P^{0'}$ gives

$$\begin{aligned} \langle \gamma_1 Sf, \chi \rangle &= \langle \gamma_1 Sf, \gamma_0 T\chi \rangle - \langle \gamma_1 T\chi, \gamma_0 Sf \rangle \\ &= \int_\Omega (Sf \cdot P^0 T\chi - P^0 Sf \cdot T\chi) \, dx = -\int_\Omega f T\chi \, dx. \end{aligned} \tag{3.10}$$

Now (3.9) holds for all $\varphi \in H^1_P(\Omega)$ due to Lemma 3.3, in particular for $\varphi = Sf$; hence

$$\langle \gamma_1 Sf, \chi \rangle = \langle \psi, \gamma_0 Sf \rangle = 0.$$

This gives from (3.10)

$$\int_\Omega f T\chi \, dx = 0 \quad \text{for all } f \in L_2(\Omega).$$

Thus $T\chi = 0$ whence $\chi = \gamma_0 T\chi = 0$. From (3.9) now follows

$$\langle \psi, \gamma_0 \varphi \rangle = 0 \quad \text{for all } \varphi \in H^1(\Omega)$$

which implies $\psi = 0$ because of the surjectivity of

$$\gamma_0: H^1(\Omega) \to H^{1/2}(\Gamma). \qquad \square$$

The continuity of the simple layer potential operator, Theorem 1(i) and (iii), will follow from an extension of Gagliardo's Trace Lemma, which seems to be new.

LEMMA 3.6. *For $s \in (\frac{1}{2}, \frac{3}{2})$ the trace map*

$$\gamma_0: u \mapsto \gamma_0 u = u|_\Gamma: H^s_{\text{loc}}(\mathbb{R}^n) \to H^{s-1/2}(\Gamma) \quad \textit{is continuous.}$$

This result for $s = \frac{3}{2}$, from which the whole range $s \in (\frac{1}{2}, \frac{3}{2}]$ would follow by interpolation, is claimed by Jerison and Kenig [18]. However, there seems to be no proof available. The proof of Lemma 3.6 is given at the end of § 4.

The last tool we need is Nečas' result on the boundary regularity for the Dirichlet and Neumann problems.

LEMMA 3.7. *For $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, the mapping $\gamma_1 T: H^{1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma)$ is continuous, and $\gamma_1 Tv \in H^{-1/2+\sigma}(\Gamma)$ implies $v \in H^{1/2+\sigma}(\Gamma)$.*

Remarks. (i) Here the result for $\sigma < 0$ means, as above, the existence of a continuous extension of the map defined for $\sigma = 0$.

(ii) Nečas [23] showed that solutions of the Dirichlet problem with Dirichlet data in $H^1(\Gamma)$ have their Neumann data (i.e., conormal derivatives)in $L_2(\Gamma)$ and conversely. This is proved by applying an identity of Rellich, generalized to arbitrary second order equations by Payne and Weinberger. Thus it uses only partial integration and is completely elementary. The same argument has been used by Jerison and Kenig [16], [17] and Verchota [27], [21]. Having thus proved the result for $\sigma = \frac{1}{2}$, Nečas deduces the result for $\sigma = -\frac{1}{2}$ from a duality argument. The whole range $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$ then clearly follows by interpolation.

4. The proofs.

Proof of Theorem 1 (i) *and* (iii). By definition (2.5) we can write the simple layer potential as

$$K_0 = G \circ \gamma_0', \tag{4.1}$$

where γ_0' is the adjoint of the trace map γ_0. By Lemma 3.6 we find that $\gamma_0: H^{-s+1/2}(\Gamma) \to H^{-s}_{\text{comp}}(\mathbb{R}^n)$ is continuous for $s \in (\frac{1}{2}, \frac{3}{2})$. The operator G is a pseudodifferential operator of order -2 on $\mathbb{R}^n$, mapping $H^{-s}_{\text{comp}}(\mathbb{R}^n) \to H^{-s+2}_{\text{loc}}(\mathbb{R}^n)$ continuously for any $s \in \mathbb{R}$. Thus Theorem 1(i) follows. The continuity of the operator $A = \gamma_0 K_0$ then follows by a second application of Lemma 3.6. $\square$

Remark. If instead of Lemma 3.6, we use only the classical result (2.1), we find for theorem 1(i) only a range $\sigma \in [0, \frac{1}{2})$, and for (iii) only $\sigma = 0$ remains.

Next we use the representation formula (3.8) in order to write the double layer potential in terms of the simple layer potential. Writing (3.8) for a solution of the

Dirichlet problem $u=Tv\in H^1_P(\Omega)$ for $v\in H^{1/2}(\Gamma)$, we obtain $Tv=-K_1v+K_0\tilde{\gamma}_1Tv$; hence

(4.2) $$K_1=(-1+K_0\tilde{\gamma}_1)T.$$

This immediately implies that

(4.3) $$K_1: H^{1/2}(\Gamma)\to H^1(\Omega)\quad \text{is continuous.}$$

Thus, using (2.1) and Lemma 3.2, we obtain all statements of Theorem 1 for $\sigma=0$ (i.e., in the "energy norm").

This will suffice to prove Theorem 2. The remaining cases of Theorem 1 will be shown together with Theorem 3.

Now we prove *jump relations* for the layer potentials. We use the notation introduced above

$$[\gamma_j u]:=\gamma_j(u|_{\Omega^c})-\gamma_j(u|_\Omega)\quad \text{for } j=0,1.$$

LEMMA 4.1.

$$[\gamma_0K_0\psi]=0,\qquad [\gamma_1K_0\psi]=-\psi\quad \textit{for } \psi\in H^{-1/2}(\Gamma),$$

$$[\gamma_0K_1v]=v,\qquad [\tilde{\gamma}_1K_1v]=0\quad \textit{for } v\in H^{1/2}(\Gamma).$$

Proof. Let $\psi\in H^{-1/2}(\Gamma)$ and $u=K_0\psi\in H^1_{\mathrm{loc}}(\mathbb{R}^n)$. The equality $\gamma_0(u|_\Omega)=\gamma_0(u|_{\Omega^c})$ follows from the definition of γ_0. From (4.1) we obtain $Pu=\gamma_0'\psi$, if we apply P in the distributional sense to u. For any test function $\varphi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ we thus obtain

(4.4) $$\int_{\mathbb{R}^n} uP'\varphi\,dx=\langle Pu,\varphi\rangle=\langle\gamma_0'\psi,\varphi\rangle=\langle\psi,\gamma_0\varphi\rangle.$$

On the other hand, the second Green formula (3.7) gives

$$\int_\Omega uP'\varphi\,dx=\langle\tilde{\gamma}_1(u|_\Omega),\gamma_0\varphi\rangle-\langle\gamma_1\varphi,\gamma_0u\rangle.$$

The corresponding formula for Ω^c is

$$\int_{\Omega^c} uP'\varphi\,dx=-\langle\tilde{\gamma}_1(u|_{\Omega^c}),\gamma_0\varphi\rangle+\langle\gamma_1\varphi,\gamma_0u\rangle.$$

Adding both, we obtain with $[\gamma_0u]=[\gamma_0\varphi]=[\tilde{\gamma}_1\varphi]=0$

(4.5) $$\int_{\mathbb{R}^n} uP'\varphi\,dx=-\langle[\tilde{\gamma}_1u],\gamma_0\varphi\rangle.$$

Comparison of (4.4) and (4.5) gives $[\gamma_1u]=-\psi$, and from $[\gamma_0u]=0$ follows $[\gamma_1u]=[\tilde{\gamma}_1u]=-\psi$.

In order to show the jump relations for the double layer potential, we choose $v\in H^{1/2}(\Gamma)$ and $\varphi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and define $u=K_1v$. Then again the second Green formula gives

(4.6) $$\int_{\mathbb{R}^n} uP'\varphi\,dx=-\langle[\tilde{\gamma}_1u],\gamma_0\varphi\rangle+\langle[\gamma_0u],\gamma_1\varphi\rangle.$$

On the other hand, the definition of K_1 gives $u=K_1v=G(\gamma_1'v)$, where the compactly supported distribution on $\mathbb{R}^n$, $\gamma_1'v$ is defined by

$$\langle\gamma_1'v,\chi\rangle=\int_\Gamma v\,\partial_\nu\chi\,ds=\langle v,\gamma_1\chi\rangle\quad \text{for all } \chi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n).$$

Thus $Pu=\gamma_1'v$, implying

(4.7) $$\int_{\mathbb{R}^n} uP'\varphi\,dx=\langle Pu,\varphi\rangle=\langle v,\gamma_1\varphi\rangle.$$

Comparison of (4.6) and (4.7) gives

(4.8) $$\langle v-[\gamma_0u],\gamma_1\varphi\rangle=\langle-[\tilde{\gamma}_1u],\gamma_0\varphi\rangle\quad \text{for all } \varphi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n).$$

Finally we apply Lemma 3.5 which allows us to infer from (4.8):

$$v-[\gamma_0u]=0=[\tilde{\gamma}_1u].$$ □

Proof of Theorem 2. Choose $v\in H^{-1/2}(\Gamma)$ and define $u=-K_0v$. Then according to Lemma 4.1, we have the jump relations

(4.9) $$[\gamma_0u]=0;\ \text{hence } \gamma_0(u|_\Omega)=-Av=\gamma_0(u|_{\Omega^c}),\ \text{and } [\gamma_1u]=v.$$

Next we choose $\chi\in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with $\chi=1$ on a neighborhood of $\bar{\Omega}$ and define $u_1:=u|_\Omega$, $u_2:=\chi u|_{\Omega^c}$.

Next we add the first Green formula (3.6) for $u=v=u_1$ and its counterpart for Ω^c for $u=v=u_2$ and obtain using (4.9)

(4.10) $$\Phi_\Omega(u_1,u_1)+\Phi_{\Omega^c}(u_2,u_2)-\int_{\Omega^c}\overline{u_2}Pu_2\,dx=-\langle[\gamma_1u],\overline{\gamma_0u}\rangle=\langle v,\overline{Av}\rangle.$$

Here Φ_{Ω^c} is defined in accordance with (2.3).

Equation (4.10) now allows us to transfer the Gårding inequality for the operator P, which we assumed to hold, to the Gårding inequality on the boundary for the operator A.

The term $\int_{\Omega^c}\overline{u_2}Pu_2\,dx$ gives rise to a compact bilinear form in $v\in H^{-1/2}(\Gamma)$ because Pu_2 has compact support in Ω^c and the mapping $v\mapsto u_2$ is continuous from $H^{-1/2}(\Gamma)$ to $C^\infty(\Omega^c)$. From the continuity of the trace mapping γ_1 (Lemma 3.2) we obtain an estimate

$$\|v\|^2_{H^{-1/2}(\Gamma)}=\|\gamma_1u_2-\gamma_1u_1\|^2_{H^{-1/2}(\Gamma)}$$

(4.11) $$\leqq C(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega^c)}+\|Pu_1\|^2_{L_2(\Omega)}+\|Pu_2\|^2_{L_2(\Omega^c)}).$$

Here on the right-hand side, Pu_1 vanishes and $\|Pu_2\|^2_{L_2(\Omega^c)}$ is a compact term.

Finally, the principal part of the right-hand side of (4.11) can be estimated from above up to compact terms by the left-hand side of (4.10) due to Gårding's inequality (2.4) which we assumed to hold. Thus (2.9) is proved.

In order to prove the strong ellipticity of the operator D, i.e., estimate (2.10), we proceed analogously.

For $v\in H^{-1/2}(\Gamma)$ we define $u=K_1v$. Then we find the jump relations

(4.12) $$[\gamma_0u]=v\ \text{and}\ [\tilde{\gamma}_1u]=0;\ \text{hence } \tilde{\gamma}_1u_1=\tilde{\gamma}_1u_2=-Dv,$$

where u_1 and u_2 are defined from u as above. Then again the first Green formula gives

(4.13) $$\Phi_\Omega(u_1,u_1)+\Phi_{\Omega^c}(u_2,u_2)-\int_{\Omega^c}\overline{u_2}Pu_2\,dx=\langle Dv,\bar{v}\rangle.$$

This time the trace lemma (2.1) implies

$$(4.14)\qquad \|v\|^2_{H^{1/2}(\Gamma)}=\|\gamma_0 u_2-\gamma_0 u_1\|^2_{H^{1/2}(\Gamma)}\le C(\|u_1\|^2_{H^1(\Omega)}+\|u_2\|^2_{H^1(\Omega^c)}),$$

and again, (2.10) follows from (4.13) and (4.14) together with Gårding's inequality (2.4). □

The derivation of convergence and stability for Galerkin approximation schemes from strong ellipticity is standard by now [13], [26], as are the approximation properties of the finite element function spaces [1], [2]; thus proofs of Theorems 4 and 5 need not be given here.

Next we show regularity in the domain for the Dirichlet problem

LEMMA 4.2. *For $\sigma\in(-\frac12,\frac12)$ the mapping $T: H^{1/2+\sigma}(\Gamma)\to H_P^{1+\sigma}(\Omega)$ is continuous.*

Proof. We choose a domain B containing $\bar\Omega$ in its interior, e.g., a large enough ball. Let $\Omega_2:=B\setminus\bar\Omega$ and $T_2: v\mapsto u=T_2v$ be the solution operator of the Dirichlet problem

$$P^0u=0\quad\text{in }\Omega_2,\qquad \gamma_0u=v,\qquad u|_{\partial B}=0.$$

Now choose $v\in H^1(\Gamma)$ and define

$$u=Tv\quad\text{in }\Omega,\qquad u=T_2v\quad\text{in }\Omega_2.$$

Then the representation formula (3.8) applies and gives with $f=0$ and $[\gamma_0u]=0$

$$(4.15)\qquad u=-K_0[\overline{\gamma_1}u]+\int_{\partial B}\partial_\nu u(y)G(\cdot,y)\,ds(y)\quad\text{in }\Omega\cup\Omega_2.$$

Now we know from the boundary regularity result for the Dirichlet problem (Lemma 3.7) that there are estimates

$$\|\partial_\nu u|_{\partial B}\|_{H^{-1/2+\sigma}(\partial B)}+\|\overline{\gamma_1}Tv\|_{H^{-1/2+\sigma}(\Gamma)}+\|\overline{\gamma_1}T_2v\|_{H^{-1/2+\sigma}(\Gamma)}\le C\|v\|_{H^{1/2+\sigma}(\Gamma)}$$

even for $\sigma\in[-\frac12,\frac12]$. Hence the continuity of the simple layer potential operator, Theorem 1(i) gives with (4.15) the desired estimate

$$\|u\|_{H^{1+\sigma}(\Omega)}\le C\|v\|_{H^{1/2+\sigma}(\Gamma)}.\qquad\square$$

Remark. The endpoint result $\sigma=\frac12$ was shown by Jerison and Kenig [16] using Dahlberg's estimates for the Poisson kernel [10].

LEMMA 4.3. *For $s\in(\frac12,\frac32)$ the trace map $\gamma_1: H_P^s(\Omega)\to H^{s-3/2}(\Gamma)$ is continuous.*

Proof. For $u\in H_P^1(\Omega)$ and arbitrary $\varphi\in H^{1/2}(\Gamma)$, $v:=T\varphi$, we can apply the second Green formula (3.7) for the operator P^0 to obtain

$$\langle\gamma_1u,\varphi\rangle=\langle\gamma_0u,\gamma_1T\varphi\rangle-\int_\Omega P^0uT\varphi\,dx.$$

This can be written as

$$(4.16)\qquad \gamma_1=(\gamma_1T)'\gamma_0-T'P^0.$$

The first member on the right-hand side is continuous from $H^s(\Omega)$ to $H^{s-3/2}(\Gamma)$ due to Lemmas 3.6 and 3.7. The second member is continuous from $H^s(\Omega)$ to $H^t(\Gamma)$ for all s and $t<0$ due to Lemma 4.2. □

As a corollary, Theorem 1(iv) follows from Theorem 1(i).

Proof of Theorem 1(ii), (v), and (vi). It suffices to show (ii). If we apply (4.2), this follows from Lemmas 4.2, 4.3, and Theorem 1(i). The proof of Theorem 1 is complete. □

Proof of Theorem 3. Let $\psi\in H^{-1/2}(\Gamma)$ and $A\psi\in H^{1/2+\sigma}(\Gamma)$. We show $\psi\in H^{-1/2+\sigma}(\Gamma)$. The a priori estimate (2.11) then follows from the closed graph theorem.

Define $u=K_0\psi$. Then we have $A\psi=\gamma_0u$ and $\psi=-[\gamma_1u]$ by Lemma 4.1. Thus u solves in Ω and Ω^c the Dirichlet problem with Dirichlet data $A\psi\in H^{1/2+\sigma}(\Gamma)$. According to Lemma 3.7, the Neumann data, and hence ψ, are in $H^{-1/2+\sigma}(\Gamma)$.

Now if $B\psi\in H^{-1/2+\sigma}(\Gamma)$, then $\gamma_1(u|_\Omega)=B\psi\in H^{-1/2+\sigma}(\Gamma)$, so that also $A\psi=\gamma_0(u|_\Omega)\in H^{1/2+\sigma}(\Gamma)$ holds.

The remaining statements follow in a similar way using the double layer potential and again Lemmas 3.7 and 4.1. □

Proof of Lemma 3.6. The statement is local, so we may assume that the boundary Γ is of the form

$$\Gamma=\{(x',x_n)\in\mathbb{R}^n\mid x'\in\mathbb{R}^{n-1};\ x_n=\psi(x')\}$$

with a function $\psi:\mathbb{R}^{n-1}\to\mathbb{R}$ that is uniformly Lipschitz, i.e., $\|\text{grad}\,\psi\|_{L_\infty(\mathbb{R}^{n-1})}<\infty$.

For functions $f\in C^\infty_{\text{comp}}(\mathbb{R}^n)$ define

$$f_\psi(x',x_n):=f(x',x_n+\psi(x')).$$

We then have to show an estimate for $1<s<\frac32$

$$(4.17)\qquad \|f_\psi(\cdot,0)\|_{H^{s-1/2}(\mathbb{R}^{n-1})}\le C\|f\|_{H^s(\mathbb{R}^n)}\quad\text{for all }f\in C^\infty_{\text{comp}}(\mathbb{R}^n).$$

The problem is that in general for $s>1$, $f_\psi\notin H^s(\mathbb{R}^n)$ and therefore the usual trace lemma cannot be applied to f_ψ. We show first that the mapping $f\mapsto f_\psi$ leaves a certain anisotropic Sobolev space X^s invariant, and then that in X^s there holds the trace estimate

$$(4.18)\qquad \|f(\cdot,0)\|_{H^{s-1/2}(\mathbb{R}^{n-1})}\le C\|f\|_{X^s}\quad\text{for all }f\in C^\infty_{\text{comp}}(\mathbb{R}^n).$$

For the definition of X^s we identify a function $f\in C^\infty_{\text{comp}}(\mathbb{R}^n)$ with the $C^\infty(\mathbb{R}^{n-1})$-valued function on $\mathbb{R}$,

$$x_n\mapsto f(\cdot,x_n).$$

Thus $H^s(\mathbb{R}^n)=H^s(\mathbb{R};L_2(\mathbb{R}^{n-1}))\cap L_2(\mathbb{R};H^s(\mathbb{R}^{n-1}))$. We define

$$X^s:=H^s(\mathbb{R};L_2(\mathbb{R}^{n-1}))\cap H^{s-1}(\mathbb{R};H^1(\mathbb{R}^{n-1})).$$

If $\hat f$ is the Fourier transform of f, we define the norm in X^s by

$$\|f\|^2_{X^s}:=\int_{-\infty}^{\infty}\int_{\mathbb{R}^{n-1}}\{(1+|\xi_n|)^{2s}+(1+|\xi_n|)^{2(s-1)}\cdot(1+|\xi'|)^2\}|\hat f(\xi',\xi_n)|^2\,d\xi'\,d\xi_n.$$

Thus

$$(4.19)\qquad \|f\|_{X^s}\le C\|f\|_{H^s(\mathbb{R}^n)}.$$

If we denote by $\tilde f(x',\xi_n)$ the Fourier transform of f with respect to the last variable, we have

$$\|f\|^2_{H^t(\mathbb{R};H^r(\mathbb{R}^{n-1}))}=\int_{-\infty}^{\infty}(1+|\xi_n|)^{2t}\|\tilde f(\cdot,\xi_n)\|^2_{H^r(\mathbb{R}^{n-1})}\,d\xi_n.$$

Now we have

$$(\widetilde{f_\psi})(x',\xi_n)=e^{i\psi(x')\xi_n}\tilde f(x',\xi_n).$$

Hence

$$(4.20)\qquad \|f_\psi\|_{H^t(\mathbb{R};L_2(\mathbb{R}^{n-1}))}=\|f\|_{H^t(\mathbb{R};L_2(\mathbb{R}^{n-1}))}\quad\text{for all }t\in\mathbb{R}.$$

For $k=1, \cdots, n-1$ we have

$$(\widetilde{\partial_k f_\psi})(x', \xi_n) = e^{i\psi(x')\xi_n}(\widetilde{\partial_k f})(x', \xi_n) + i\xi_n(\partial_k\psi)(x')\, e^{i\psi(x')\xi_n}\tilde{f}(x', \xi_n).$$

This implies that

$$\|(\widetilde{\partial_k f_\psi})(\cdot, \xi_n)\|^2_{L_2(\mathbb{R}^{n-1})} \leqq \|(\widetilde{\partial_k f})(\cdot, \xi_n)\|^2_{L_2(\mathbb{R}^{n-1})} + \xi_n^2 \|\operatorname{grad} \psi\|^2_{L_\infty(\mathbb{R}^{n-1})} \|\tilde{f}(\cdot, \xi_n)\|^2_{L_2(\mathbb{R}^{n-1})}.$$

Hence

$$\|f_\psi\|^2_{H^t(\mathbb{R};\, H^1(\mathbb{R}^{n-1}))} \leqq \|f\|^2_{H^t(\mathbb{R};\, H^1(\mathbb{R}^{n-1}))} + C\|f\|^2_{H^{t+1}(\mathbb{R};\, L_2(\mathbb{R}^{n-1}))} \tag{4.21}$$

for all $t \in \mathbb{R}$.

Formulae (4.20) and (4.21) together imply the estimate

$$\|f_\psi\|_{X^s} \leqq \|f\|_{X^s} \quad \text{for all } s \in \mathbb{R}. \tag{4.22}$$

Next we show (4.18). We use the fact that with

$$m(\xi', \xi_n) := (1+|\xi_n|)^{2s} + (1+|\xi_n|)^{2s-2} \cdot (1+|\xi'|)^2$$

we have

$$\int_{-\infty}^{\infty} m(\xi', \xi_n)^{-1}\, d\xi_n = C_s(1+|\xi'|)^{1+2s} < \infty \quad \text{for } 1 \leqq s < \tfrac{3}{2}.$$

Hence using the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\|f_\psi(\cdot, 0)\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})} & \int_{\mathbb{R}^{n-1}} (1+|\xi'|)^{2s-1} \left| \int_{-\infty}^{\infty} \widehat{f_\psi}(\xi', \xi_n)\, d\xi_n \right|^2 d\xi' \\
&\leqq \int_{\mathbb{R}^{n-1}} (1+|\xi'|)^{2s-1} \left(\int_{-\infty}^{\infty} m(\xi', \xi_n)^{-1}\, d\xi_n \right) \\
&\quad \times \left(\int_{-\infty}^{\infty} m(\xi', \xi_n) |\widehat{f_\psi}(\xi', \xi_n)|^2\, d\xi_n \right) d\xi' \\
&= C \int_{\mathbb{R}^n} m(\xi', \xi_n) |\widehat{f_\psi}(\xi', \xi_n)|^2\, d\xi_n\, d\xi' = C\|f_\psi\|^2_{X^s}.
\end{aligned}$$

This together with the estimates (4.22) and (4.19) gives the desired estimate (4.17). □

5. Concluding remarks. (i) Along the same lines as presented here it is also possible to easily deduce invertibility results for integral equations involving the operators A, B, C, and D. Note that, for instance, by Theorem 2 the operators A and D are Fredholm operators of index 0 in the energy norm spaces. Thus if we assume injectivity, which in turn can be inferred from positivity of the bilinear form Φ_Ω, we obtain bijectivity. For the operator A this holds for the case of the Laplace equation and the standard fundamental solution in dimension $n \geqq 3$ and for $n=2$ if the analytic capacity of Γ is different from one. By Theorem 3 and duality arguments, bijectivity holds for the whole range of Sobolev spaces given in Theorem 1(ii). In this way we get results about the *solvability* of the boundary value problems by means of the boundary integral equations. Theorems 4 and 5 then are really statements about the numerical solution of boundary value problems by means of the so-called boundary element method [28]. In practice, this method is frequently used to solve (also mixed) boundary value problems of three-dimensional linear elasticity on domains with corners

and edges [25], [3]. For these problems, the present paper yields convergence proofs and asymptotic error estimates for Galerkin methods.

(ii) If the domain is more regular than merely Lipschitz, e.g., a smooth image of a polyhedron, then higher regularity results should be possible and they should improve with higher dimension. For the Dirichlet problem this is well known, but for the boundary integral equations higher regularity has been studied, to the best of the author's knowledge, only in the case of plane domains (see [6], [8], and the literature quoted therein).

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ON THE SHARPNESS OF WEYL'S ESTIMATES FOR EIGENVALUES OF SMOOTH KERNELS, II*

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Abstract. The estimate $\lambda_n = o(1/n)$ obtained by H. Weyl (1912) for the nth largest in modulus eigenvalue λ_n of any symmetric Fredholm operator on $L^2[0, 1]^2$ with kernel in $C^1[0, 1]^4$ is shown to be best possible in the sense that for any increasing sequence $\alpha_n \to \infty$ there exist such operators whose nth eigenvalue is not $o(1/n\alpha_n)$. The construction of the counterexample makes use of Rudin–Shapiro polynomials. The corresponding result for positive definite operators is proved with a simpler counterexample. The methods generalise to the case $L^2[0, 1]^m (m \geqq 3)$ without further difficulty.

Key words. eigenvalue, operator, kernel, asymptotics

AMS(MOS) subject classification. 45C

1. Introduction. If $K(x, t) = \overline{K(t, x)} \in L^2[a, b]^2$ then

$$(Tf)(x) = \int_a^b K(x, t)f(t)\,dt$$

defines a compact symmetric operator T on the Hilbert space $L^2[a, b]$. Such an operator T has a real null sequence $(\lambda_n)_{n \geqq 1}$ of eigenvalues which we can assume has been enumerated so that

$$|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n| \geqq \cdots.$$

H. Weyl showed in [4] that if $K(x, t) \in C^1[a, b]^2$, i.e., $K(x, t)$ has continuous partial derivatives, then $\lambda_n = o(1/n^{3/2})$. We showed in [1] that if $K(x, t) \in PDC^1[a, b]^2$, i.e., $K(x, t)$ is positive definite and $\in C^1[a, b]^2$, then $\lambda_n = o(1/n^2)$.

Similar results are true for operators of the form

$$(Tf)(x, y) = \int_a^b \int_a^b K(x, y, t, u)f(t, u)\,dt\,du$$

where $K(x, y, t, u) = \overline{K(t, u, x, y)} \in L^2[a, b]^4$ and $f(t, u) \in L^2[a, b]^2$. The estimates are $\lambda_n = o(1/n)$ for $K(x, y, t, u) \in C^1[a, b]^4$ and $\lambda_n = o(1/n^{3/2})$ for $K(x, y, t, u) \in PDC^1[a, b]^4$. The proofs are similar to those for kernels in two variables.

In [2] we considered the sharpness of the estimates for two variable kernels. Here we consider four variable kernels.

2. Double Fourier series. Any $k(t, u) \in L^2[0, 1]^2$ has a double Fourier series

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn}\, e^{2\pi i(mt+nu)}$$

where

$$c_{mn} = \int_0^1 \int_0^1 k(t, u)\, e^{-2\pi i(mt+nu)}\,dt\,du$$

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