GLC Project I

"Corner Singularities and Analytic Regularity for Linear Elliptic Systems"

Part I: Smooth domains

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Foreword to Part I

This is a preliminary version of the first part of a book project that will consist of four parts. This first part was basically finished more than a year ago. We are making it available in electronic form now, because there is a demand for some of the technical tools it provides, in particular a detailed presentation of analytic elliptic regularity estimates in the neighborhood of smooth boundary points. We hope to be able to finish the whole project soon and to publish all four parts, but in the meantime this first part can be used as a starting point for proofs of elliptic regularity estimates in more complicated situations.

The origin of this book project is twofold. In the beginning there were two separate needs we perceived as not being satisfied by the currently available literature on corner singularities:

On one hand, we saw a demand for a general introduction into the regularity results for elliptic boundary value problems on domains with corners and edges and into the main techniques for their analysis. This demand comes in particular from applied mathematicians, engineers and numerical analysts for whom the precise knowledge of the asymptotic behavior near singular points of the geometry can be of crucial importance, the availability of norm estimates in function spaces often having a direct impact on the efficiency of models and algorithms, while the presence of certain singularities can show their limitations.

On the other hand, the analysis of high order approximation methods, in particular the hp version of the finite element method, requires regularity results in *spaces of analytic functions*, and these were not available for sufficiently general classes of piecewise smooth domains, including polyhedral domains in three dimensions.

We found that these two motivations complement each other in a natural way: Analytic estimates appear as the ultimate goal of regularity estimates in various situations, and the way to their achievement can give useful guidance for shaping the presentation of the tools even for the analysis of finite regularity. Conversely, the technical difficulty of establishing analytic regularity results imposes a systematic approach beginning with simple but technically complete presentations of simple situations, and then successively building on these to proceed to more and more complicated situations.

Level 1 of complexity is constituted by smooth domains, and this is the subject of this first part of the book. Not only a refined understanding of the results, but also of their proofs in the smooth case is necessary if one wants to go on to more complicated domains such as corner or edge domains, or eventually, polyhedral domains.

We found that the proof of results on elliptic regularity in spaces of analytic functions that were obtained in the late 1950s for smooth domains can be improved in a decisive way. This improvement consists in turning the proof of analytic regularity into the proof of a family of higher order *a priori estimates* that are in a characteristic way uniform with respect to the derivation order. This type of estimates has a simple scaling behavior under coordinate transformations and can therefore be transported to corner or edge domains by a simple procedure, the *dyadic partition* technique. In our proof of these analytic a priori estimates, besides the classical Morrey-Nirenberg techniques of nested open sets and difference quotients, a new ingredient is a Cauchy-type estimate for coordinate transformations that is based on the Faà di Bruno formula for derivatives of composite functions.

This first part can also serve as a general introduction into the subject of regularity for linear elliptic systems with smooth coefficients on smooth domains. We treat regularity in L^2-based Sobolev spaces for a general class of second order elliptic systems and corresponding boundary operators that cover, in particular, many elliptic problems in variational form. Starting from the regularity of the variational solution, we follow the improvement of the regularity of the solution as the regularity of the data is raised, first in the interval of low regularity between H^1 and H^2, and then starting from H^2, going to ever higher regularity and finally to analytic regularity. Supported by the discussion of many examples, some of them new, such as the variational formulation of the electromagnetic impedance problem, we hope to provide new insight into this classical subject that is still very much alive.

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Introduction (February 2010)

Prehistory of this book

Questions of boundary regularity were already studied at the beginning of the 20th century in the framework of conformal mappings, where the appearance of singularities at corner points of the boundary – related to non-integer powers of the complex variable – is quite obvious. An asymptotic expansion of the conformal mapping at a corner point was given as early as 1911 by Lichtenstein [55, 56].

In the 1950s, the modern theory of elliptic boundary value problems was developed, culminating in the classical papers by Agmon, Douglis and Nirenberg [4, 5] on the regularity of solutions of boundary value problems for linear elliptic systems on smooth domains in Hölder and Sobolev spaces. Around the same time, analytic regularity was proved by Morrey and Nirenberg [70]. There exist many books presenting this basic elliptic theory, some of them focussing on the fundamental theory of linear elliptic boundary value value problems, from the classical treatises by Lions and Magenes [57, 58, 59], up to more recent ones [94, 91, 6], but many of them rather concentrating on numerous generalizations like nonlinear and degenerate elliptic or non-elliptic problems [69, 66, 38, 47, 48, 49].

In the field of domains with corner points, one work stands out, namely the 1967 paper by V. Kondrat'ev [52], which is still a standard reference for results on regularity in weighted Sobolev spaces, on the decomposition into regular and singular parts, and for the technique of Mellin transformation on domains with conical points. Kondrat'ev's results and techniques will be presented in Part II of this book.

In the past four decades, a large literature about corner problems has developed. Out of the many developments, let us mention five which have been of particular importance for the genesis of this book:

1. V. Maz'ya has contributed, together with many coworkers such as B. Plamenevskii, S. Nazarov, V. Kozlov, J. Rossmann, a large body of results. Among them are results about the estimates of coefficients of singularities in L^p Sobolev spaces and Hölder spaces, about problems with non-smooth and singular coefficients, about domains with edges and variable coefficients, about strips free of poles of the Mellin resolvent, about estimates of Green functions, and these are often the strongest and most general available results of their kind. Many of theses papers will be quoted at the appropriate places in our book. While we are not striving for maximal generality in this book, and we are therefore often not using the most general framework studied by Maz'ya and coworkers, we have found and adopted many useful ideas for the presentation of the basics and for nice proofs in the book by Kozlov-Maz'ya-Rossmann [53].

2. B.-W. Schulze has been leading a long-standing project to study pseudodifferential operators on piecewise smooth manifolds. These are tools appearing in the reduction to the boundary of elliptic boundary value problems on domains with piecewise smooth boundaries, thus allowing to analyze singularities at corners and edges. The emphasis in that project is on generality of ideas and on the interaction of these ideas with the general theory of partial differential operators and with domains of mathematics such as global analysis, topology, differential geometry. Although the main results of the project have been published in several books [83, 86, 87, 88, 89, 73], they are not easily accessible to the non-specialist readership we have in mind for our book.

3. P. Grisvard started writing on corner problems in the 1970s. His work culminated in two influential books [42, 43] which are accessible to a large readership and are still points of reference for many techniques and results on corner singularities, in particular on two-dimensional mixed boundary value problems. There exists now, however, a body of knowledge, for example about variable coefficients, about edge singularities, about higher-dimensional corners, that is not covered by Grisvard's books. It is one of the objectives of the present book to become a similar point of reference for some of these more modern techniques and results. In the wake of Grisvard's work, S. Nicaise began studying boundary value problems on polygonal ramified spaces [78] and then interface problems on polygonal domains [79], and we are building on this to include results on interface and transmission problems in our book.

4. M. Dauge's Lecture Notes volume [32] contains results about asymptotics near polyhedral corners and about regularity in standard non-weighted Sobolev spaces that are not available elsewhere in the literature. This 1988 book is currently out of print, and it has a reputation of being difficult to read, so that there is a real need for a new presentation of these results. This is one of the two main motivations for writing the present book. It is intended to contain the principal results and techniques of [32] and to present them in a more easily accessible manner, by trying at the same time to be more explicit in the construction of the technical tools and to show a broader landscape of applications and examples in which these results are embedded.

5. I. Babuška introduced in the 1980s the analysis of a type of higher degree approximations of solutions of elliptic boundary value problems with singular points on the boundary, the hp version of the finite element method. This method uses polynomials of increasing degree to approximate the smooth parts of the solution, and it takes care of corner singularities by employing very strong mesh refinements near the boundary singularities. It turns out that in this way one can obtain exponential convergence of the approximations, which makes these methods extremely efficient if they are applicable [11]. The fast convergence is based on the analyticity of the solutions, and in a series of papers, B. Guo and I. Babuška proved the appropriate kind of regularity in spaces of weighted analytic functions, their "countably normed spaces", for some classes of domains and examples [9, 10]. This covers the classical boundary value problems on two-dimensional piecewise analytic domains. For the practically important case of three-dimensional domains with edges and polyhedral corners [44], a corresponding proof of analytic regularity results has not yet been published. Such a proof needs to be based on a solid foundation of careful proofs of analytic regularity for problems posed on the constituents of a polyhedral corner domain, namely first domains with smooth boundaries, then conical points, then smooth edges, and finally finite edges. It is the second main motivation for writing the present book to close the gap by laying this foundation and building on it to give a complete proof of analytic regularity of solutions of elliptic boundary value problem on domains with conical points and with edges in any dimension, and on domains with polyhedral corners in three dimensions. As a byproduct, we obtain the higher order finite regularity in anisotropic weighted Sobolev spaces that is needed for other high degree approximation methods [16, 17].

How to read this book

Inherent to the material presented in this book is a natural linear hierarchy of complexity: Later parts require the techniques and results of earlier parts in order to be fully understood.

This hierarchy is determined already by the geometric structure of the piecewise smooth domains we are considering, and it is reflected by the presence of increasing levels of complexity of the tools necessary to analyze the elliptic operators and the solutions of boundary value problems on these domains. This hierarchy is therefore unavoidable, and it poses a problem for the readability of the book: Each general theorem presented – and the more interesting ones naturally appear towards the end of the book – requires, in principle, reading everything that comes before.

In order to mitigate this technical difficulty, we chose a special structure for each of the parts and chapters of the book: Besides a specific introduction, there is always an enlarged table of contents and a special section called “Essentials”. The aim of the “Essentials” section is to give quick access to all of the notations, to the most important ideas and to the main results of the corresponding chapter. In this way, reading the details of a later chapter will be possible without reading all the details of the preceding chapters: Reading only the “Essentials” of the earlier chapters should be sufficient. For some of the chapters, in particular those devoted to concrete examples, this will not quite suffice, because in order to fully understand the technical details in the discussion of an example, one will have to consult the details of the techniques presented in the chapter treating the general theory corresponding to the example. In this case, the “Essentials” section can help to quickly find the corresponding theorems and proofs where these techniques are introduced.

The four parts of the book correspond to four levels of complexity of the set of singular points of piecewise smooth domains: The lowest level corresponds to smooth domains whose points are either interior points or points on smooth boundaries. The next level are corner domains in which isolated singular points can appear. In the third part on edge domains, singular points are allowed that lie on smooth curves, or more generally, on smooth submanifolds. In the last part, corner points and edges are both present, and edges can meet at polyhedral corners.

In each part, a general framework is introduced and analyzed in a sequence of chapters that correspond to and are determined by the evolution of mathematical ideas. Besides these, there are sections and chapters discussing examples that serve as illustrations to the general techniques and results. Some of the examples go beyond the current framework, showing how the presented ideas can be applied in more general settings.

A delicate question we were faced with from the beginning and which accompanied us during the whole work on this book was: ***How to choose the right level of generality?*** From observing the existing literature, we see that choosing a level of generality that is too low will not allow to understand all the ideas necessary for the complete analysis of important examples or will simply exclude them. Choosing too much generality, on the other hand, will make the text difficult to read, will lead to the inclusion of a lot of material that is perhaps not essential, and may make the distance from the general theory to the concrete examples so large that individual examples require separate constructions that render the general theory less useful.

There are three categories of objects for which this question has to be answered: Domains, operators, and function spaces. Let us explain the choices we made for this book and the guidelines leading to these choices:

Our main guideline is twofold: We have a certain number of examples in mind which we consider essential and which should be covered as completely as possible by the general framework, but we want to keep the overall level of generality as low as possible.

a. Domains. Here the examples we have in mind are: Domains with conical points in any dimension, all piecewise smooth domains in two dimensions, circular and straight edges, polyhedra in three dimensions.

There are classes of domains which need to be excluded, because their analysis either does not exist or is too different from any possible general framework: Domains with outward cusps and some other degeneracies of the tangent structure. We made an effort, however, to allow inward cusps, cracks, and, for example, two-dimensional curved polygons with angles π and 2π. Such domains are not locally diffeomorphic to Lipschitz polygons or Lipschitz cones, so that we need to introduce extensions both of the notion of local diffeomorphisms by using polar coordinates ("blow up" of corners) and of the notion of smooth domains to allow double points on the boundary ("unfolded boundary" of crack domains).

Another class of domains which we exclude here are general smooth edges with variable angles. There exists a literature on these to which we even contributed [24, 25, 26, 27, 28], but we do not include this theory here, simply for reasons of space.

For the class of corner domains of polyhedral type, we restrict the discussion to genuine polyhedra in three dimensions, that is domains with plane faces and straight edges.

b. Operators. The examples we want to cover are the classical boundary value problems of mathematical physics, in particular Dirichlet, Neumann and Robin problems for the Laplace equation and the standard boundary value problems of linear elasticity theory and electrodynamics.

It turns out that even this modest collection of examples is not completely covered by the so-called general theory of elliptic boundary value problems available in the literature. The main reason that led us to define our own class of elliptic boundary value problems was that we wanted to cover problems given in variational form, and among them, for instance, the electromagnetic scattering problem described by the vector Helmholtz equation with the perfect conductor “electric” boundary condition, which means vanishing tangential component of the field on the boundary.

The standard setting of linear elliptic boundary value problems is the class of Agmon-Douglis-Nirenberg (ADN) elliptic systems with covering boundary conditions. This class allows to treat, in addition to the examples mentioned above, other classical examples like the standard boundary value problems for the Stokes system in fluid dynamics and various boundary value problems for biharmonic functions. It has, however, two drawbacks: There is no natural class of problems given in variational form that corresponds to ADN systems, and it is in its simplest form essentially a local theory. There exist global formulations involving the notion of vector bundles, but this is a level of abstraction that we did not want to impose on our intended readership. In the book [94], for example, which treats elliptic boundary value problems on smooth domains and manifolds, vector bundles are introduced as a tool for globally formulating ADN elliptic systems, but then the first definition of the term “elliptic boundary value problem” appears not before page 392. Often in this general theory, the definition of boundary conditions is local in a sense that will exclude our electromagnetic scattering problem mentioned above. The issue in a nutshell: For a 3-component vector field, the condition that its tangential component vanishes on the boundary of a ball in $\mathbb{R}^3$ corresponds to how many boundary conditions, 2 or 3? Answer: 2 locally, but 3 globally, unless you are willing to consider the boundary trace as a section of a vector bundle that can be split into the tangent bundle and the normal bundle. But then, how do you define variational formulations using these vector bundles? And what will happen to them near singular boundary points?

Our solution to this problem consists of two choices:

First, we restrict the general discussion in this book to *second order* systems, elliptic in the classical sense of Petrovski. This covers many of the standard examples, including problems in variational form, but it excludes other important examples such the Stokes system and also higher order operators such as the bilaplacian. We will, however, show how such examples can be analyzed with a simple extension of the techniques presented in our general framework. The restriction to second order systems considerably simplifies the formulation of boundary conditions which are grouped into first order and zero order conditions in a natural way.

Second, we introduce the notion of *projector fields* on the boundary that are used in the definition of the boundary conditions. In our example above, the projection on the tangent plane is a projector field of rank 2 defined on 3-component vector fields on the boundary. This notion is very well adapted to variational formulations, where essential boundary conditions are often formulated using such projections. It also allows a lot of flexibility not only for overcoming the topological obstruction we hinted at, but it helps also with the corner asymptotics, by writing, for example, the operator of the normal derivative near a conical point not as a differential operator with coefficients that are discontinuous at the corner, but as the composition of the projection on the normal component with the gradient which is a differential operator with constant coefficients.

Further choices are made in the definitions of "admissible operators", where we go beyond operators with smooth coefficients in order to include at least the images of operators with smooth coefficients under diffeomorphisms that are smooth in polar coordinates, but not necessarily smooth in Cartesian coordinates. Again, in order to keep the presentation as simple as possible, we do not try to include the most general class of singular coefficients that could be treated with the same techniques.

c. Function spaces. The most visible choice is that we use only Sobolev spaces over L^2 in this book. One reason for this is the motivation by linear problems in variational form for which the space of initial regularity is H^1. Another reason is the motivation by the aim of proving analytic regularity. For analytic estimates, one has to prove regularity estimates in all orders of derivation, with constants that depend in a controlled way on the order. On the other hand, because of Sobolev embedding theorems, the L^p space in which the derivatives are measured does not matter. One obtains the same analytic regularity results whether one builds on L^1, L^2, or L^∞ estimates. A corollary is that one can as well use the spaces that are most simple to handle, namely the scales of Hilbert Sobolev spaces.

Another consequence of the motivation by analytic estimates is that we put the emphasis on Sobolev regularity of integer order. Fractional Sobolev spaces do appear in our results, but in general more as an afterthought, and we do not try to treat the most general possible range for these spaces of fractional regularity.

There is one point where we do insist on more generality than what is often presented in the literature on corner problems: We treat two classes of weighted Sobolev spaces, not only the spaces with homogeneous norms introduced by Kontratiev ("K-weighted spaces") and the corresponding analytic class A_β, but also spaces with inhomogeneous norms ("J-weighted spaces") which contain in particular the ordinary non-weighted Sobolev spaces. There is a seemingly innocent difference in the way how we write the weight index for the scale of J-spaces, compared to the identical scale of W-spaces as used by Maz'ya and collaborators, but this is crucial for the definition of the right class B_β of weighted analytic functions that generalizes the Babuška-Guo spaces.

Part I

Smooth domains

Introduction to Part I

A boundary value problem in a domain Ω is made of an interior partial differential equation

$$L\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega$$

and, if the boundary of Ω is not empty, of boundary conditions.

We choose to present, in the main course of this book, the situation where the interior operator L is a linear system of partial differential equations of order 2. The ellipticity of such a system implies the validity of interior estimates with a gain of two derivatives. Complementing the interior elliptic equations by "covering" boundary conditions of order one

$$T\mathbf{u} = \mathbf{g} \quad \text{on} \quad \partial\Omega$$

or of order zero

$$D\mathbf{u} = \mathbf{h} \quad \text{on} \quad \partial\Omega$$

allows to obtain this regularity shift by 2 on the whole domain, up to the boundary.

The proof of this regularity shift is based on Fourier analysis of model problems with constant coefficients, on the space $\mathbb{R}^n$ or the torus $\mathbb{T}^n$ for interior estimates, and the half-space $\mathbb{R}^n_+$ or the periodic half-space $\mathbb{T}^n_+$ for boundary estimates. A Fredholm theory (finite dimensional kernel and cokernel, Fredholm alternative) is deduced in the framework of standard Hilbertian Sobolev spaces.

The refined technique of nested open sets combined with Faà di Bruno formulas for the change of variables allows a Cauchy-type analytic control of the growth of derivatives when their order tends to infinity.

Plan of Part I

This part is made of five chapters. The exposition of regular elliptic problems, together with Sobolev and analytic estimates, is organized into three chapters. We develop examples and extensions in the next two chapters.

Chapter 1: After recalling Sobolev spaces and stating a Faà di Bruno formula for analytic change of variables, we prove refined *interior estimates* for elliptic second order systems. We deduce the Fredholm property for such problems when posed on smooth manifolds without boundaries. We also deduce the analytic regularity of their solutions when all data are analytic.

Chapter 2: To take properly the boundary into account, we recall the classical notion of *covering boundary conditions* and introduce some generalizations in order to include cases where global trivialization of the domain or boundary conditions is impossible. We prove *boundary estimates*, which allow to extend the Fredholm theory and analytic regularity properties to bounded domains with smooth boundary.

Chapter 3: We consider boundary value problems defined by a *coercive variational formulation*. Such problems enter the class of elliptic problems with covering boundary conditions studied in Chapter 2. But their solutions are proved to exist in *variational spaces* for which first order derivatives are required to be square integrable, whereas the classical elliptic regularity starts from the assumption that second order derivatives are square integrable. We prove that variational solutions enjoy the same local regularity properties in higher Sobolev and analytic norms as those of general elliptic problems if the data are regular.

Chapter 4: Classical examples: The Laplace operator Δ, with Dirichlet, Neumann or Robin boundary conditions; The Lamé elasticity system for isotropic materials and more general elasticity systems for anisotropic materials, with different boundary conditions (hard clamped, stress free, etc.; The regularized Maxwell equations with perfectly or imperfectly conducting boundary conditions; Other examples from physics (Reissner-Mindlin plate model, piezoelectric system).

Chapter 5: As a natural generalization of elliptic second order systems, we study second order transmission problems: The model situation along an interface can be described as a boundary value problem for a larger system. We prove optimal piecewise Sobolev and analytic regularity along smooth interfaces.

Chapter 1

Interior estimates and analytic hypoellipticity

Introduction

This chapter is devoted to interior a priori estimates and analytic hypoellipticity for linear elliptic systems of partial differential equations. We want to keep the presentation as simple as possible, while covering the main applications of interest, and we focus therefore on elliptic systems of second order, but we allow any space dimension. We give an overview of basic techniques and results that for the most part are not new and have even been known since the late 1950s. When presenting them here, we have several aims in mind:

a) Chapter 1 can be read as an introduction to the analysis of elliptic systems, and then Chapter 2 as an introduction to elliptic boundary value problems. The main results about a-priori estimates, regularity of the solution and Fredholm properties in Sobolev spaces that hold for smooth domains constitute the model for analogous, but more complicated, results valid on non-smooth domains that will be the main subject of later parts of the book.

b) The results about regularity up to the boundary on smooth domains imply local regularity results near smooth parts of the boundary of non-smooth domains, and these are necessary ingredients to the overall description of the regularity results on domains with corners and edges. Some global regularity results on domains with conical singularities can even be obtained directly by homogeneity arguments from results for smooth domains.

c) Similarly, the analyticity up to the boundary near analytic parts of the boundary is a necessary ingredient in the description of the analytic regularity of solutions in the presence of corners and edges;

d) The Fourier and localization techniques for constructing parametrices and for obtaining a-priori estimates and Fredholm properties, described here for smooth domains, will be an essential building block for the analysis on domains with edges.

e) The basic techniques of *nested open sets* and *difference quotients* which are introduced here, are the key techniques for obtaining analytic estimates not only on smooth domains but also on non-smooth domains in later parts of this book.

A priori estimates are bounds for derivatives of a solution by certain norms of the right hand side (possibly up to lower order derivatives of the solution). A certain uniformity of such estimates with respect to the order of the derivatives will imply what is called the *analytic hypoellipticity*: The solution is analytic if the right hand side is analytic. In this chapter, we present the principal building blocks of a proof of elliptic regularity and Fredholm properties of elliptic operators on compact manifolds without boundary. Then we use the technique of nested open sets to deduce analytic estimates leading to analytic hypoellipticity.

Plan of Chapter 1

- **§1** Definition of functional spaces (Sobolev spaces and analytic classes). A Faà di Bruno type formula for the composition by an analytic map.
- **§2** Definition of ellipticity for model and general systems. Local inverse operators (parametrices). Basic a priori estimates.
- **§3** Higher order a priori estimates. Fredholm theorem in Sobolev spaces.
- **§4** Basic a priori estimates in nested concentric balls, taking the difference of radii into account.
- **§5** Order-independent estimates in nested concentric balls in the case of an operator with *constant coefficients* (estimates are simpler to state and to prove in this case).
- **§6** Order-independent estimates in nested concentric balls in the general case of variable (analytic) coefficients.
- **§7** Analytic regularity of solutions of elliptic problems with analytic coefficients and data.

Essentials

The subject of the first part of this book is the presentation of classical results on the regularity of solutions of linear elliptic partial differential equations. Whereas the second chapter will discuss elliptic boundary value problems, this first chapter develops the results that do not involve the boundary of the domain. Boundaries are irrelevant for two classes of results: First, for *local* estimates and regularity results, where properties of the solution are obtained on a subdomain Ω_1 compactly contained in Ω_2, under hypotheses on the given data in the larger domain Ω_2; and second, for *global* results on compact manifolds without boundary. One can pass between global and local results by techniques of *localization*, and local results can be combined to give global results by using finite coverings. Passages in both directions will be used to move forward from simpler to more general situations.

We present results on both finite regularity and analytic regularity. Regularity is measured by *Sobolev* norms, and we use the standard definition of the Sobolev space $\mathsf{H}^s(\Omega)$ on a domain Ω in $\mathbb{R}^n$ or in a smooth n-dimensional manifold, $n \geq 1$. The space $\mathsf{A}(\Omega)$ of analytic functions on a bounded domain Ω is characterized by Cauchy-type estimates for all Sobolev norms:

$$\frac{1}{k!}|u|_{k;\,\Omega} \leq c^{k+1} \quad \text{for some } c > 0 \text{ and all } k \in \mathbb{N}\,. \tag{1.a}$$

Here instead of the Sobolev semi-norm $|u|_{k;\,\Omega}$ we could take the Sobolev norm $\|u\|_{k;\,\Omega}$ or, in fact, any Sobolev norm of order k based on L^p instead of L^2, for any $1 \leq p \leq \infty$, and in view of the Sobolev embedding theorems we would always get the same space of analytic functions. This is one of the reasons why we will restrict ourselves here to the simplest case of the Sobolev spaces based on the Hilbert space L^2.

Thanks to a Faà-di-Bruno formula, we obtain the finite Cauchy-type estimate (1.22) for an analytic change of variable $\mathbf{g} : \Omega \to \Omega'$, which we write in the form:

$$\frac{1}{k!}|f \circ \mathbf{g}|_{k;\,\Omega} \;\leq\; c_{\mathbf{g}}^{k+1} \;\sum_{\ell=0}^{k} \frac{1}{\ell!}|f|_{\ell;\,\mathbf{g}(\Omega)}\;, \tag{1.b}$$

with a constant $c_{\mathbf{g}}$ which depends on $\mathbf{g}$ only. This allows to handle easily localization arguments.

The main object in this chapter is an $N \times N$ second order system $L = (L_{jj'})_{1\leq j,j'\leq N}$ of linear partial differential operators with smooth coefficients. The partial differential equation $L\mathbf{u} = \mathbf{f}$ corresponds to the $N \times N$ system of equations

$$\sum_{j'=1}^{N} L_{jj'}(\mathbf{x}; \mathrm{D}_{\mathbf{x}})\, u_{j'} = \sum_{j'=1}^{N} \sum_{|\alpha|\leq 2} a_{jj'}^{\alpha}(\mathbf{x})\, \partial_{\mathbf{x}}^{\alpha} u_{j'} = f_j, \quad j = 1, \ldots, N, \tag{1.c}$$

with the variable $\mathbf{x} = (x_1, \ldots, x_n)$ in Ω, the data $\mathbf{f} = (f_1, \ldots, f_N)$ and the unknown $\mathbf{u} = (u_1, \ldots, u_N)$. Here the coefficients $a_{ij}^{\alpha}(\mathbf{x})$ are $\mathscr{C}^\infty$ functions on Ω and $\mathrm{D}_{\mathbf{x}} = (-i\partial_{x_1}, \ldots, -i\partial_{x_n})$.

The analysis of equation (1.c) begins with the study of a *model* situation where the operator L has constant coefficients and thus is translation invariant, and the domain is also invariant under a group of translations.

As *model operators* we have the systems $L(\mathrm{D}_{\mathbf{x}})$ with constant coefficients a_{ij}^{α}. Via Fourier transformation, the differential operator $L(\mathrm{D}_{\mathbf{x}})$ becomes the multiplication operator by the matrix $L(\boldsymbol{\xi})$, the *symbol* of L. The condition of *ellipticity* appears naturally as the condition of invertibility of the symbol $L^{\mathsf{pr}}(\boldsymbol{\xi})$ of its principal part for all non-zero values of the Fourier variable $\boldsymbol{\xi}$:

Definition 1.A *Let* $L = \big(\sum_{|\alpha|\le 2} a^{\alpha}_{jj'}\,\partial^{\alpha}_{\mathbf{x}}\big)_{1\le j,j'\le N}$ *be a second order system with constant coefficients. It is called elliptic if the matrix*

$$L^{\mathsf{pr}}(\boldsymbol{\xi}) := \Big(\sum_{|\alpha|=2} a^{\alpha}_{jj'}\,(i\boldsymbol{\xi})^{\alpha}\Big)_{1\le j,j'\le N} \tag{1.d}$$

is invertible for all $\boldsymbol{\xi}\in\mathbb{R}^n\setminus\{0\}$.

This is the classical definition of ellipticity of a system of partial differential operators in the Petrovski sense.

As a *model domain* we choose here the n-torus, the compact manifold without boundary $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. We prefer this to the often-used model domain $\mathbb{R}^n$, because the latter has a kind of "boundary" at infinity which frequently makes itself felt by effects foreign to the local/global estimates we are looking for.

On $\mathbb{T}^n$ one can, at least formally, solve the partial differential equation with constant coefficients $L\mathbf{u} = \mathbf{f}$ by Fourier series. Unique solvability is then equivalent to the invertibility of the symbol $L(\mathbf{p})$ for integer frequencies $\mathbf{p}\in\mathbb{Z}^n$. This property alone does not, however, imply stability of the solution in terms of Sobolev norms. What is missing is precisely the ellipticity condition, which can be interpreted as asymptotic invertibility for frequencies on the "infinite sphere" $\mathbb{S}^{n-1}$.

For the torus $\mathbb{T}^n$, there is thus a very simple characterization of ellipticity in terms of invertibility between Sobolev spaces, and it easily leads to a complete regularity theory for operators with constant coefficients, summarized in the following theorem. Here the Sobolev spaces $\mathbf{H}^s$ and analytic classes $\mathbf{A}$ are simply the N-component vector version of H^s and A.

Theorem 1.B *Let* L *be a second order system with constant coefficients on the torus* $\mathbb{T}^n$.

(i) The operator L *is an isomorphism from* $\mathbf{H}^2(\mathbb{T}^n)$ *onto* $\mathbf{L}^2(\mathbb{T}^n)$ *if and only if* L *is elliptic and in addition* $L(\mathbf{p})$ *is invertible for any* $\mathbf{p}\in\mathbb{Z}^n$.

(ii) The system L *is elliptic if and only if there exists another second order system* $\tilde L$ *with constant coefficients that has the same principal part as* L *and is such that* $\tilde L$ *defines an isomorphism from* $\mathbf{H}^2(\mathbb{T}^n)$ *onto* $\mathbf{L}^2(\mathbb{T}^n)$.

(iii) If the assumption of (ii) is satisfied, then $\tilde L$ *is an isomorphism between* $\mathbf{H}^{s+2}(\mathbb{T}^n)$ *and* $\mathbf{H}^s(\mathbb{T}^n)$ *for any order* $s\in\mathbb{R}$ *and also from the analytic class* $\mathbf{A}(\Omega)$ *onto itself.*

(iv) If L *is elliptic, then* L *is a Fredholm operator of index zero between* $\mathbf{H}^{s+2}(\mathbb{T}^n)$ *and* $\mathbf{H}^s(\mathbb{T}^n)$ *for any* $s\in\mathbb{R}$*, and its kernel and cokernel do not depend on* s*, are in fact represented by analytic functions. There holds the analytic hypoellipticity in the form: If* $\mathbf{u}\in\mathscr{D}'(\mathbb{T}^n)^N$ *is solution of* $L\mathbf{u}=\mathbf{f}$ *with* $\mathbf{f}\in\mathbf{A}(\mathbb{T}^n)$*, then there holds also* $\mathbf{u}\in\mathbf{A}(\mathbb{T}^n)$.

This theorem is in a certain sense the ideal form of all elliptic regularity theorems that will appear later on. In our presentation, it is also a tool for getting local a priori estimates that will then lead to local regularity results for systems with non-constant coefficients and eventually to global regularity results on compact manifolds without boundary.

For a general system L as in (1.c), we consider at any point $\mathbf{x}_0 \in \Omega$ the model operator $L(\mathbf{x}_0; \mathrm{D}_\mathbf{x})$ with coefficients frozen in $\mathbf{x}_0$:

$$L(\mathbf{x}_0; \mathrm{D}_\mathbf{x}) = \big(L_{jj'}(\mathbf{x}_0; \mathrm{D}_\mathbf{x})\big)_{j,j'} \quad \text{with} \quad L_{jj'}(\mathbf{x}_0; \mathrm{D}_\mathbf{x}) = \sum_{|\alpha|\le 2} a^\alpha_{jj'}(\mathbf{x}_0)\, \partial^\alpha_\mathbf{x}\,.$$

Definition 1.C *The system L is said to be* elliptic *on Ω if for all $\mathbf{x}_0 \in \Omega$, the frozen operator $L(\mathbf{x}_0; \mathrm{D}_\mathbf{x})$ is elliptic.*

Then the continuous inverse $\tilde{L}(\mathrm{D}_\mathbf{x})^{-1}$ from $\mathbf{L}^2(\mathbb{T}^n)$ to $\mathbf{H}^2(\mathbb{T}^n)$ associated with any frozen operator $L(\mathbf{x}_0; \mathrm{D}_\mathbf{x})$ by virtue of the preceding theorem is the essential building block for the construction of a local *parametrix* (inverse modulo compact operators) of L on any open domain Ω in $\mathbb{R}^n$ that is sufficiently small so that it can be considered as a subdomain of $\mathbb{T}^n$. The construction of the parametrix uses also a *dilation* argument exploiting the fact that very close to a point the operator L differs only by a small perturbation from its principal part frozen in that point. This is a technique – sometimes called "blowup" – that will appear repeatedly later on, because we will always consider geometries where tangent cones exist, so that at least asymptotically the domain is dilation invariant.

Once a parametrix is constructed, one has local a priori estimates in $\mathbf{H}^2$, such as (1.32)

$$\|\mathbf{u}\|_{2;\,\mathcal{B}^*_{\mathbf{x}_0}} \le A_0\big(\|L\mathbf{u}\|_{0;\,\mathcal{B}^*_{\mathbf{x}_0}} + \|\mathbf{u}\|_{1;\,\mathcal{B}^*_{\mathbf{x}_0}}\big). \tag{1.e}$$

which is valid for all $\mathbf{u} \in \mathbf{H}^2_0(\mathcal{B}^*_{\mathbf{x}_0})$ on a sufficiently small ball $\mathcal{B}^*_{\mathbf{x}_0}$ centered at $\mathbf{x}_0$.

By estimating difference quotients, one can get higher order a priori estimates, and one can write them in the form used in Theorem 1.3.3:

$$\|\mathbf{u}\|_{k+2;\,\Omega_1} \le c\big(\|L\mathbf{u}\|_{k;\,\Omega_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\big) \tag{1.f}$$

This is valid for any bounded domains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$ and any $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ satisfying $L\mathbf{u} \in \mathbf{H}^k(\Omega_2)$. The constant c here may depend on k, Ω_1, Ω_2 and L, but not on $\mathbf{u}$.

From such local a priori estimates it is easy to obtain global results about *finite regularity* and Fredholm properties in Sobolev spaces on smooth manifolds with boundary.

In order to get *analytic* regularity, we use the technique of nested open sets. This relies on estimates similar to (1.f), but with precise control over the way how the constant c depends on the relative position of the two domains Ω_1 and Ω_2. The starting point is again the $\mathbf{H}^2$ a priori estimate (1.e), from which one can deduce a more precise $\mathbf{H}^2$ estimate as in (1.42):

$$\sum_{|\alpha|\le 2} \rho^{|\alpha|}\|\partial^\alpha_\mathbf{x}\mathbf{u}\|_{B_{R-|\alpha|\rho}} \le A_1\Big(\rho^2\|L\mathbf{u}\|_{B_{R-\rho}} + \sum_{|\alpha|\le 1} \rho^{|\alpha|}\|\partial^\alpha_\mathbf{x}\mathbf{u}\|_{B_{R-|\alpha|\rho}}\Big).$$

Here $\|\cdot\|_B$ is the $\mathbf{L}^2$ norm on B, and the $B_{R-k\rho}$ $(k = 0, 1, 2)$ are three concentric balls of radius $R - k\rho$, so that ρ denotes the distance between their successive boundaries. The constant A_1 here does not depend on R and ρ, as long as $0 < \rho < R/2$ and R remains below some sufficiently small threshold R_*.

By again using difference quotients and carefully estimating commutators between derivatives and multiplication by the smooth coefficients of the operator L, one obtains higher order a priori estimates in nested open sets.

– For the constant coefficient case, they are written in the form of estimate (1.43) which considers again norms on a finite number of concentric balls of successive distance ρ:

$$\sum_{|\alpha|\le k+2} \rho^{|\alpha|} \|\partial_{\mathbf{x}}^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} \le \sum_{|\beta|\le k} A^{k+1-|\beta|}\, \rho^{2+|\beta|} \|\partial_{\mathbf{x}}^\beta L\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + A^{k+1} \sum_{|\alpha|\le 1} \rho^{|\alpha|} \|\partial_{\mathbf{x}}^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} . \tag{1.g}$$

– For the variable coefficient case, the estimates involve a continuous range of concentric balls. They culminate in estimate (1.46) which can be written in detailed form as

$$\max_{0<\rho\le \frac{R}{2(k+1)}} \max_{|\delta|=k+2} \rho^{k+2} \|\partial_{\mathbf{x}}^\delta \mathbf{u}\|_{B_{R-(k+2)\rho}} \le A^{k+1} \Big(\sum_{\ell=0}^{k} A^{-\ell} \max_{0<\rho\le \frac{R}{2(\ell+1)}} \max_{|\delta|=\ell} \rho^{2+\ell} \|\partial_{\mathbf{x}}^\delta L\mathbf{u}\|_{B_{R-(\ell+1)\rho}} + \max_{0<\rho\le \frac{R}{2}} \max_{|\delta|=1} \rho \|\partial_{\mathbf{x}}^\delta \mathbf{u}\|_{B_{R-\rho}} + \|\mathbf{u}\|_{B_R} \Big). \tag{1.h}$$

From these estimates in nested open sets, it is not hard to obtain Cauchy-type estimates for Sobolev norms on two fixed nested domains $\Omega_1 \subset\subset \Omega_2$, such as estimate (1.51)

$$\frac{1}{k!}\, |\mathbf{u}|_{k;\,\Omega_1} \le A^{k+1} \Big(\sum_{\ell=0}^{k-2} \frac{1}{\ell!}\, |L\mathbf{u}|_{\ell;\,\Omega_2} + \sum_{\ell=0}^{1} |\mathbf{u}|_{\ell;\,\Omega_2} \Big), \tag{1.i}$$

valid for all $k \in \mathbb{N}$ with a constant A depending only on Ω_1, Ω_2 and the coefficients of L, but not on k or $\mathbf{u}$.

The final result of this chapter can be summarized as follows.

Theorem 1.D *Let L be an elliptic second order $N \times N$ system with smooth coefficients.*

(i) For any two bounded domains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$ and any $s \ge 0$, if $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ satisfies $L\mathbf{u} = \mathbf{f} \in \mathbf{H}^s(\Omega_2)$, then $\mathbf{u} \in \mathbf{H}^{s+2}(\Omega_1)$ and one has the estimates (1.f) with a constant c independent of $\mathbf{u}$.

(ii) If the coefficients of L are analytic, then one has the more precise estimates (1.i) for any $k \in \mathbb{N}$ with a constant A independent of k, and if $L\mathbf{u} \in \mathbf{A}(\Omega_2)$, then $\mathbf{u} \in \mathbf{A}(\Omega_1)$.

(iii) If L is elliptic on a smooth compact manifold Ω without boundary, then the above estimates hold with $\Omega_1 = \Omega_2 = \Omega$, and L is a Fredholm operator between $\mathbf{H}^{s+2}(\Omega)$ and $\mathbf{H}^s(\Omega)$ for any $s \in \mathbb{R}$; its kernel and cokernel do not depend on s and are represented by $\mathscr{C}^\infty$ functions; if Ω is analytic and L has analytic coefficients, they are represented by analytic functions.

1.1 Classical function spaces

In this section we recall the definitions of some classical spaces, namely the Sobolev spaces and spaces of analytic functions, together with their main properties. We assume that the reader is familiar with the basic theory of Sobolev spaces, and we refer to the standard literature [2, 57, 91] for more details. We mainly consider two model domains for the definition of Sobolev spaces: The non-periodic case $\mathbb{R}^n$ and the periodic case $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, and subdomains of these. We also mention the extension to subdomains of more general smooth manifolds.

In Lemma 1.1.1 we prove precise estimates for analytic coordinate transformations which seem to be new in this form.

By n we denote the space dimension, $n \geq 1$, and by $\mathbf{x} = (x_1, \ldots, x_n)$ the Cartesian coordinates in $\mathbb{R}^n$ or $\mathbb{T}^n$.

1.1.a Sobolev spaces

Throughout, we will use only Hilbert Sobolev spaces based on L^2. All the material presented in this section is covered in detail by Chapter 4 of [91].

• **Sobolev spaces on** $\mathbb{R}^n$ of any real order s are defined via Fourier transformation: Let the Fourier transform $\mathscr{F}$ of u be given by

$$(\mathscr{F}u)(\boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, u(\mathbf{x})\, \mathrm{d}\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^n. \tag{1.1}$$

Here $\mathbf{x} \cdot \boldsymbol{\xi} = x_1\xi_1 + \ldots + x_n\xi_n$. Then the Sobolev norm of order $s \in \mathbb{R}$ is defined as

$$\|u\|_s = \left(\int_{\mathbb{R}^n} (1 + |\boldsymbol{\xi}|^2)^s \, |(\mathscr{F}u)(\boldsymbol{\xi})|^2 \, \mathrm{d}\boldsymbol{\xi}\, . \right)^{1/2} \tag{1.2}$$

The Sobolev space $\mathsf{H}^s(\mathbb{R}^n)$ consists of all tempered distributions for which the norm (1.2) is finite. These spaces are also called *Bessel potential spaces*. Unless specially mentioned, we will think of complex-valued functions, but there are examples where all functions are assumed to be real-valued.

• **Sobolev spaces of integral order on domains of** $\mathbb{R}^n$ are defined in the distributional sense. We understand by domain an open connected subset, as usual. Since our main emphasis later on will be on regularity of higher order and analytic regularity, we will be working chiefly with Sobolev spaces of *integral* order $m \in \mathbb{N}$[1] and the related trace spaces. We will therefore often use the classical definition of these spaces via L^2 norms of (distributional) derivatives.

Let Ω be a domain in $\mathbb{R}^n$. We denote by $\mathscr{C}^0(\overline{\Omega})$ the space of continuous functions up to the boundary of Ω equipped with the L^∞ norm, and by $\mathscr{C}^\infty(\overline{\Omega})$ the space of functions infinitely differentiable up to the boundary of Ω. Finally $\mathscr{C}_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω.

We denote by $\|\cdot\|_\Omega$ the L^2 norm over Ω for scalar functions, that is

$$\|u\|_\Omega = \Big(\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\Big)^{1/2}.$$

For a vector function $\mathbf{u} = (u_1, \dots, u_N)$, we set

$$\|\mathbf{u}\|_\Omega = \Big(\sum_{i=1}^N \|u_i\|_\Omega^2\Big)^{1/2}.$$

The Sobolev space of index m on Ω, for $m \in \mathbb{N}$, is defined as the space $\mathsf{H}^m(\Omega)$ of all functions $u \in \mathsf{L}^2(\Omega)$ such that all their partial derivatives ∂^α of order $|\alpha|$ less than m belong to $\mathsf{L}^2(\Omega)$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$ its length and

$$\partial_{\mathbf{x}}^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u\,.$$

The norm and semi-norm in $\mathsf{H}^m(\Omega)$ are denoted by $\|\cdot\|_{m;\,\Omega}$ and $|\cdot|_{m;\,\Omega}$, respectively, and given by

$$\|u\|_{m;\,\Omega} = \Big(\sum_{|\alpha|\le m} \|\partial_{\mathbf{x}}^\alpha u\|_\Omega^2\Big)^{1/2} \quad \text{and} \quad |u|_{m;\,\Omega} = \Big(\sum_{|\alpha|= m} \|\partial_{\mathbf{x}}^\alpha u\|_\Omega^2\Big)^{1/2}. \tag{1.3}$$

For a vector function $\mathbf{u}$, the corresponding norm is still denoted by $\|\cdot\|_{m;\,\Omega}$:

$$\|\mathbf{u}\|_{m;\,\Omega} = \Big(\sum_{i=1}^N \|u_i\|_{m;\,\Omega}^2\Big)^{1/2}.$$

Thus the notation $\|\cdot\|_\Omega$ for the L^2 norm is an alternative to $\|\cdot\|_{0;\,\Omega}$.

The two norms for functions u defined on $\mathbb{R}^n$, $\|u\|_m$ via (1.2) and $\|u\|_{m;\,\mathbb{R}^n}$ via (1.3) are equivalent with universal constants not depending on the space dimension. With the Parseval equality one can easily obtain

$$\|u\|_{m;\,\mathbb{R}^n}^2 \le \|u\|_m^2 \le \frac{2^m}{m+1}\|u\|_{m;\,\mathbb{R}^n}^2 \ .$$

[1] We use the convention $\mathbb{N} = \{0, 1, 2, \dots\}$.

Thus the equivalence is not uniform in m, but the growth of the constant is sufficiently slow so that it will not influence the definition of our analytic spaces (see inequalities (1.18)).

For $m \in \mathbb{N}$, the closure of $\mathscr{C}_0^\infty(\Omega)$ in $\mathsf{H}^m(\Omega)$ is denoted by $\mathsf{H}_0^m(\Omega)$, and $\mathsf{H}^{-m}(\Omega)$ is defined as the dual space of $\mathsf{H}_0^m(\Omega)$.

If Ω satisfies some minimal regularity property, namely that Ω is bounded and Lipschitz[2], or is unbounded and a uniformly Lipschitz epigraph[3] then

1. The extension by zero of any function belonging to $\mathsf{H}_0^m(\Omega)$ defines an element of $\mathsf{H}^m(\mathbb{R}^n)$,
2. Functions in $\mathsf{H}^m(\Omega)$ have extensions in $\mathsf{H}^m(\mathbb{R}^n)$. Therefore the space of restrictions to Ω of elements of $\mathsf{H}^m(\mathbb{R}^n)$ coincides with $\mathsf{H}^m(\Omega)$.

On the other hand, if Ω has cracks, or even if Ω is only "smooth in the extended sense" as defined in Section 2.8, then these two properties do not hold any more.

• **Sobolev spaces of general real order on domains of $\mathbb{R}^n$.** For a real non-integral $s > 0$, the space $\mathsf{H}^s(\Omega)$ is defined by interpolation [57] between $\mathsf{H}^{[s]}(\Omega)$ and $\mathsf{H}^{[s]+1}(\Omega)$, with $[s]$ the integral part of s. If Ω is bounded and Lipschitz, or is unbounded and a uniformly Lipschitz epigraph, then the norm of $\mathsf{H}^s(\Omega)$ is equivalent to its Sobolev-Slobodeckii norm defined as follows

$$\|u\|_{s;\Omega} = \Big(\|u\|^2_{[s];\Omega} + \sum_{|\alpha|=[s]} |\partial^\alpha_{\mathbf{x}} u|^2_{s-[s];\Omega} \Big)^{1/2}. \tag{1.4}$$

Here for $0 < \sigma < 1$, the semi-norm $|v|_{\sigma;\Omega}$ is defined by

$$|v|^2_{\sigma;\Omega} = \int\!\!\int_{\Omega\times\Omega} \frac{|v(\mathbf{x}) - v(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|^{n+2\sigma}}\, \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{x}\,.$$

NB. For the class of extended smooth domains with double points on the boundary which we introduce later on in section 2.8, the same equivalence still holds, provided the semi-norm $|v|_{\sigma;\Omega}$ is defined by

$$|v|^2_{\sigma;\Omega} = \int\!\!\int_{\Omega\times\Omega} \frac{|v(\mathbf{x}) - v(\mathbf{x}')|^2}{d(\mathbf{x},\mathbf{x}')^{n+2\sigma}}\, \mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{x}, \tag{1.5}$$

where $d(\mathbf{x},\mathbf{x}')$ denotes the geodesic distance between $\mathbf{x}$ and $\mathbf{x}'$ inside Ω. For bounded Lipschitz domains, $d(\mathbf{x},\mathbf{x}')$ is equivalent to $|\mathbf{x}-\mathbf{x}'|$.

The space $\mathsf{H}_0^s(\Omega)$ is defined as the closure of $\mathscr{C}_0^\infty(\Omega)$ in $\mathsf{H}^s(\Omega)$. We define another space $\widetilde{\mathsf{H}}^s(\Omega)$ by Hilbert space interpolation between $\mathsf{H}_0^{[s]}(\Omega)$ and $\mathsf{H}_0^{[s]+1}(\Omega)$ and then $\mathsf{H}^{-s}(\Omega)$ as the dual space of $\widetilde{\mathsf{H}}^s(\Omega)$. It is known [57] that the space $\widetilde{\mathsf{H}}^s(\Omega)$ is contained in $\mathsf{H}_0^s(\Omega)$, but if $\Omega \neq \mathbb{R}^n$, it may happen that $\widetilde{\mathsf{H}}^s(\Omega)$ is strictly smaller than $\mathsf{H}_0^s(\Omega)$: When Ω is bounded Lipschitz, there holds

[2] This means that Ω has a Lipschitz boundary, see [42, Def. 1.2.1.1] for a definition.

[3] This condition is equivalent to the exterior uniform cone condition, and also to the interior uniform cone condition, see [42, Chap. 1].

1. For $s \notin \mathbb{N} + \frac{1}{2}$, $\widetilde{\mathsf{H}}^s(\Omega) = \mathsf{H}^s_0(\Omega)$.
2. For $s \in \mathbb{N} + \frac{1}{2}$, the extension by zero of any function belonging to $\widetilde{\mathsf{H}}^s(\Omega)$ defines an element of $\mathsf{H}^s(\mathbb{R}^n)$, while this is not true for $\mathsf{H}^s_0(\Omega)$. The space $\widetilde{\mathsf{H}}^{\frac{1}{2}}(\Omega)$ is also denoted by $H^{\frac{1}{2}}_{00}(\Omega)$ in [57].

Finally, $\widetilde{\mathsf{H}}^{-s}(\Omega)$ is defined as the dual space of $\mathsf{H}^s(\Omega)$. Here we use Triebel's notation [93, Section 4.8]. If Ω is bounded Lipschitz, we have the identity for all $s \in \mathbb{R}$

$$\widetilde{\mathsf{H}}^s(\Omega) = \{u \in \mathsf{H}^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}.$$

Then the two dual families $\mathsf{H}^s(\Omega)$ and $\widetilde{\mathsf{H}}^s(\Omega)$ $(s \in \mathbb{R})$ are interpolation scales, the former consisting of spaces of distributions in Ω, the latter of distributions in $\mathbb{R}^n$, for any $s \in \mathbb{R}$ without exception.

• **Periodic Sobolev spaces** on the n-torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ are defined by the periodic Fourier transformation (Fourier series). Let

$$(\mathscr{F}_{\mathsf{per}} u)(\mathbf{p}) = \hat{u}(\mathbf{p}) = (2\pi)^{-n/2} \int_{\mathbb{T}^n} \mathrm{e}^{-i\mathbf{x}\cdot\mathbf{p}}\, u(\mathbf{x})\, \mathrm{d}\mathbf{x}, \quad \mathbf{p} \in \mathbb{Z}^n. \tag{1.6}$$

The Sobolev norm of order $s \in \mathbb{R}$ is defined by

$$\|u\|_{s;\mathbb{T}^n} = \Big(\sum_{\mathbf{p}\in\mathbb{Z}^n} (1 + |\mathbf{p}|^2)^s |\hat{u}(\mathbf{p})|^2 \Big)^{1/2}, \tag{1.7}$$

and the Sobolev space $\mathsf{H}^s(\mathbb{T}^n)$ is defined as the Hilbert space of all n-ply 2π-periodic distributions on $\mathbb{R}^n$ for which this norm is finite.

In the periodic case, since $\mathbb{T}^n$ has an empty boundary, $\mathsf{H}^{-s}(\mathbb{T}^n)$ is in a natural way the dual space of $\mathsf{H}^s(\mathbb{T}^n)$, for all $s \in \mathbb{R}$.

If Ω is a domain in $\mathbb{R}^n$ such that its closure is contained in an open cube of side length not more than 2π, then Ω can also be considered in a natural way as a subdomain of $\mathbb{T}^n$, and the two definitions of the Sobolev space $\mathsf{H}^s(\Omega)$, by restriction from $\mathbb{R}^n$ and by restriction from $\mathbb{T}^n$, are equivalent[4]. We will often use this identification later on.

• **Sobolev spaces on smooth manifolds.** One can extend all the above definitions to the situation where Ω is a bounded subdomain of a *smooth manifold* M of dimension n with or without boundary. A canonical example that will play an important role later on is the case of M being the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Another important case is given by regular parts of the boundary of a domain in $\mathbb{R}^n$.

Sobolev spaces on a smooth compact manifold M without boundary are defined by a coordinate covering and a corresponding partition of unity: Let $\mathcal{U}_j \subset M$ $(j = 1, \dots, J)$ be a finite family of open sets covering M and ϕ_j $(j = 1, \dots, J)$ $\mathscr{C}^\infty$ diffeomorphisms

[4] Equivalence constants depend on s and also on the size of Ω. We will use these equivalences for low regularities only.

from $\mathcal{U}_j$ to the open unit ball $B \subset \mathbb{R}^n$ – which we can also consider as a subset of $\mathbb{T}^n$ if we wish. Let $\psi_j \in \mathscr{C}_0^\infty(\mathcal{U}_j)$ $(j = 1, \dots, J)$ be such that $\sum \psi_j \equiv 1$ on M.

For $u \in \mathscr{C}^\infty(M)$, the functions $(\psi_j\, u) \circ \phi_j^{-1}$ belong to $\mathscr{C}_0^\infty(B)$, and one can define the Sobolev norm

$$\|u\|_{s;\, M} = \Big(\sum_{j=1}^{J} \|(\psi_j\, u) \circ \phi_j^{-1}\|^2_{s;\, \mathbb{R}^n} \Big)^{1/2} \tag{1.8}$$

The Sobolev space $\mathsf{H}^s(M)$ is the completion of $\mathscr{C}^\infty(M)$ under this norm.

We define similarly Sobolev spaces on a smooth compact manifold M *with* boundary. In the latter case, some of the diffeomorphisms ϕ_j act from $\mathcal{U}_j$ to the open unit half-ball $B_+ \subset \mathbb{R}^n_+$.

If Ω is a subdomain of a smooth compact manifold M, with or without boundary, the spaces $\mathsf{H}^m(\Omega)$, $\mathsf{H}^m_0(\Omega)$, $\mathsf{H}^s(\Omega)$, $\mathsf{H}^s_0(\Omega)$, $\tilde{\mathsf{H}}^s(\Omega)$, and $\mathsf{H}^{-s}(\Omega)$ are defined as before, using local maps.

These definitions are invariant in the sense that the norm defined in (1.8) does, of course, depend on the choice of the coordinate covering, but that the passage to a different covering yields an *equivalent* norm. The latter is proved by using some basic tools of the calculus of Sobolev spaces, namely the boundedness of the operators defined by coordinate transformations and by multiplication with smooth functions.

We note that if we consider the torus $\mathbb{T}^n$ as a compact manifold without boundary and define the Sobolev spaces $\mathsf{H}^s(\mathbb{T}^n)$ by localizing and transport from the unit ball in $\mathbb{R}^n$, we get new definitions, different from but equivalent to the global definitions given above in (1.7) via the periodic Fourier transformation.

The norm of $H^s(M)$ can be equivalently defined by restrictions rather than by cut-off, that is, when M is without boundary, by

$$\|u\|_{s;\, M} = \Big(\sum_{j=1}^{J} \|\big(u\big|_{\mathcal{U}_j}\big) \circ \phi_j^{-1}\|^2_{s;\, B} \Big)^{1/2} \tag{1.9}$$

instead of (1.8). In the case of a manifold with boundary, some of the terms in the right hand side of (1.9) are replaced with contributions on the half-ball B_+ instead of the ball B. The expression (1.9) is more convenient for the definition of analytic classes on analytic manifolds.

Considering for $n \geq 2$ a particular case of smooth manifolds of dimension $n-1$, namely the boundary $\partial\Omega$ of a smooth bounded domain Ω of dimension n, we have defined now Sobolev spaces of any order on the boundary $\partial\Omega$. This will be needed when we consider traces on $\partial\Omega$ of functions defined on Ω, see Section 2.1.

• **Additional basic properties of Sobolev spaces,** see for instance [2, 61, 76, 93].

(i) Poincaré Inequality: For bounded Lipschitz Ω there exists a constant $c_{m,\Omega}$:

$$\|u\|_{m-1;\,\Omega} \leq c_{m,\Omega}\, |u|_{m;\,\Omega}\,, \quad \forall u \in \mathsf{H}^m_0(\Omega). \tag{1.10}$$

More generally, if E is a closed subspace of $H^m(\Omega)$ such that its intersection with the space $\mathbb{P}^{m-1}$ of polynomials of degree $\le m-1$ is reduced to $\{0\}$, there exists a constant $c = c_E$:

$$\|u\|_{m-1;\,\Omega} \le c\,|u|_{m;\,\Omega}\,, \quad \forall u \in E. \tag{1.11}$$

This is a consequence of the *Bramble-Hilbert Theorem* [15], which states that the quotient norm in $\mathsf{H}^m(\Omega)/\mathbb{P}^{m-1}$ is bounded by the semi-norm

$$\|u\|_{\mathsf{H}^m(\Omega)/\mathbb{P}^{m-1}} \le c\,|u|_{m;\,\Omega}\,, \quad \forall u \in \mathsf{H}^m(\Omega). \tag{1.12}$$

Another form of this estimate is given in [69]: if $u \in \mathsf{H}^m(\Omega)$ has zero averages for all its partial derivatives of order $\le m-1$, its norm in $\mathsf{H}^m(\Omega)$ is bounded by its semi-norm.

(ii) Sobolev embedding Theorem: If Ω is a bounded Lipschitz domain, then:

$$\forall s > m + \tfrac{n}{2},\ m \in \mathbb{N}: \quad \mathsf{H}^s(\Omega) \subset \mathscr{C}^m(\overline{\Omega}), \quad \text{with continuous embedding.} \tag{1.13}$$

Thus, in particular,

$$\bigcap_{k\in\mathbb{N}} \mathsf{H}^k(\Omega) = \mathscr{C}^\infty(\overline{\Omega}). \tag{1.14}$$

(iii) Rellich compact embedding Theorem: For a bounded Lipschitz[5] domain Ω,

$$\text{the embedding} \quad \mathsf{H}^t(\Omega) \subset \mathsf{H}^s(\Omega) \quad \text{is compact if } t > s. \tag{1.15}$$

1.1.b Analytic functions

Besides the Sobolev spaces, our most important classical function space is the **class of analytic functions** on Ω which we introduce now. The definition we will mostly use is:

$$\mathsf{A}(\Omega) = \Big\{ u \in \bigcap_{k\in\mathbb{N}} \mathsf{H}^k(\Omega) \mid \exists c > 0\ \forall k \in \mathbb{N}: \ |u|_{k;\,\Omega} \le c^{k+1} k! \Big\}, \tag{1.16}$$

and $\mathbf{A}(\Omega) = \mathsf{A}(\Omega)^N$. The defining sequence of estimates in (1.16) can be written equivalently – more precisely but a little less conveniently – as

$$\exists c_1 > 0\ \exists c_0 > 0\ \forall k \in \mathbb{N}: \ |u|_{k;\,\Omega} \ \le\ c_1\, c_0^k\, k!\,. \tag{1.17}$$

The class of functions satisfying (1.17) for fixed c_0 and arbitrary c_1 then forms a linear subspace of $\mathsf{A}(\Omega)$, giving $\mathsf{A}(\Omega)$ in a natural way the structure of an inductive limit of Banach spaces.

Other equivalent sequences of estimates defining the same class $\mathsf{A}(\Omega)$ are

$$\exists c_2 > 0\ \exists c_0 > 0\ \forall k \in \mathbb{N}: \ \|u\|_{k;\,\Omega} \ \le\ c_2\, c_0^k\, k!\,, \tag{1.18}$$

and for a bounded Lipschitz domain Ω:

[5] It is easy to see that the statements *(i)* and *(iii)* extend to domains which are a finite union of bounded Lipschitz domains.

$$\exists c_3 > 0\ \exists c_4 > 0\ \forall \alpha \in \mathbb{N}^n\ \forall \mathbf{x} \in \Omega:\ |\partial^\alpha u(\mathbf{x})| \le c_3\, c_4^{|\alpha|}\, |\alpha|!\,. \tag{1.19}$$

The relations between the constants c_1, c_2, c_3 and between c_0 and c_4 depend only on the dimension n and the domain Ω.

The equivalence of the uniform estimates (1.19) with (1.17) is a consequence of the Sobolev embedding (1.13). They imply that the Taylor series of an element u of $\mathsf{A}(\Omega)$ is convergent in any point $\mathbf{x}_0 \in \overline{\Omega}$, in the sense that there exists a ball B centered in $\mathbf{x}_0$ such that the Taylor series of u at $\mathbf{x}_0$,

$$\sum_{\alpha \in \mathbb{N}^n} \frac{(\mathbf{x} - \mathbf{x}_0)^\alpha}{\alpha!}\, \partial^\alpha u(\mathbf{x}_0),$$

converges on B. Here $\alpha! = \alpha_1! \ldots \alpha_n!$ and $(\mathbf{x} - \mathbf{x}_0)^\alpha = (x_1 - x_{0,1})^{\alpha_1} \ldots (x_n - x_{0,n})^{\alpha_n}$.

Moreover, its sum coincides with $u(\mathbf{x})$ for any $\mathbf{x} \in \overline{\Omega} \cap B$, which means that $\mathsf{A}(\Omega)$ is nothing else than the class of real analytic functions up to the boundary of Ω. The "modulus of analyticity" c_0 or c_4 in (1.17)-(1.19) gives an estimate for the inverse of the smallest radius of convergence of the Taylor series.

We will prove a result on the behavior of the above analytic estimates under coordinate transformations. This will then allow us to extend the definitions of the analytic classes to the situation where Ω is a subdomain of an *analytic manifold* M of dimension n without boundary. We assume that Ω coincides with M (case without boundary) or is an analytic subdomain of M (case with boundary). The canonical example where M is the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ is suitable. Other examples that will play role here are boundary manifolds of smooth domains. Another example of a compact manifold without boundary is the torus $\mathbb{T}^n$ which has the particularity of admitting a global system of coordinates, so that the analytic classes on $\mathbb{T}^n$ or its subdomains can be defined by (1.16) directly.

In general, the analytic class $\mathsf{A}(\Omega)$ will be defined locally, by an analytic atlas of coordinate maps on M. The analyticity of M corresponds to the condition that the coordinate transformations between different local coordinate systems belonging to the atlas are analytic diffeomorphisms between domains in $\mathbb{R}^n$. The definition of $\mathsf{A}(\Omega)$ is independent of the choice of the analytic atlas, owing to the obvious fact that the composition of an analytic function f by an analytic map $\mathbf{g}$ is analytic. One way to show this is by estimating derivatives of composite maps. For this, we have the following precise quantitative result, concerning the preservation of analytic-type estimates of *finite* order:

Lemma 1.1.1 *Let $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$, let $k \in \mathbb{N}$ be fixed, and let $\mathbf{g} : \Omega \to \Omega'$ be of class $\mathscr{C}^k(\overline{\Omega})$ and $f : \Omega' \to \mathbb{R}$ of class $\mathscr{C}^k(\overline{\Omega'})$.*

(i) We assume that there are constants $C_0 > 0$, $C_1 \ge 1$ such that the components $g_1, \ldots, g_m$ of $\mathbf{g}$ satisfy the estimates

$$\forall \alpha \in \mathbb{N}^n,\ |\alpha| \le k,\ \forall \mathbf{x} \in \Omega: \quad |\partial^\alpha g_j(\mathbf{x})| \le C_1\, C_0^{|\alpha|}\, |\alpha|! \tag{1.20}$$

Then the composite function $F = f \circ \mathbf{g} \in \mathscr{C}^k(\overline{\Omega})$ *satisfies with* $C_2 = (m+1)C_0C_1$*:*

$$\forall \alpha \in \mathbb{N}^n,\ |\alpha| \le k,\ \forall \mathbf{x} \in \Omega: \quad |\partial^\alpha F(\mathbf{x})| \ \le\ C_2^{|\alpha|}\,|\alpha|!\,\sum_{\ell=0}^{|\alpha|} \frac{1}{\ell!} \max_{|\beta|=\ell} |(\partial^\beta f)(\mathbf{g}(\mathbf{x}))| \tag{1.21}$$

(ii) If in addition $m = n$ *and* $\mathbf{g}$ *is a diffeomorphism from* $\overline{\Omega}$ *to* $\overline{\Omega'}$*, so that the mapping* $f \mapsto f \circ \mathbf{g}$ *is a bounded linear operator from* $\mathsf{L}^2(\Omega')$ *to* $\mathsf{L}^2(\Omega)$ *whose norm we denote by* $M_{\mathbf{g}}$*, then we have the estimate between Sobolev seminorms with* $C_3 = \sqrt{n}C_2$

$$\frac{1}{k!}|F|_{k;\,\Omega} \ \le\ M_{\mathbf{g}}\,C_3^k\,\sum_{\ell=0}^{k} \frac{1}{\ell!}|f|_{\ell;\,\Omega'}\ . \tag{1.22}$$

Proof: It is clear that we need to prove (1.21) only for $|\alpha| = k \ge 1$. For this, we first establish a multi-dimensional version of the formula of FAÀ DI BRUNO [1, p.823]. We claim that for $|\alpha| = k$ and $\mathbf{y} = \mathbf{g}(\mathbf{x})$ we have

$$\partial^\alpha F(\mathbf{x}) \ =\ \sum_{\ell=1}^{k} \sum_{\substack{j_1,\ldots,\,j_\ell\\ \alpha_1,\ldots,\,\alpha_\ell}} \gamma^{n,k,\ell}_{\boldsymbol{j},\,\alpha_1,\ldots,\alpha_\ell} \big(\partial_{y_{j_1}\cdots y_{j_\ell}} f(\mathbf{y})\big) \prod_{i=1}^{\ell} \big(\partial^{\alpha_i} g_{j_i}(\mathbf{x})\big)\ . \tag{1.23}$$

Here the second sum is extended over all integers $j_1,\ldots,j_\ell \in \{1,\ldots,m\}$ and over all non-zero multiindices $\alpha_1,\ldots,\alpha_\ell \in \mathbb{N}^n$ (not to be mistaken for the components of α) which satisfy

$$\alpha_1 + \cdots + \alpha_\ell = \alpha\ .$$

The coefficients $\gamma^{n,k,\ell}_{\boldsymbol{j},\,\alpha_1,\ldots,\alpha_\ell}$ are non-negative integers which may depend on n, k, ℓ, $\boldsymbol{j} = (j_1,\ldots,j_\ell)$, $\alpha_1,\ldots,\alpha_\ell$, but are otherwise independent of f, $\mathbf{g}$, Ω, Ω'.

In fact, formula (1.23) can be proved simply by induction on $|\alpha|$, and the proof remains easy (and is left to the reader) as long as we do not need explicit expressions for the combinatorial coefficients $\gamma^{n,k,\ell}_{\boldsymbol{j},\alpha_1,\ldots,\alpha_\ell}$. Fortunately, for our purposes we do not need the coefficients explicitly, but we are going to show the equality

$$\sum_{\ell=1}^{k} \sum_{\substack{j_1,\ldots,\,j_\ell\\ \alpha_1,\ldots,\,\alpha_\ell}} \gamma^{n,k,\ell}_{\boldsymbol{j},\,\alpha_1,\ldots,\alpha_\ell}\,\ell!\,\prod_{i=1}^{\ell}|\alpha_i|!\ =\ m(m+1)^{k-1}\,k! \tag{1.24}$$

where the second sum is extended over the same set of indices as in (1.23).

For the proof of (1.24), we use (1.23) for a particular choice of f and $\mathbf{g}$: Let us temporarily use the shorthand notation $|\mathbf{x}| = \sum_{i=1}^{n} x_i$ for $\mathbf{x} \in \mathbb{R}^n$. Then we set

$$
\begin{aligned}
g_j(\mathbf{x}) &= \frac{1}{1-|\mathbf{x}|} = \sum_{\beta\in\mathbb{N}^n} \frac{|\beta|!}{\beta!}\mathbf{x}^\beta \quad \text{for } \mathbf{x} \text{ near } 0\,, \\
f(\mathbf{y}) &= \frac{1}{1-|\mathbf{y}-\mathbf{1}|} = \frac{1}{m+1-|\mathbf{y}|} \quad \text{for } \mathbf{y} \text{ near } \mathbf{1}=(1,\dots,1)\,. \\
\text{Then } F(\mathbf{x}) &= \frac{1}{m+1-|\mathbf{g}(\mathbf{x})|} = \frac{1}{m+1} + \frac{m}{m+1}\,\frac{1}{1-(m+1)|\mathbf{x}|}\,.
\end{aligned}
$$

We compute for $\alpha_j, \alpha \in \mathbb{N}^n$, $|\alpha| \geq 1$:

$$
\partial^{\alpha_j} g_j(0) = |\alpha_j|!\,; \quad \partial_{y_{j_1}\cdots y_{j_\ell}} f(\mathbf{1}) = \ell!\,; \quad \partial^\alpha F(0) = \frac{m}{m+1}(m+1)^{|\alpha|}\,|\alpha|!\,.
$$

If we insert this in (1.23), we find (1.24). Since all terms in (1.24) are positive, we can estimate the sum of the terms with a fixed ℓ also by the right hand side of (1.24).

Coming now back to general f and $\mathbf{g}$ satisfying (1.20), we insert the estimate

$$
\Big|\prod_{i=1}^{\ell} \partial^{\alpha_i} g_{j_i}(\mathbf{x})\Big| \leq C_1^\ell C_0^{\sum|\alpha_i|} \prod_{i=1}^{\ell} |\alpha_i|!
$$

into (1.24) and obtain

$$
\left| \sum_{\substack{j_1,\dots,j_\ell \\ \alpha_1,\dots,\alpha_\ell}} \gamma^{n,k,\ell}_{\boldsymbol{j},\alpha_1,\dots,\alpha_\ell} \big(\partial_{y_{j_1}\cdots y_{j_\ell}} f(\mathbf{y})\big) \prod_{i=1}^{\ell} \big(\partial^{\alpha_i} g_{j_i}(\mathbf{x})\big) \right| \;\leq\; (m+1)^k\, k!\, C_1^\ell\, C_0^k\, \frac{1}{\ell!}\, M(\ell, f, \mathbf{x})
$$

where

$$
M(\ell, f, \mathbf{x}) \;=\; \max_{|\beta|=\ell} |(\partial^\beta f)(\mathbf{g}(\mathbf{x}))|\,.
$$

This implies with $C_2 = (m+1)C_0C_1$:

$$
|\partial^\alpha F(\mathbf{x})| \;\leq\; k!\, C_2^k \sum_{\ell=0}^{k} \frac{1}{\ell!}\, M(\ell, f, \mathbf{x})
$$

hence (1.21).

For the proof of (1.22), we take L^2 norms in (1.21) and obtain

$$
\|\partial^\alpha F\|_\Omega \;\leq\; k!\, C_2^k \sum_{\ell=0}^{k} \frac{1}{\ell!}\, \|M(\ell, f, \cdot)\|_\Omega \;\leq\; k!\, C_2^k\, M_{\mathbf{g}} \sum_{\ell=0}^{k} \frac{1}{\ell!} |f|_{\ell;\Omega'}\,.
$$

The extra factor $\sqrt{n}$ in C_3 takes care of the sum over the multiindices α of length k. $\square$

1.2 Elliptic operators and basic estimates

In this section we introduce elliptic systems of second order and prove their fundamental regularizing property: Any solution of a second order elliptic system with an $\mathbf{L}^2$ right hand side is *locally* in $\mathbf{H}^2$. Moreover, an elliptic system on a smooth compact manifold Ω without boundary defines a Fredholm[6] operator from $\mathbf{H}^2(\Omega)$ into $\mathbf{L}^2(\Omega)$. Classical references for the theory of linear elliptic systems are [4, 5, 3, 76, 94].

All the problems we consider have associated *model* situations with simplified structure and certain invariance properties. In the case of the analysis of local properties of elliptic systems on a smooth domain, our model problems have constant coefficients and can be set on $\mathbb{R}^n$ or on the periodic domain $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. For the introduction of the basic theory, we follow the approach of [53] and prefer $\mathbb{T}^n$ to $\mathbb{R}^n$ because of the possibility of constructing model *isomorphisms* in a simple and explicit way.

1.2.a Model problems with constant coefficients on the torus

We define partial differential operators with constant coefficients and their symbol.

Definition 1.2.1 *We denote by $\mathrm{D}_{\mathbf{x}}$ the row of first order operators $(-i\partial_{x_1}, \ldots, -i\partial_{x_n})$. For $d \in \mathbb{N}$, let $P = P(\mathrm{D}_{\mathbf{x}})$ be a partial differential operator of order d with constant coefficients on $\mathbb{T}^n$:*

$$P(\mathrm{D}_{\mathbf{x}}) = \sum_{|\alpha| \le d} p^\alpha \, \partial_{\mathbf{x}}^\alpha. \tag{1.25}$$

- ⋆ *The **symbol** of P is the function $P(\boldsymbol{\xi})$ defined for $\boldsymbol{\xi} \in \mathbb{R}^n$ by:*

$$P(\boldsymbol{\xi}) = \sum_{|\alpha| \le d} p^\alpha (i\boldsymbol{\xi})^\alpha \quad \textit{with} \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_n), \quad \boldsymbol{\xi}^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}.$$

- ⋆ *The **principal part** $P^{\mathsf{pr}} = P^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}})$ of P is defined as*

$$P^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}}) = \sum_{|\alpha| = d} p^\alpha \partial_{\mathbf{x}}^\alpha.$$

  which means that we only keep the derivatives of maximal order d.

With the discrete Fourier transformation $\mathscr{F}_{\mathsf{per}}$ (1.6), we have the fundamental formula

$$\big(\mathscr{F}_{\mathsf{per}} P(\mathrm{D}_{\mathbf{x}}) u\big)(\mathbf{p}) = P(\mathbf{p})(\mathscr{F}_{\mathsf{per}} u)(\mathbf{p}), \quad \forall \mathbf{p} \in \mathbb{Z}^n.$$

Let now $L = (L_{ij})_{1 \le i,j \le N}$ be a $N \times N$ system of order 2 with *constant* coefficients. This means that for each i and j we can write

$$L_{ij}(\mathrm{D}_{\mathbf{x}}) u = \sum_{|\alpha| \le 2} a_{ij}^\alpha \, \partial_{\mathbf{x}}^\alpha u.$$

[6] An operator is said to be Fredholm if it has a finite-dimensional kernel and a closed range of finite codimension.

The equation $L\mathbf{u} = \mathbf{f}$ corresponds to the $N \times N$ system of partial differential equations

$$\sum_{j=1}^{N} L_{ij} u_j = f_i, \quad i = 1, \dots, N.$$

Let $k \in \mathbb{N}$. As an obvious consequence of the definitions, $\mathbf{u} \in \mathbf{H}^{k+2}(\mathbb{T}^n)$ implies $L\mathbf{u} \in \mathbf{H}^k(\mathbb{T}^n)$ and there holds, with a constant c independent of $\mathbf{u}$:

$$\|\mathbf{f}\|_{k;\,\mathbb{T}^n} \le c\,\|\mathbf{u}\|_{k+2;\,\mathbb{T}^n}\,.$$

Conversely, if $\mathbf{f} \in \mathbf{H}^k(\mathbb{T}^n)$, then the existence and uniqueness of a solution $\mathbf{u} \in \mathbf{H}^{k+2}(\mathbb{T}^n)$ of the equation

$$L\mathbf{u} = \mathbf{f} \quad \text{in} \quad \mathbb{T}^n, \tag{1.26}$$

is contingent on a special property of L, the *ellipticity*. We start with the following fundamental result.

Proposition 1.2.2 *Let $L(\mathrm{D}_{\mathbf{x}})$ be a system of order 2 with constant coefficients on $\mathbb{T}^n$, and let $L^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}})$ denote its principal part.*

(i) Let us assume that the following two-part condition is satisfied:

$$\begin{aligned} &\text{(a)} \qquad \forall \boldsymbol{\xi} \in \mathbb{S}^{n-1}, \quad L^{\mathsf{pr}}(\boldsymbol{\xi}) \ \textit{is invertible} \\ &\text{(b)} \qquad \forall \mathbf{p} \in \mathbb{Z}^n, \quad\ \ L(\mathbf{p}) \ \textit{is invertible.} \end{aligned} \tag{1.27}$$

Then for all $\mathbf{f} \in \mathbf{L}^2(\mathbb{T}^n)$, there exists a unique solution $\mathbf{u} \in \mathbf{H}^2(\mathbb{T}^n)$ to $L\mathbf{u} = \mathbf{f}$.

(ii) Conversely, if the problem $L\mathbf{u} = \mathbf{f}$ is uniquely solvable in $\mathbf{H}^2(\mathbb{T}^n)$ for all $\mathbf{f} \in \mathbf{L}^2(\mathbb{T}^n)$, then (1.27) holds.

Proof: *(i)* We assume (1.27). Since the function $\boldsymbol{\xi} \mapsto L^{\mathsf{pr}}(\boldsymbol{\xi})$ is continuous on the compact set $\mathbb{S}^{n-1}$, there exists $C_a > 0$ such that

$$\forall \boldsymbol{\xi} \in \mathbb{S}^{n-1}, \quad \|L^{\mathsf{pr}}(\boldsymbol{\xi})^{-1}\|_{\mathcal{L}_n} \le C_a.$$

Here $\|\cdot\|_{\mathcal{L}_n}$ is the norm of the endomorphisms of $\mathbb{C}^n$. Since the function $\boldsymbol{\xi} \mapsto L^{\mathsf{pr}}(\boldsymbol{\xi})^{-1}$ is homogeneous of degree -2 we deduce from the previous inequality that

(1) $$\forall \boldsymbol{\xi} \in \mathbb{R}^n \setminus \{0\}, \quad \|L^{\mathsf{pr}}(\boldsymbol{\xi})^{-1}\|_{\mathcal{L}_n} \le C_a |\boldsymbol{\xi}|^{-2}.$$

Let us prove that there exists $C_b > 0$ such that with $\langle \mathbf{p} \rangle = (1 + |\mathbf{p}|^2)^{1/2}$ there holds

(2) $$\forall \mathbf{p} \in \mathbb{Z}^n, \quad \|L(\mathbf{p})^{-1}\|_{\mathcal{L}_n} \le C_b \langle \mathbf{p} \rangle^{-2}.$$

Since $L(\mathbf{p}) - L^{\mathsf{pr}}(\mathbf{p}) = \mathcal{O}(\langle \mathbf{p} \rangle)$, we deduce from (1) that for all $\boldsymbol{\eta} \in \mathbb{C}^N$ and $\mathbf{p} \neq 0$

$$|\boldsymbol{\eta}||\mathbf{p}|^2 \le C_a |L^{\mathsf{pr}}(\mathbf{p})\boldsymbol{\eta}| \le C_a \Big(|L(\mathbf{p})\boldsymbol{\eta}| + C' \langle \mathbf{p} \rangle |\boldsymbol{\eta}| \Big),$$

for some positive constant C' independent of $\mathbf{p}$. This estimate yields (2) for $\langle\mathbf{p}\rangle$ large enough, say $\langle\mathbf{p}\rangle > R > 1$. The estimation (2) in the bounded set $\langle\mathbf{p}\rangle \le R$ is a direct consequence of (1.27) (b).

Let $\mathbf{f} \in \mathbf{L}^2(\mathbb{T}^n)$. Denoting by $\hat{\mathbf{f}}(\mathbf{p})$ the Fourier coefficient $\mathscr{F}_{\mathsf{per}}\mathbf{f}(\mathbf{p})$, we set for all $\mathbf{p} \in \mathbb{Z}^n$

$$\hat{\mathbf{u}}(\mathbf{p}) = L(\mathbf{p})^{-1}\hat{\mathbf{f}}(\mathbf{p}).$$

Using (2) together with the fact that $\mathbf{f}$ belongs to $\mathbf{L}^2(\mathbb{T}^n)$, we find that

$$\sum_{\mathbf{p}\in\mathbb{Z}^n} \langle\mathbf{p}\rangle^4 |\hat{\mathbf{u}}(\mathbf{p})|^2 < \infty.$$

Thus, setting

$$\mathbf{u}(\mathbf{x}) = (2\pi)^{-n/2} \sum_{\mathbf{p}\in\mathbb{Z}^n} \hat{\mathbf{u}}(\mathbf{p}) \mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}},$$

we obtain a solution $\mathbf{u} \in \mathbf{H}^2(\mathbb{T}^n)$ to the problem $L\mathbf{u} = \mathbf{f}$. This solution is clearly unique.

(ii) From the unique solvability in $\mathbf{H}^2(\mathbb{T}^n)$ of the equation $L\mathbf{u} = \mathbf{f}$ for all $\mathbf{f} \in \mathbf{L}^2(\mathbb{T}^n)$ results the estimate (invoking the closed graph theorem)

$$\|\mathbf{u}\|_{\mathbf{H}^2(\mathbb{T}^n)} \le C \|\mathbf{f}\|_{\mathbf{L}^2(\mathbb{T}^n)}.$$

Using this estimate for the special choice $\mathbf{u}(\mathbf{x}) = \boldsymbol{\eta}\,\mathrm{e}^{i\mathbf{p}\cdot\mathbf{x}}$, $\boldsymbol{\eta} \in \mathbb{C}^N$, we find

$$|\boldsymbol{\eta}|\langle\mathbf{p}\rangle^2 \le C|L(\mathbf{p})\boldsymbol{\eta}|.$$

We conclude that (1.27) (b) holds, together with the uniform estimates (2). Reversing the steps of the proof of *(i)*, we find that there exists $R' > 0$, $C' > 0$ such that:

(3) $$\forall \mathbf{p} \in \mathbb{Z}^n,\ \langle\mathbf{p}\rangle > R', \quad \|L^{\mathsf{pr}}\big(\tfrac{\mathbf{p}}{|\mathbf{p}|}\big)^{-1}\|_{\mathcal{L}_n} \le C'.$$

Then (1.27) (a) is a consequence of the density in $\mathbb{S}^{n-1}$ of the set

$$\Big\{\frac{\mathbf{p}}{|\mathbf{p}|}\ ;\ \ \mathbf{p} \in \mathbb{Z}^{n-d},\ \ |\mathbf{p}| \ge R\Big\}.$$

□

We deduce from Proposition 1.2.2 the following important corollary which is fundamental for the definition of ellipticity and also for the construction of parametrices for elliptic problems.

Corollary 1.2.3 *Let L be a system of order 2 with constant coefficients on $\mathbb{T}^n$, and let L^{pr} denote its principal part. We assume that it is* **elliptic***, that is $L^{\mathsf{pr}}(\boldsymbol{\xi})$ is invertible for all $\boldsymbol{\xi}$ in the unit sphere $\mathbb{S}^{n-1}$ – condition* (1.27) (a)*. Then there holds*

- *(i) There exists a system $\tilde{L}$ of order 2 with constant coefficients and the same principal part as L, and such that $\tilde{L}$ defines an isomorphism from $\mathbf{H}^2(\mathbb{T}^n)$ to $\mathbf{L}^2(\mathbb{T}^n)$.*
- *(ii) The system L defines a Fredholm operator from $\mathbf{H}^2(\mathbb{T}^n)$ onto $\mathbf{L}^2(\mathbb{T}^n)$.*

Proof: *(i)* With $\vec{\frac{1}{2}} = (\frac{1}{2},\ldots,\frac{1}{2}) \in \mathbb{R}^n$, we set, cf. [53, Rem. 2.1.1]:

$$\tilde{L}(\mathrm{D}_{\mathbf{x}}) = L^{\mathsf{pr}}\big(\mathrm{D}_{\mathbf{x}} + \vec{\tfrac{1}{2}}\big).$$

Then $\tilde{L}(\mathbf{p}) = L^{\mathsf{pr}}(\mathbf{p} + \vec{\frac{1}{2}})$. For all $\mathbf{p} \in \mathbb{Z}^n$, $\mathbf{p} + \vec{\frac{1}{2}}$ is not zero. Therefore the ellipticity condition, which implies that $L^{\mathsf{pr}}(\boldsymbol{\xi})$ is invertible for all $\boldsymbol{\xi} \neq 0$, yields condition (1.27) (b) for $\tilde{L}$. Proposition 1.2.2 gives the invertibility of $\tilde{L}$.

(ii) Since $L^{\mathsf{pr}} = \tilde{L}^{\mathsf{pr}}$, and $\mathbb{T}^n$ is compact, the difference $L - \tilde{L}$ is compact from $\mathbf{H}^2(\mathbb{T}^n)$ into $\mathbf{L}^2(\mathbb{T}^n)$. Thus *(ii)* is a consequence of *(i)*. □

Remark 1.2.4 If L is elliptic and homogeneous of order 2, that is $L = L^{\mathsf{pr}}$, then the condition (1.27) (b) is never satisfied, because $L(\mathbf{0}) = 0$. The kernel and cokernel consist of constants $\mathbf{u} \in \mathbb{C}^N$. △

Remark 1.2.5 The simple case $n = 1$ is included in the results of this chapter. In this case, we have a second order system of ordinary differential equations, and the ellipticity condition (1.27) (a) is equivalent to the invertibility of the matrix multiplying the second derivative. △

1.2.b Local a priori estimates for problems with smooth coefficients

Let Ω be a open set in $\mathbb{R}^n$ or in $\mathbb{T}^n$. By $\mathbf{x}$ we denote Cartesian coordinates.

Definition 1.2.6 *Let $P = P(\mathbf{x};\mathrm{D}_{\mathbf{x}})$ be a partial differential operator of order d with smooth coefficients on $\overline{\Omega}$:*

$$P(\mathbf{x};\mathrm{D}_{\mathbf{x}}) = \sum_{|\alpha|\le d} p^\alpha(\mathbf{x})\,\partial^\alpha_{\mathbf{x}}, \quad p^\alpha \in \mathscr{C}^\infty(\overline{\Omega}). \tag{1.28}$$

- ⋆ *Its principal part is $P^{\mathsf{pr}}(\mathbf{x};\mathrm{D}_{\mathbf{x}}) = \sum_{|\alpha|=d} p^\alpha(\mathbf{x})\,\partial^\alpha_{\mathbf{x}}$.*
- ⋆ *Let $\mathbf{x}_0 \in \overline{\Omega}$. The principal part of the operator P **frozen at** $\mathbf{x}_0$ is the partial differential operator $P^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})$, homogeneous with constant coefficients:*

$$P^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}}) = \sum_{|\alpha|=d} p^\alpha(\mathbf{x}_0)\,\partial^\alpha_{\mathbf{x}}.$$

- ⋆ *The **principal symbol** of P is the function $(\mathbf{x},\boldsymbol{\xi}) \mapsto P^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi})$ defined for $\mathbf{x} \in \overline{\Omega}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$ by the symbol in $\boldsymbol{\xi}$ of its principal part:*

$$P^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi}) = \sum_{|\alpha|=d} p^\alpha(\mathbf{x})(i\boldsymbol{\xi})^\alpha \quad \textit{with} \quad \boldsymbol{\xi} = (\xi_1,\ldots,\xi_n).$$

Let $L = \big(L_{ij}\big)_{1\le i,j\le N}$ be a $N \times N$ system of order 2 with smooth coefficients in $\overline{\Omega}$:

$$L_{ij}(\mathrm{D}_{\mathbf{x}})u = \sum_{|\alpha|\le 2} a^\alpha_{ij}(\mathbf{x})\,\partial^\alpha_{\mathbf{x}}u, \quad a^\alpha_{ij} \in \mathscr{C}^\infty(\overline{\Omega}), \quad i,j \in \{1,\ldots,n\}. \tag{1.29}$$

If Ω is bounded, then it is clear that the operator L is continuous from $\mathbf{H}^{k+2}(\Omega)$ to $\mathbf{H}^{k}(\Omega)$ for any $k \in \mathbb{N}$. The converse estimate, namely of $\mathbf{u}$ by $\mathbf{f}$ globally in Ω, is much more difficult, and even impossible if Ω has a non-empty boundary. Nevertheless, if Ω_1 and Ω_2 are subdomains of Ω such that the closure $\overline{\Omega}_1$ is included in Ω_2 (which implies that the distance between $\partial\Omega_1$ and $\partial\Omega_2$ is positive), and *if L is elliptic*, then it is possible, knowing *a priori* a solution $\mathbf{u}$ of the equation $L\mathbf{u} = \mathbf{f}$ on Ω_2, to give an estimate of the Sobolev norms of $\mathbf{u}$ over the smaller domain Ω_1 by the norms of $\mathbf{f}$ over Ω_2 and the $\mathbf{H}^1$ norm of $\mathbf{u}$ over Ω_2.

The property of ellipticity is defined by means of the principal part frozen in each point, and for the latter it coincides with the notion of ellipticity introduced in Corollary 1.2.3.

Definition 1.2.7 *Let $L(\mathbf{x};\mathrm{D}_\mathbf{x})$ be a $N \times N$ system of second order operators as in (1.29).*

- $\star$ *Its **principal part** L^{pr} is the system (L^{pr}_{ij}), with $L^{\mathsf{pr}}_{ij}(\mathbf{x};\mathrm{D}_\mathbf{x}) = \sum_{|\alpha|=2} a^\alpha_{ij}(\mathbf{x})\,\partial^\alpha_\mathbf{x}$.*
- $\star$ *Its **principal part frozen at $\mathbf{x}_0$** is the system $L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_\mathbf{x})$ with constant coefficients*

$$L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_\mathbf{x}) = \big(L^{\mathsf{pr}}_{ij}(\mathbf{x}_0;\mathrm{D}_\mathbf{x})\big) \quad \textit{with} \quad L^{\mathsf{pr}}_{ij}(\mathbf{x}_0;\mathrm{D}_\mathbf{x}) = \sum_{|\alpha|=2} a^\alpha_{ij}(\mathbf{x}_0)\,\partial^\alpha_\mathbf{x}.$$

- $\star$ *The **principal symbol** of L is the matrix valued function*

$$\overline{\Omega} \times \mathbb{R}^n \ni (\mathbf{x},\boldsymbol{\xi}) \mapsto L^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi}) := \big(L^{\mathsf{pr}}_{ij}(\mathbf{x};\boldsymbol{\xi})\big) \in \mathcal{L}_n.$$

- $\star$ *Let $\mathbf{x}_0 \in \overline{\Omega}$. The system L is called **elliptic** at $\mathbf{x}_0$ if*

$$\forall \boldsymbol{\xi} \in \mathbb{S}^{n-1}: \quad L^{\mathsf{pr}}(\mathbf{x}_0;\boldsymbol{\xi}) \textit{ is invertible.} \tag{1.30}$$

- $\star$ *The system L is called elliptic **on** $\overline{\Omega}$ if it is elliptic at $\mathbf{x}_0$ for all $\mathbf{x}_0 \in \overline{\Omega}$.*

Example 1.2.8 *(i)* The simplest example of a *scalar* elliptic operator (i.e. with $N = 1$) is the Laplace operator $\Delta = \partial_1^2 + \ldots + \partial_n^2$. Its symbol is $-(\xi_1^2 + \ldots + \xi_n^2) = -|\boldsymbol{\xi}|^2$.
(ii) A standard example of elliptic system with $N = n$ is the Lamé system which can be written as $\mu\Delta\mathbb{I}_n + (\lambda+\mu)\nabla\,\mathrm{div}$, where λ and μ are real numbers, $\mathbb{I}_n$ is the identity matrix of dimension n, ∇ is the gradient operator and div is the divergence, its adjoint operator. For isotropic linear elasticity, λ and μ are positive, and L is elliptic. More generally, L is elliptic for any couple (λ,μ) with $\mu \neq 0$ and $\lambda \neq -2\mu$, see [63]. $\triangle$

We will obtain the Fredholm theory for elliptic systems with variable coefficients, by considering them as perturbations of the constant coefficient case. Using Corollary 1.2.3, we first prove the existence of “local parametrices”, a kind of approximate two-sided inverses. For this, the continuity of coefficients is sufficient.

Proposition 1.2.9 *Let L be a second order system with $\mathscr{C}^0$ coefficients, elliptic at $\mathbf{x}_0$. There exists a ball $\mathcal{B}_{\mathbf{x}_0}$ centered at $\mathbf{x}_0$ and an operator $\mathfrak{E}_{\mathbf{x}_0}$ continuous from $\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ into*

$\mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0})$ *such that for any* $\psi', \psi'' \in \mathscr{C}_0^\infty(\mathcal{B}_{\mathbf{x}_0})$ *with* $\psi''\psi' = \psi'$*:*

$$L\psi''\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} = \psi'\mathbf{f} + K\mathbf{f}, \quad \forall \mathbf{f} \in \mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0}), \tag{1.31a}$$

$$\psi''\mathfrak{E}_{\mathbf{x}_0}\psi' L\mathbf{u} = \psi'\mathbf{u} + K'\mathbf{u}, \quad \forall \mathbf{u} \in \mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0}), \tag{1.31b}$$

where $K : \mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0}) \to \mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ *and* $K' : \mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0}) \to \mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0})$ *are compact operators. In fact,* K *is continuous from* $\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ *to* $\mathbf{H}^1(\mathcal{B}_{\mathbf{x}_0})$*, and* K' *from* $\mathbf{H}^1(\mathcal{B}_{\mathbf{x}_0})$ *to* $\mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0})$*.*

Proof: As will be used in next chapters, let us denote by $\underline{L}_{\mathbf{x}_0}$ the principal part of L frozen at $\mathbf{x}_0$:

$$\underline{L}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{x}}) = L^{\mathsf{pr}}(\mathbf{x}_0; \mathrm{D}_{\mathbf{x}}).$$

Since L is elliptic at $\mathbf{x}_0$, the system $\underline{L}_{\mathbf{x}_0}$ satisfies the assumptions of Corollary 1.2.3. Therefore there exists an operator $\tilde{L}$ with constant coefficients on $\mathbb{T}^n$ which has the same principal part as L at $\mathbf{x}_0$ and is invertible from $\mathbf{H}^2(\mathbb{T}^n)$ onto $\mathbf{L}^2(\mathbb{T}^n)$. We choose $R_* \in (0, \pi)$ so that $B_{R_*}(\mathbf{x}_0)$, the ball of center $\mathbf{x}_0$ and radius R, can be considered as a subset of $\mathbb{T}^n$. We then have the estimate for all R, $0 < R \le R_*$

(1) $$\|L^{\mathsf{pr}}\mathbf{u} - \underline{L}_{\mathbf{x}_0}\mathbf{u}\|_{0;\,B_R(\mathbf{x}_0)} \le c \max_{\substack{i,j;\,|\alpha|=2\\ |\mathbf{x}-\mathbf{x}_0|\le R}} |a^\alpha_{ij}(\mathbf{x}) - a^\alpha_{ij}(\mathbf{x}_0)|\, \|\mathbf{u}\|_{2;\,B_R(\mathbf{x}_0)}\,.$$

Here the constant c does not depend on R. Choose $\psi \in \mathscr{C}_0^\infty(B_1(\mathbf{0}))$ with values in $[0,1]$ and such that $\psi \equiv 1$ on $B_{1/2}(\mathbf{0})$. From (1) we deduce

(2) $$\|\psi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{R}\Big)(L^{\mathsf{pr}}\mathbf{u} - \underline{L}_{\mathbf{x}_0}\mathbf{u})\|_{0;\,\mathbb{T}^n} \le c \max_{\substack{i,j;\,|\alpha|=2\\ |\mathbf{x}-\mathbf{x}_0|\le R}} |a^\alpha_{ij}(\mathbf{x}) - a^\alpha_{ij}(\mathbf{x}_0)|\, \|\mathbf{u}\|_{2;\,\mathbb{T}^n}\,.$$

Since we have assumed that the coefficients of L are continuous, this estimate implies that

(3) $$\|\psi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{R}\Big)(L^{\mathsf{pr}}\mathbf{u} - \underline{L}_{\mathbf{x}_0}\mathbf{u})\|_{0;\,\mathbb{T}^n} \le \gamma(R)\|\mathbf{u}\|_{2;\,\mathbb{T}^n}\,,$$

for some non-decreasing function $\gamma > 0$, which tends to 0 as $R \to 0$ and is independent of u. We now choose $R = R_0$ small enough so that

(4) $$\gamma(R_0)\|\tilde{L}^{-1}\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{T}^n),\mathbf{H}^2(\mathbb{T}^n))} \le \tfrac{1}{2}.$$

We define the operator $\mathfrak{L}$ on $\mathbb{T}^n$ by

(5) $$\mathfrak{L} = \tilde{L} + \psi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{R_0}\Big)(L^{\mathsf{pr}} - \underline{L}_{\mathbf{x}_0}).$$

From (3)-(5) we deduce that $\mathfrak{L}$ is invertible from $\mathbf{H}^2(\mathbb{T}^n)$ onto $\mathbf{L}^2(\mathbb{T}^n)$. We set

(6) $$\mathcal{B}_{\mathbf{x}_0} = B_{R_0/2}(\mathbf{x}_0).$$

By construction we have $\mathfrak{L}^{\mathsf{pr}}(\mathbf{x};\mathrm{D}_{\mathbf{x}}) = L^{\mathsf{pr}}(\mathbf{x};\mathrm{D}_{\mathbf{x}})$ for all $\mathbf{x}\in\mathcal{B}_{\mathbf{x}_0}$, hence

(7) $$(L-\mathfrak{L})\big|_{\mathcal{B}_{\mathbf{x}_0}} \quad \text{is an operator of order } 1.$$

Let $\mathfrak{P}_0$ be the operator of extension by zero from $\mathcal{B}_{\mathbf{x}_0}$ into $\mathbb{T}^n$, continuous from $\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ into $\mathbf{L}^2(\mathbb{T}^n)$. We define our operator $\mathfrak{E}_{\mathbf{x}_0}$ for $\mathbf{f}\in\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ by

$$\mathfrak{E}_{\mathbf{x}_0}\mathbf{f} = \big(\mathfrak{L}^{-1}\mathfrak{P}_0\mathbf{f}\big)\big|_{\mathcal{B}_{\mathbf{x}_0}} \in \mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0}).$$

Thus

(8) $$\mathfrak{L}\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} = \psi'\mathbf{f},\ \forall\mathbf{f}\in\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0}) \quad\text{and}\quad \mathfrak{E}_{\mathbf{x}_0}\mathfrak{L}\psi'\mathbf{u} = \psi'\mathbf{u},\ \forall\mathbf{u}\in\mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0}).$$

From (6)-(8) we can deduce (1.31a)-(1.31b). Indeed from (8), we have

$$L\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} = \psi'\mathbf{f} + (L-\mathfrak{L})\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f},$$

and therefore with the commutator $[L,\psi''] = L\psi'' - \psi''L$

$$\begin{aligned} L\psi''\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} &= \psi''L\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} + [L,\psi'']\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} \\ &= \psi''\psi'\mathbf{f} + \psi''(L-\mathfrak{L})\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} + [L,\psi'']\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} \\ &= \psi'\mathbf{f} + \psi''(L-\mathfrak{L})\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} + [L,\psi'']\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f}. \end{aligned}$$

This shows (1.31a) with

(9) $$K\mathbf{f} = \psi''(L-\mathfrak{L})\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f} + [L,\psi'']\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{f}.$$

The operator K is continuous from $\mathbf{L}^2(\mathcal{B}_{\mathbf{x}_0})$ to $\mathbf{H}^1(\mathcal{B}_{\mathbf{x}_0})$, because both $\psi''(L-\mathfrak{L})$ and $[L,\psi'']$ are differential operators of order one.

The identity (1.31b) is proved in a similar manner: From (8) we have

$$\begin{aligned} \mathfrak{E}_{\mathbf{x}_0}\psi'L\mathbf{u} &= \mathfrak{E}_{\mathbf{x}_0}L\psi'\mathbf{u} + \mathfrak{E}_{\mathbf{x}_0}[\psi',L]\mathbf{u} \\ &= \mathfrak{E}_{\mathbf{x}_0}\mathfrak{L}\psi'\mathbf{u} + \mathfrak{E}_{\mathbf{x}_0}(L-\mathfrak{L})\psi'\mathbf{u} + \mathfrak{E}_{\mathbf{x}_0}[\psi',L]\mathbf{u} \\ &= \psi'\mathbf{u} + \mathfrak{E}_{\mathbf{x}_0}(L-\mathfrak{L})\psi'\mathbf{u} + \mathfrak{E}_{\mathbf{x}_0}[\psi',L]\mathbf{u}. \end{aligned}$$

This shows (1.31b) with

(10) $$K'\mathbf{u} = \psi''\mathfrak{E}_{\mathbf{x}_0}(L-\mathfrak{L})\psi'\mathbf{u} + \psi''\mathfrak{E}_{\mathbf{x}_0}[\psi',L]\mathbf{u}.$$

□

Proposition 1.2.9 allows us to prove the basic local estimate which we are going to use as a starting point for the proof of elliptic regularity (the gain of two orders of derivation) and of analytic hypoellipticity.

Corollary 1.2.10 *Let L be a $N\times N$ system of second order partial differential operators with smooth coefficients on $\overline{\Omega}$. We assume that L is elliptic at $\mathbf{x}_0\in\Omega$. Then there exists a ball $\mathcal{B}^*_{\mathbf{x}_0}$ centered at $\mathbf{x}_0$ and a constant $A_0>0$ such that for all $\mathbf{u}\in\mathbf{H}^2_0(\mathcal{B}^*_{\mathbf{x}_0})$ there* *holds*

$$\|\mathbf{u}\|_{2;\,\mathcal{B}^*_{\mathbf{x}_0}} \le A_0\big(\|L\mathbf{u}\|_{0;\,\mathcal{B}^*_{\mathbf{x}_0}} + \|\mathbf{u}\|_{1;\,\mathcal{B}^*_{\mathbf{x}_0}}\big). \tag{1.32}$$

Proof: Let $\mathcal{B}^*_{\mathbf{x}_0}$ be a ball with center $\mathbf{x}_0$ strictly smaller than the ball $\mathcal{B}_{\mathbf{x}_0}$ found in Proposition 1.2.9. We choose the cut-off functions ψ' and ψ'' as in that Proposition and such that $\psi' \equiv 1$ on $\mathcal{B}^*_{\mathbf{x}_0}$. The relation (1.31b) together with the continuity of $\mathfrak{E}_{\mathbf{x}_0}$ implies for any $\mathbf{u} \in \mathbf{H}^2_0(\mathcal{B}^*_{\mathbf{x}_0})$ the estimate

$$\|\mathbf{u} + K'\mathbf{u}\|_{2;\,\mathcal{B}^*_{\mathbf{x}_0}} \le c\|L\mathbf{u}\|_{0;\,\mathcal{B}^*_{\mathbf{x}_0}}.$$

Using the continuity of K' from $\mathbf{H}^1(\mathcal{B}_{\mathbf{x}_0})$ to $\mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0})$, we obtain estimate (1.32). □

Remark 1.2.11 The converse statement also holds: If the system satisfies the estimate (1.32), then L is elliptic at $\mathbf{x}_0$, see Theorem 3.2.4 of [53]. △

1.2.c Problems with smooth coefficients on a compact manifold

Let M be a smooth n-dimensional manifold. This means that there is an "atlas" of coordinate maps, that is a covering of M by open sets $\mathcal{U}_j$ and associated bijections ϕ_j from $\mathcal{U}_j$ to the open unit ball $B \subset \mathbb{R}^n$ such that the coordinate transformations

$$\phi_j \circ \phi_i^{-1}: \quad \phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \phi_j(\mathcal{U}_i \cap \mathcal{U}_j)$$

are $\mathscr{C}^\infty$ diffeomorphisms between open subsets of B. A function u on M is of class $\mathscr{C}^\infty$ if all its "expressions in local coordinates" $\phi_j^* u := u \circ \phi_j^{-1}$ are $\mathscr{C}^\infty$ functions on B. Similarly one can introduce the notion of manifolds of class $\mathscr{C}^k$ for $k \ge 2$ by requiring that the coordinate transformations are of regularity $\mathscr{C}^k$.

A differential operator with smooth coefficients on M can be defined as an operator mapping $\mathscr{C}^\infty(M)$ to $\mathscr{C}^\infty(M)$ such that all its expressions in local coordinates are differential operators with smooth coefficients on M. In local coordinates, one can then define the notions of principal part, principal part frozen at a point and principal symbol as in Definition 1.2.7. The objects so defined will at first depend on the choice of local coordinates. Although it is possible to define underlying invariant differential-geometric objects, we will not make the effort to do so, because the main concept of *ellipticity* and the main results, such as regularity, will nevertheless be independent of the choice of local coordinates.

For the case of the notion of ellipticity at a point, it is easy to see that the principal symbols of two different expressions in local coordinates of the same differential operator are related by a similarity transformation, so that they are simultaneously invertible or not. The following definition therefore makes sense.

Definition 1.2.12 *Let L be an $N \times N$ system of partial differential operators of order 2 with smooth coefficients on the smooth manifold M.*

- ⋆ *L is said to be elliptic at the point $\mathbf{x} \in M$ if for any local coordinate map $\phi_j : \mathcal{U}_j \to B$ with $\mathbf{x} \in \mathcal{U}_j$, the representation of L in local coordinates $\phi_j^* L (\phi_j^*)^{-1}$ is elliptic at the point $\phi_j(\mathbf{x}) \in B$.*

- ⋆ *L is said to be elliptic on a subset Ω of M if it is elliptic at any point $x \in \Omega$.*

By transport via local coordinate maps, the local parametrix construction of Proposition 1.2.9 and the local a priori estimate (1.32) of Corollary 1.2.10 remain valid if Ω is an open subset of a smooth manifold M. The only change in the formulation of these two propositions is in the interpretation of the term "ball". The "ball" $\mathcal{B}_{\mathbf{x}_0}$ centered at $\mathbf{x}_0 \in \Omega \subset M$ now has to be understood as the image under ϕ_j^{-1} of a ball $B \subset \mathbb{R}^n$ centered at $\phi_j(\mathbf{x}_0)$.

Proposition 1.2.9 now implies the Fredholm property for elliptic systems of order 2 in the case of a manifold without boundary. For basic facts about Fredholm operators, see [41, 94].

Theorem 1.2.13 *Let Ω be a smooth compact manifold without boundary. Let L be a $N \times N$ system of second order operators with smooth coefficients, elliptic over Ω. Then L defines a Fredholm operator from $\mathbf{H}^2(\Omega)$ to $\mathbf{L}^2(\Omega)$.*

Proof: We extract a finite covering $\{\mathcal{B}_{\mathbf{x}} : \mathbf{x} \in X\}$ of Ω from the covering by all balls $\mathcal{B}_{\mathbf{x}_0}$, $\mathbf{x}_0 \in \Omega$, provided by Proposition 1.2.9. Choosing an atlas on the manifold Ω, we can assume that every ball $\mathcal{B}_{\mathbf{x}}$ for $\mathbf{x} \in X$ is contained in one coordinate domain and can therefore be considered as a subset of $\mathbb{R}^n$. Let $\psi'_{\mathbf{x}}$ be a smooth partition of unity on Ω subordinate to the covering $\{\mathcal{B}_{\mathbf{x}} : \mathbf{x} \in X\}$. Let $\psi''_{\mathbf{x}}$ be a smooth cut-off function such that $\psi''_{\mathbf{x}}\psi'_{\mathbf{x}} = \psi'_{\mathbf{x}}$, and with support contained in $\mathcal{B}_{\mathbf{x}}$ Then we set

$$E = \sum_{\mathbf{x} \in X} \psi''_{\mathbf{x}} \mathfrak{E}_{\mathbf{x}} \psi'_{\mathbf{x}}.$$

The operator E is continuous from $\mathbf{L}^2(\Omega)$ into $\mathbf{H}^2(\Omega)$ and is a global parametrix of L, which means that

$$LE = \mathbb{I} + K,\ K \text{ compact in } \mathbf{L}^2(\Omega), \quad EL = \mathbb{I} + K',\ K' \text{ compact in } \mathbf{H}^2(\Omega).$$

From this two identities, we see that E is both a left and a right regularizer for L. Therefore, by Atkinson's well-known theorem in [8], see also [64, Ch. 1], we find that L is Fredholm. ☐

If Ω has a boundary (which is our main subject of interest), we will need estimates *up to the boundary*. As we will explain in the next chapter, the existence of such estimates is ensured by complementing the system L by a set of N *"covering" boundary conditions* on $\partial\Omega$. Then similar estimates as (1.32) hold up to the boundary $\partial\Omega$ in a neighborhood of any point $\mathbf{x}_0$ where $\partial\Omega$ is sufficiently smooth.

1.3 Interior regularity of solutions in Sobolev spaces

1.3.a Model elliptic systems on the torus

We start with the situation of model systems on $\mathbb{T}^n$: For elliptic systems in the periodic case, it is rather easy to prove a result on Sobolev and analytic regularity, which prefigures what we will find later on in many different, more complicated, situations. We have here in a nutshell the statements that will be labelled "regularity theorem", "shift theorem", and "analytic hypoellipticity".

Although it addresses a rather particular situation, we choose to present the proof because of its simplicity. We start with finite regularity in Sobolev spaces and note that we get uniform estimates in Sobolev norms.

Theorem 1.3.1 *Let L be a system of order 2 with constant coefficients on $\mathbb{T}^n$. Let $\mathbf{u} \in \mathscr{D}'(\mathbb{T}^n)^N$ be such that $L\mathbf{u} = \mathbf{f}$ with $\mathbf{f} \in \mathbf{H}^s(\mathbb{T}^n)$ for a real number s.*

(i) We suppose that (1.27) (a) & (b) hold. Then $\mathbf{u} \in \mathbf{H}^{s+2}(\mathbb{T}^n)$, and there exists a constant C_L independent of s and $\mathbf{f}$ such that we have the estimates

$$\|\mathbf{u}\|_{s+2;\,\mathbb{T}^n} \le C_L \|\mathbf{f}\|_{s;\,\mathbb{T}^n}. \tag{1.33}$$

(ii) We suppose that (1.27) (a) holds. Then $\mathbf{u} \in \mathbf{H}^{s+2}(\mathbb{T}^n)$, and there exist two constants C_L and M_L independent of s and $\mathbf{f}$ such that we have the following a priori estimates for all $\ell < s+2$:

$$\|\mathbf{u}\|_{s+2;\,\mathbb{T}^n} \le C_L\big(\|\mathbf{f}\|_{s;\,\mathbb{T}^n} + M_L^{s+2-\ell}\|\mathbf{u}\|_{\ell;\,\mathbb{T}^n}\big). \tag{1.34}$$

Using the estimates (1.34) for all $s \in \mathbb{N}$ and with $\ell = 0$, we obtain immediately the "analytic shift theorem" for elliptic systems with constant coefficients on the torus.

Corollary 1.3.2 *Let L be an elliptic system of order 2 with constant coefficients on $\mathbb{T}^n$, that is condition (1.27) (a) holds. Let $\mathbf{u} \in \mathscr{D}'(\mathbb{T}^n)^N$ be such that $L\mathbf{u} = \mathbf{f}$ with $\mathbf{f}$ in the analytic class $\mathbf{A}(\mathbb{T}^n)$. Then $\mathbf{u}$ belongs to the same analytic class.*

Proof of Theorem 1.3.1: *(i)* When conditions (1.27) (a) & (b) hold, the estimate (1.33) is a consequence of the expression (1.7) of the norm in $\mathrm{H}^s(\mathbb{T}^n)$ by Fourier coefficients and of the uniform bound on the inverse of the symbol

(1) $$\|L(\mathbf{p})^{-1}\|_{\mathcal{L}_n} \le C_b\langle\mathbf{p}\rangle^{-2},$$

valid for all $\mathbf{p} \in \mathbb{Z}^n$ as shown in the proof of Proposition 1.2.2. Then (1.33) holds with $C_L = C_b$.

(ii) If condition (1.27) (a) holds, we have shown in the same proof that there exists R such that (1) is still valid for $\langle\mathbf{p}\rangle \ge R$. Thus we use (1) to bound the Fourier coefficients $\hat{\mathbf{u}}(\mathbf{p})$

of $\mathbf{u}$ for $\langle \mathbf{p} \rangle \geq R$. For $\langle \mathbf{p} \rangle < R$, we simply write

$$\langle \mathbf{p} \rangle^{s+2} |\hat{\mathbf{u}}(\mathbf{p})| \leq R^{s+2-\ell} \langle \mathbf{p} \rangle^{\ell} |\hat{\mathbf{u}}(\mathbf{p})|,$$

which proves the estimate (1.34) with $M_L = R$. □

1.3.b General elliptic systems

Let now L be a $N \times N$ general system of second order operators with smooth coefficients, elliptic over Ω. Elliptic regularity theorems (also called *shift theorems*) give answers to the following questions

(i) If Ω is compact without boundary and if $\mathbf{u} \in \mathbf{H}^2(\Omega)$ satisfies $L\mathbf{u} \in \mathbf{H}^k(\Omega)$ with $k > 0$, does $\mathbf{u}$ belong to $\mathbf{H}^{k+2}(\Omega)$?

(ii) The localized version of *(i)*: For any subdomains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2 \subset \Omega$ and any $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ such that $L\mathbf{u} \in \mathbf{H}^k(\Omega_2)$, does $\mathbf{u}$ belong to $\mathbf{H}^{k+2}(\Omega_1)$?

(iii) If Ω is compact without boundary and if $\mathbf{u} \in \mathbf{H}^2(\Omega)$ satisfies $L\mathbf{u} \in \mathbf{A}(\Omega)$, does $\mathbf{u}$ belong to the analytic class $\mathbf{A}(\Omega)$?

(iv) The localized version of *(iii)*: For Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2 \subset \Omega$ and $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ such that $L\mathbf{u} \in \mathbf{A}(\Omega_2)$, does $\mathbf{u}$ belong to $\mathbf{A}(\Omega_1)$?

The answer is "Yes" to the four questions, and the whole subsequent part of Chapter 1 is devoted to the proof of this.

Answers to questions *(i)* and *(ii)* can be given by two different approaches:

- Combining Corollary 1.2.3 with Theorem 1.3.1, we can construct local parametrices from $\mathbf{H}^k(\mathcal{B}_\mathbf{x})$ into $\mathbf{H}^{k+2}(\mathcal{B}_\mathbf{x})$, like in Proposition 1.2.9.
- Relying on the basic $\mathbf{H}^2$-$\mathbf{L}^2$ estimates provided by Corollary 1.2.10, one can iteratively apply a priori estimates to derivatives of $\mathbf{u}$, at any order.

In this section, we are going to prove interior regularity estimates in Sobolev spaces, giving a positive answer to questions *(i)* and *(ii)*, using the method of local parametrices. In contrast, concerning questions *(iii)* and *(iv)*, the iterative approach allows a better control of constants with respect to the derivatives order k, which is necessary to prove regularity in analytic classes. This will be done in the remaining sections of this chapter.

The main result on finite Sobolev regularity follows, first in integer order Sobolev spaces, then in positive real order Sobolev spaces.

Theorem 1.3.3 *Let k be a positive integer. Let Ω be a domain in $\mathbb{R}^n$ or, more generally, in a $\mathscr{C}^{k+2}$ manifold of dimension n. Let L be a $N \times N$ system of second order operators elliptic on Ω. We assume that the coefficients of L belong to $\mathscr{C}^k(\Omega)$.*

(i) For any bounded subdomains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2 \subset \Omega$, if $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ satisfies $L\mathbf{u} \in \mathbf{H}^k(\Omega_2)$, then $\mathbf{u}$ belongs to $\mathbf{H}^{k+2}(\Omega_1)$ with the estimate

$$\|\mathbf{u}\|_{k+2;\,\Omega_1} \leq c\big(\|L\mathbf{u}\|_{k;\,\Omega_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\big),$$

where the positive constant c depends on k, Ω_1, Ω_2 and L, but not on $\mathbf{u}$.

(ii) We assume now that Ω *is a compact manifold without boundary. If* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *satisfies* $L\mathbf{u} \in \mathbf{H}^k(\Omega)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{H}^{k+2}(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{k+2;\,\Omega} \leq c\big(\|L\mathbf{u}\|_{k;\,\Omega} + \|\mathbf{u}\|_{1;\,\Omega}\big).$$

In addition, L *defines a Fredholm operator from* $\mathbf{H}^{k+2}(\Omega)$ *to* $\mathbf{H}^k(\Omega)$*, denoted by* L_k*. The kernel and the cokernel of* L_k *do not depend on* k*.*

Before proving this theorem, we deduce the following corollary which states the extension to non-integral Sobolev exponents.

Corollary 1.3.4 *Under the conditions of Theorem 1.3.3, let* s *be real,* $0 \leq s \leq k$*.*

(i) For Ω_1*,* Ω_2 *as above, if* $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ *satisfies* $L\mathbf{u} \in \mathbf{H}^s(\Omega_2)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{H}^{s+2}(\Omega_1)$ *with the estimate*

$$\|\mathbf{u}\|_{s+2;\,\Omega_1} \leq c(\|L\mathbf{u}\|_{s;\,\Omega_2} + \|\mathbf{u}\|_{1;\,\Omega_2}). \tag{1.35}$$

(ii) If Ω *is a compact manifold without boundary and* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *satisfies* $L\mathbf{u} \in \mathbf{H}^s(\Omega)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{H}^{s+2}(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{s+2;\,\Omega} \leq c(\|L\mathbf{u}\|_{s;\,\Omega} + \|\mathbf{u}\|_{1;\,\Omega}). \tag{1.36}$$

In addition, L *defines a Fredholm operator* $L_s : \mathbf{H}^{s+2}(\Omega) \mapsto \mathbf{H}^s(\Omega)$*. The kernel and the cokernel of* L_s *do not depend on* s*.*

Proof: Let ℓ be the integral part of s. From *(ii)* of Theorem 1.3.3, L_ℓ and $L_{\ell+1}$ are Fredholm, and have the same kernels and cokernels. By the fundamental theorem of Hilbert space interpolation, we deduce that L_s is also Fredholm, with the same kernels and cokernels. This proves *(ii)* for all $s \in [0, k]$ and we deduce *(i)* by localization. □

The proof of Theorem 1.3.3 relies on refined properties of local parametrices $\mathfrak{E}_{\mathbf{x}_0}$, along the lines of Proposition 1.2.9. Let us prove these first.

Proposition 1.3.5 *Let* k *be a positive integer. Let* L *be a second order system with* $\mathscr{C}^k$ *coefficients, elliptic at* $\mathbf{x}_0$*. There exist a ball* $\mathcal{B}_{\mathbf{x}_0}$ *centered at* $\mathbf{x}_0$ *and an operator* $\mathfrak{E}_{\mathbf{x}_0}$ *(the local parametrix) enjoying the same properties as in Proposition 1.2.9, with, moreover:*

- ⋆ $\mathfrak{E}_{\mathbf{x}_0}$ *is continuous from* $\mathbf{H}^k_0(\mathcal{B}_{\mathbf{x}_0})$ *into* $\mathbf{H}^{k+2}(\mathcal{B}_{\mathbf{x}_0})$*,*
- ⋆ *The compact operators* K *and* K' *satisfy the following continuity properties:* K *is continuous from* $\mathbf{H}^k(\mathcal{B}_{\mathbf{x}_0})$ *to* $\mathbf{H}^{k+1}(\mathcal{B}_{\mathbf{x}_0})$*, and* K' *from* $\mathbf{H}^{k+1}(\mathcal{B}_{\mathbf{x}_0})$ *to* $\mathbf{H}^{k+2}(\mathcal{B}_{\mathbf{x}_0})$*.*

Proof: We start from the same operator $\mathfrak{L}$ on $\mathbb{T}^n$ defined by equation (5) in the proof of Proposition 1.2.9:

$$\mathfrak{L} = \tilde{L} + \psi\Big(\frac{\mathbf{x}-\mathbf{x}_0}{R_0}\Big)(L^{\mathsf{pr}} - \underline{L}_{\mathbf{x}_0}).$$

From its construction, $\mathfrak{L}$ is an isomorphism from $\mathbf{H}^2(\mathbb{T}^n)$ onto $\mathbf{L}^2(\mathbb{T}^n)$. Then we have to apply a difference quotient argument to deduce that $\mathfrak{L}$ is also an isomorphism from $\mathbf{H}^{k+2}(\mathbb{T}^n)$ onto $\mathbf{H}^k(\mathbb{T}^n)$ (see Lemma 1.3.6 below).

Next, we consider the operator of extension by zero from $\mathcal{B}_{\mathbf{x}_0}$ into $\mathbb{T}^n$ as a continuous operator $\mathfrak{P}_k$ from $\mathbf{H}^k_0(\mathcal{B}_{\mathbf{x}_0})$ into $\mathbf{H}^k(\mathbb{T}^n)$. We define now our operator $\mathfrak{E}_{\mathbf{x}_0}$ for $\mathbf{f} \in \mathbf{H}^k_0(\mathcal{B}_{\mathbf{x}_0})$ by

$$\mathfrak{E}_{\mathbf{x}_0}\mathbf{f} = \left(\mathfrak{L}^{-1}\mathfrak{P}_k\mathbf{f}\right)\big|_{\mathcal{B}_{\mathbf{x}_0}} \in \mathbf{H}^{k+2}(\mathcal{B}_{\mathbf{x}_0}).$$

The end of the proof relies on the same arguments as in the case $k = 0$, considered in Proposition 1.2.9. □

Lemma 1.3.6 *We assume that L is a $N \times N$ second order system with $\mathscr{C}^k$ coefficients on the torus $\mathbb{T}^n$ which is an isomorphism from $\mathbf{H}^2(\mathbb{T}^n)$ onto $\mathbf{L}^2(\mathbb{T}^n)$. Then L defines also an isomorphism from $\mathbf{H}^{k+2}(\mathbb{T}^n)$ onto $\mathbf{H}^k(\mathbb{T}^n)$.*

Proof: It is clearly enough to prove that for any $\mathbf{f} \in \mathbf{H}^k(\mathbb{T}^n)$, the solution $\mathbf{u} \in \mathbf{H}^2(\mathbb{T}^n)$ of the equation $L\mathbf{u} = \mathbf{f}$, belongs to $\mathbf{H}^{k+2}(\mathbb{T}^n)$.

Let $k = 1$ and let us prove that $\mathbf{u}$ belongs to $\mathbf{H}^3(\mathbb{T}^n)$. Let $j \in \{1, \ldots, n\}$. For $h \in \mathbb{T}$, we introduce the difference quotient $\Delta^h_j(\mathsf{v})$ in the direction of x_j of a function $\mathsf{v} \in \mathsf{L}^2(\mathbb{T}^n)$ as

$$\Delta^h_j(\mathsf{v}) : \quad \mathbb{T}^n \ni \mathbf{x} \longmapsto \frac{\mathsf{v}(\mathbf{x} + h\mathbf{e}_j) - \mathsf{v}(\mathbf{x})}{h}.$$

Here $\mathbf{e}_j$ is the unit vector in the direction of x_j. There hold the implications

$$\begin{aligned} &\partial_{x_j}\mathsf{v} \in \mathsf{L}^2(\mathbb{T}^n) \quad \Longrightarrow \quad \Delta^h_j(\mathsf{v}) \text{ bounded in } \mathsf{L}^2(\mathbb{T}^n) \text{ as } h \to 0, \\ &\Delta^h_j(\mathsf{v}) \text{ bounded in } \mathsf{L}^2(\mathbb{T}^n) \text{ as } h \to 0 \quad \Longrightarrow \quad \partial_{x_j}\mathsf{v} \in \mathsf{L}^2(\mathbb{T}^n). \end{aligned} \tag{1.37}$$

Since $\mathbf{u}$ belongs to $\mathbf{H}^2(\mathbb{T}^n)$, by linearity $\Delta^h_j(\mathbf{u})$ belongs to $\mathbf{H}^2(\mathbb{T}^n)$. From the estimate

$$\|\mathbf{v}\|_{2;\mathbb{T}^n} \le C\|L\mathbf{v}\|_{0;\mathbb{T}^n}$$

we deduce the uniform estimate in $h \in \mathbb{T}$:

$$\text{(1)} \qquad \sum_{|\alpha|=2} \|\partial^\alpha_{\mathbf{x}}\Delta^h_j(\mathbf{u})\|_{\mathbb{T}^n} \le C\|L\Delta^h_j(\mathbf{u})\|_{\mathbb{T}^n}.$$

But, denoting by $\Delta^{-h}_j L$ the second order operator with coefficients $\Delta^{-h}_j(a^\alpha_{ij})$, we have

$$\text{(2)} \qquad \|L\Delta^h_j(\mathbf{u})\|_{\mathbb{T}^n} \le \|\Delta^h_j(L\mathbf{u})\|_{\mathbb{T}^n} + \|(\Delta^{-h}_j L)\mathbf{u}\|_{\mathbb{T}^n}.$$

Since the coefficients of L are $\mathscr{C}^1$, those of $\Delta^{-h}_j L$ are bounded with respect to h. Therefore

$$\text{(3)} \qquad \|(\Delta^{-h}_j L)\mathbf{u}\|_{\mathbb{T}^n} \le C\|\mathbf{u}\|_{2;\mathbb{T}^n}.$$

Since, moreover, $L\mathbf{u}$ belongs to $\mathbf{H}^1(\mathbb{T}^n)$, we obtain, owing to (1.37), that the right hand side of (1) is uniformly bounded for $h \in \mathbb{T}$. Therefore, by (1.37) again, we obtain that $\partial_{x_j}\partial^\alpha_{\mathbf{x}}\mathbf{u} \in \mathrm{L}^2(\mathbb{T}^n)$ for $j = 1, \dots, n$. Whence $\mathbf{u} \in \mathbf{H}^3(\mathbb{T}^n)$. The proof proceeds in a similar way for $k \geq 2$. □

Proof of Theorem 1.3.3: *(i)* Let us assume that $k = 1$ and take $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ such that $L\mathbf{u} \in \mathbf{H}^1(\Omega_2)$. Let $\mathbf{x}_0 \in \overline{\Omega}_1$. Using Proposition 1.2.9, we choose the ball $\mathcal{B}_{\mathbf{x}_0}$ small enough so that it is contained in a domain $\Omega_2' \subset\subset \Omega_2$. Let $\mathcal{B}^*_{\mathbf{x}_0}$ the ball with center $\mathbf{x}_0$ and radius half of $\mathcal{B}_{\mathbf{x}_0}$. Then we take the cut-off functions ψ' and ψ'' so that $\psi' \equiv 1$ on $\mathcal{B}^*_{\mathbf{x}_0}$. Relation (1.31b) together with the continuity of $\mathfrak{E}_{\mathbf{x}_0} : \mathbf{H}^1_0(\mathcal{B}_{\mathbf{x}_0}) \to \mathbf{H}^3(\mathcal{B}_{\mathbf{x}_0})$ (Proposition 1.3.5) implies that $\psi'(\mathbf{u} + K'\mathbf{u})$ belongs to $\mathbf{H}^3(\mathcal{B}_{\mathbf{x}_0})$ with the estimate

$$\|\mathbf{u} + K'\mathbf{u}\|_{3;\,\mathcal{B}^*_{\mathbf{x}_0}} \leq C\|L\mathbf{u}\|_{1;\,\mathcal{B}_{\mathbf{x}_0}}.$$

Using the continuity of $K' : \mathbf{H}^2(\mathcal{B}_{\mathbf{x}_0}) \to \mathbf{H}^3(\mathcal{B}_{\mathbf{x}_0})$, we deduce

$$\|\mathbf{u}\|_{3;\,\mathcal{B}^*_{\mathbf{x}_0}} \leq C\big(\|L\mathbf{u}\|_{1;\,\mathcal{B}_{\mathbf{x}_0}} + \|\mathbf{u}\|_{2;\,\mathcal{B}_{\mathbf{x}_0}}\big).$$

We extract from the balls $\mathcal{B}^*_{\mathbf{x}_0}$, $\mathbf{x}_0 \in \overline{\Omega}_1$, a finite covering of Ω_1, and find that $\mathbf{u} \in \mathbf{H}^3(\Omega_1)$ with the estimate

$$\|\mathbf{u}\|_{3;\,\Omega_1} \leq C\big(\|L\mathbf{u}\|_{1;\,\Omega_2'} + \|\mathbf{u}\|_{2;\,\Omega_2'}\big). \tag{1}$$

Since $\Omega_2' \subset\subset \Omega_2$, we prove in the same way

$$\|\mathbf{u}\|_{2;\,\Omega_2'} \leq C\big(\|L\mathbf{u}\|_{0;\,\Omega_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\big). \tag{2}$$

From (1) and (2) we deduce the estimate

$$\|\mathbf{u}\|_{3;\,\Omega_1} \leq C\big(\|L\mathbf{u}\|_{1;\,\Omega_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\big). \tag{3}$$

The general case $k \geq 2$ follows by induction since the above arguments show that the result for $k-1$ and Proposition 1.3.5 imply the result for k.

(ii) The regularity and estimate are obvious consequences of *(i)*. The Fredholm property is a consequence of the regularity and of the Fredholm property from $\mathbf{H}^2(\Omega)$ into $\mathbf{L}^2(\Omega)$ (Theorem 1.2.13). □

1.4 Basic nested a priori estimates

We present now estimates in some "weighted" semi-norms, in our way towards analytic estimates. We start with a more local situation. We fix a point in Ω and suppose that it coincides with the origin $\mathbf{0}$ of $\mathbb{R}^n$. Let B_R be the ball of center $\mathbf{0}$ and radius R. If L is elliptic at $\mathbf{0}$, then according to Corollary 1.2.10, there exists a positive radius R_* such that the interior estimate (1.32) is valid for all $\mathbf{u} \in \mathbf{H}^2_0(B_{R_*})$.

To proceed further, we can forget the definition of ellipticity and take one consequence of ellipticity, namely the interior $\mathbf{H}^2$ estimate (1.32), as a starting point: For all $\mathbf{u} \in \mathbf{H}^2_0(B_{R_*})$:

$$\sum_{|\alpha|\le 2} \|\partial^\alpha \mathbf{u}\|_{B_{R_*}} \le A_0\Big(\|L\mathbf{u}\|_{B_{R_*}} + \sum_{|\alpha|\le 1} \|\partial^\alpha \mathbf{u}\|_{B_{R_*}}\Big), \tag{1.38}$$

where the positive constant A_0 does not depend on $\mathbf{u}$. As a matter of fact, estimate (1.38) is the *unique foundation* of the local estimates of higher order on which we will base the proofs of analytic regularity. For this reason, (1.38) will be our sole hypothesis for the following statements (lemmas and propositions 1.4.1 to 1.6.3). It is true that (1.38) holds if and only if L is elliptic at $\mathbf{0}$, but from a technical point of view it is interesting to notice that everything can be deduced from a single estimate (and from the control of derivatives of the coefficients of L in the case of variable coefficients).

Without much difficulty one can deduce from (1.38) some a priori estimates of the $\mathbf{H}^{k+2}$ norm of $\mathbf{u}$ by the $\mathbf{H}^k$ norm of $L\mathbf{u}$, but without control of the constants with respect to k. Let us explain this first. For $0 < R' < R \le R_*$, let $\phi \in \mathscr{C}^\infty_0(B_R)$ be a cut-off function such that $\phi \equiv 1$ on $B_{R'}$. Let $\mathbf{u} \in \mathbf{H}^2(B_{R_*})$. Then $\phi\mathbf{u}$ belongs to $\mathbf{H}^2_0(B_{R_*})$. The estimate (1.38) applied to $\phi\mathbf{u}$ gives

$$\sum_{|\alpha|\le 2} \|\partial^\alpha \mathbf{u}\|_{B_{R'}} \le c_0(R,R')\Big(\|L\mathbf{u}\|_{B_R} + \sum_{|\alpha|\le 1} \|\partial^\alpha \mathbf{u}\|_{B_R}\Big). \tag{1.39}$$

We may apply the above estimate to any $\partial^\beta\mathbf{u}$, $|\beta| \le k$, instead of $\mathbf{u}$ and find

$$\sum_{|\alpha|\le k+2} \|\partial^\alpha \mathbf{u}\|_{B_{R'}} \le c_k(R',R)\Big(\sum_{|\alpha|\le k}\|\partial^\alpha L\mathbf{u}\|_{B_R} + \sum_{|\alpha|\le k+1} \|\partial^\alpha \mathbf{u}\|_{B_R}\Big).$$

Iterating and composing similar estimates for $k, k-1, \dots, 1$ associated with intermediate radii we find eventually

$$\sum_{|\alpha|\le k+2} \|\partial^\alpha \mathbf{u}\|_{B_{R'}} \le c'_k(R',R)\Big(\sum_{|\alpha|\le k}\|\partial^\alpha L\mathbf{u}\|_{B_R} + \sum_{|\alpha|\le 1} \|\partial^\alpha \mathbf{u}\|_{B_R}\Big).$$

The *analytic estimates* needed for answering questions *(iii)* and *(iv)* require controlling the behavior of the constant $c'_k(R,R')$ with respect to R', R and k. In fact, in the end we would only need to know its behavior with respect to k, but in order to get there, we have to use intermediate radii, and this requires a precise control of the blow up of $c'_k(R,R')$ for R' near R. This will be done by means of a special family of cut-off functions $\chi_{R,\rho}$ which we introduce now. Let χ be a smooth function in $\mathscr{C}^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $(-\infty, 0)$ and $\chi \equiv 0$ on $[1, +\infty)$. Let R and ρ be such that $0 < R \le R_*$ and $0 < \rho < R$, and define the cut-off function

$$\chi_{R,\rho}: \quad \mathbf{x} \longmapsto \chi\left(\frac{|\mathbf{x}| - R + \rho}{\rho}\right) \tag{1.40}$$

which equals 1 in $B_{R-\rho}$ and 0 outside B_R. We note the following important bound on the derivatives of $\chi_{R,\rho}$

$$\exists D > 0, \quad \forall R \in (0, R_*],\ \forall \rho \in (0, R),\ \forall \alpha,\ |\alpha| \le 2, \quad |\partial^\alpha \chi_{R,\rho}| \le D\rho^{-|\alpha|}. \tag{1.41}$$

Using these cut-off functions, we first prove a more precise version of the a priori estimates (1.39), where the distance $\rho := R - R'$ between $B_{R'}$ and the boundary of B_R acts as a parameter. It turns out that it is natural to consider seminorms of the form $\rho^{|\alpha|}\|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}}$.

Lemma 1.4.1 *Let L be a $N \times N$ second order system with $\mathscr{C}^0(\overline{\Omega})$ coefficients. We assume that estimate (1.38) holds for all $\mathbf{u} \in \mathbf{H}^2_0(B_{R_*})$. Let $\mathbf{u}$ be any function in $\mathbf{H}^2(B_R)$ with $R \le R_*$ and let $\rho \in (0, \frac{R}{2})$. We have*

$$\sum_{|\alpha|\le 2} \rho^{|\alpha|}\|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} \le A_1\Big(\rho^2\|L\mathbf{u}\|_{B_{R-\rho}} + \sum_{|\alpha|\le 1}\rho^{|\alpha|}\|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}}\Big), \tag{1.42}$$

where the positive constant A_1 is independent of R and ρ.

Proof: Applying estimate (1.38) to the function $\chi_{R,\rho}\mathbf{u}$ and using the bounds (1.41) for the derivatives of $\chi_{R,\rho}$ we find that

$$\sum_{|\alpha|=2} \|\partial^\alpha \mathbf{u}\|_{B_{R-\rho}} \le A_0'\Big(\|L\mathbf{u}\|_{B_R} + \sum_{|\alpha|=1}\rho^{-1}\|\partial^\alpha \mathbf{u}\|_{B_R} + \rho^{-2}\|\mathbf{u}\|_{B_R}\Big).$$

Multiplying by ρ^2 and applying this estimate for $R-\rho$ instead of R, we obtain

$$\rho^2\sum_{|\alpha|=2} \|\partial^\alpha \mathbf{u}\|_{B_{R-2\rho}} \le A_0'\Big(\rho^2\|L\mathbf{u}\|_{B_{R-\rho}} + \sum_{|\alpha|=1}\rho^{1}\|\partial^\alpha \mathbf{u}\|_{B_{R-\rho}} + \|\mathbf{u}\|_{B_{R-\rho}}\Big).$$

Adding $\sum_{|\alpha|\le 1}\rho^{|\alpha|}\|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}}$ on both sides, we deduce (1.42). □

1.5 Nested a priori estimates for constant coefficients

Since the constant coefficient case allows much simpler proofs, we begin with this case, leaving the variable coefficient case for a second step in the next section.

With the parameter dependent local interior estimate (1.42) we are ready to prove estimates for all derivatives.

Proposition 1.5.1 *We assume that L is a $N \times N$ second order system with constant coefficients and that estimate (1.38) holds for all $\mathbf{u} \in \mathbf{H}^2_0(B_{R_*})$. There exists a constant*

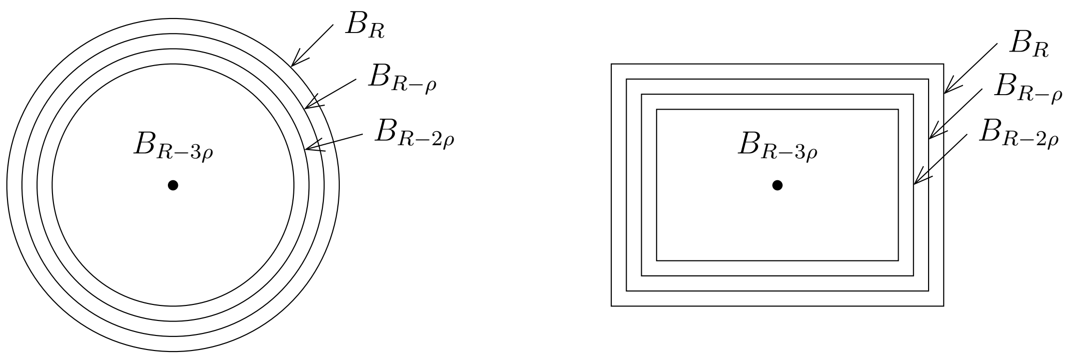


Figure 1.1: Nested neighborhoods for interior estimates (balls and rectangles)

$A \geq 1$ *such that for all* $R \in (0, R_*]$, *for all* $\rho \in (0, \frac{R}{k+2}]$ *and for all* $k \in \mathbb{N}$ *with* $\mathbf{u} \in \mathbf{H}^{k+2}(B_R)$ *there holds*

$$\sum_{|\alpha|\leq k+2} \rho^{|\alpha|} \|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} \leq \sum_{|\beta|\leq k} A^{k+1-|\beta|}\, \rho^{2+|\beta|} \|\partial^\beta L\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + A^{k+1} \sum_{|\alpha|\leq 1} \rho^{|\alpha|} \|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}}. \quad (1.43)$$

Proof: Let β be any multi-index of length $|\beta| = k$. We use estimate (1.42) with $\partial^\beta \mathbf{u}$ instead of $\mathbf{u}$ and with $R - |\beta|\rho$ instead of R, which gives

$$\sum_{|\alpha|\leq 2} \rho^{|\alpha|} \|\partial^\alpha \partial^\beta \mathbf{u}\|_{B_{R-|\beta|\rho-|\alpha|\rho}} \leq A_1 \Big(\rho^2 \|L(\partial^\beta \mathbf{u})\|_{B_{R-\rho-|\beta|\rho}} + \sum_{|\alpha|\leq 1} \rho^{|\alpha|} \|\partial^\alpha \partial^\beta \mathbf{u}\|_{B_{R-|\beta|\rho-|\alpha|\rho}} \Big).$$

Since L *has constant coefficients, it commutes with* ∂^β, thus $L(\partial^\beta \mathbf{u}) = \partial^\beta L\mathbf{u}$, which gives

$$\sum_{|\alpha|\leq 2} \rho^{|\alpha|} \|\partial^{\alpha+\beta} \mathbf{u}\|_{B_{R-|\beta|\rho-|\alpha|\rho}} \leq A_1 \Big(\rho^2 \|\partial^\beta L\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + \sum_{|\alpha|\leq 1} \rho^{|\alpha|} \|\partial^{\alpha+\beta} \mathbf{u}\|_{B_{R-|\beta|\rho-|\alpha|\rho}} \Big).$$

Multiplying both sides by $\rho^{|\beta|}$ and summing over all β such that $|\beta| = k$, we obtain in particular

$$\sum_{|\alpha|=k+2} \rho^{|\alpha|} \|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} \leq A_1 \Big(\sum_{|\beta|=k} \rho^{2+|\beta|} \|\partial^\beta L\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + \sum_{|\alpha|=k,k+1} \rho^{|\alpha|} \|\partial^\alpha \mathbf{u}\|_{B_{R-|\alpha|\rho}} \Big), \quad (1.44)$$

where the constant A_1 is still the same as in (1.42) (therefore independent of $k \geq 0$). The proof of (1.43) now follows by an induction argument over k: For $k = 0$, (1.43) holds with any $A \geq A_1$ by (1.42), and (1.44) allows to go from $k-1$ to k as soon as $A \geq A_1 + 1$. □

Note. This class of arguments is known as the *nested open set technique*, by reference to the nested balls $B_R \supset B_{R-\rho} \supset \ldots \supset B_{R-(k+2)\rho}$, see Fig. 1.1.

Remark 1.5.2 If convenient, we can choose other "standard" domains than balls. An example are boxes of the form, see Fig. 1.1,

$$B_R = (a_1, b_1) \times \ldots \times (a_n, b_n), \quad \text{with} \quad R = \min_{1 \le j \le n} \frac{b_j - a_j}{2}.$$

In this situation, a corresponding suitable definition for the nested domains $B_{R-\rho}$ is

$$B_{R-\rho} = (a_1 + \rho, b_1 - \rho) \times \ldots \times (a_n + \rho, b_n - \rho).$$

Estimates (1.43) are still valid for such domains because the construction of a generic cut-off function $\chi_{R,\rho}$ with the same properties is possible: Translating coordinates, we may suppose that $a_j = -b_j$ and, instead of (1.40), we take

$$\chi_{R,\rho} : \mathbf{x} \longmapsto \chi\left(\frac{|x_1| - b_1 + \rho}{\rho}\right) \times \ldots \times \chi\left(\frac{|x_n| - b_n + \rho}{\rho}\right).$$

If polar or spherical coordinates are applicable, it is also possible to use domains which are "rectangular" in these coordinates *provided the origin of such coordinates does not belong to the closure of the largest domain under consideration*, see later, Ch.2.

The only essential feature of such a sequence of domains is the possibility of being associated with cut-off functions $\chi_{R,\rho}$ such that the bound (1.41) holds. △

1.6 Nested a priori estimates for variable coefficients

If the coefficients a^α_{ij} in (1.29) are no longer constant, then the *commutators*

$$[L, \partial^\beta] := L\partial^\beta - \partial^\beta L \quad \text{and} \quad [L, \Delta^h_j] := L\Delta^h_j - \Delta^h_j L$$

of L with ∂^β and Δ^h_j are no more zero. Nevertheless, their orders are strictly less than the sum of the orders of the operators involved: The order of $[L, \partial^\beta]$ is at most $|\beta|+1$ and $[L, \Delta^h_j]$ tends to a second order operator as $h \to 0$. Using these simple arguments, it is easy to extend Lemma 1.3.6 to the variable coefficient case, in contrast with Proposition 1.5.1 which does not immediately carry over in its current form.

We will have to estimate commutator norms of the form

$$\rho^{2+|\beta|} \| a(\mathbf{x}) \partial^\beta \partial^\alpha u - \partial^\beta (a(\mathbf{x}) \partial^\alpha u) \|_{B_{R-\rho-|\beta|\rho}},$$

where β is an arbitrary multiindex, α satisfies $|\alpha| \le 2$, and ρ is any positive number such that $\rho \le \frac{R}{2(|\beta|+2)}$. In order to prepare the introduction of the weighted semi-norms on u which allow to bound quantities like above, we start by investigating bounds for typical terms

$$N := \rho^b \|a(\mathbf{x})\partial^\beta\partial^\alpha u - \partial^\beta(a(\mathbf{x})\partial^\alpha u)\|_{B_{R-(b-1)\rho}} \quad \text{with} \quad b := |\alpha| + |\beta| \ \text{ and } \ \rho < \frac{R}{2b},$$

where the coefficient a satisfies the analytic estimates

$$\forall \gamma \in \mathbb{N}^n, \quad |\partial^\gamma a| \le M^{|\gamma|+1}|\gamma|! \ \text{ on } \ B_R\,.$$

Here M is a positive constant independent of γ.

Commutator estimate: We use the Leibniz formula

$$\partial^\beta(a\,\partial^\alpha u) = \sum_{\gamma\le\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!}\ \partial^\gamma a\ \partial^{\beta-\gamma}\partial^\alpha u$$

together with the combinatorial inequality

$$\frac{\beta!}{\gamma!(\beta-\gamma)!} \le \frac{|\beta|!}{|\gamma|!\,(|\beta|-|\gamma|)!}$$

and find, since the contribution of $\gamma = 0$ is absent from N:

$$\begin{aligned} N &\le \rho^b \sum_{1\le|\gamma|,\ \gamma\le\beta} M^{|\gamma|+1}\frac{|\beta|!}{(|\beta|-|\gamma|)!}\ \|\partial^{\beta-\gamma}\partial^\alpha u\|_{B_{R-(b-1)\rho}} \\ &\le \rho^b \sum_{g=1}^{|\beta|}\ \sum_{|\gamma|=g,\ \gamma\le\beta} M^{g+1}\frac{|\beta|!}{(|\beta|-g)!}\ \max_{|\delta|=|\beta-\gamma+\alpha|}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\ . \end{aligned}$$

By induction on n, we check that $\sum_{|\gamma|=g} \le (g+1)^{n-1}$. This bound implies that

$$\begin{aligned} N &\le \rho^b \sum_{g=1}^{b-|\alpha|}(g+1)^{n-1}M^{g+1}\frac{(b-|\alpha|)!}{(b-|\alpha|-g)!}\ \max_{|\delta|=b-g}\|\partial^\delta u\|_{B_{R-(b-1)\rho}} \\ &\le \rho^b \sum_{d=|\alpha|}^{b-1}(b-d+1)^{n-1}M^{b-d+1}\frac{(b-|\alpha|)!}{(d-|\alpha|)!}\ \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\ . \end{aligned}$$

A contribution for $d = 0$ appears only if $|\alpha| = 0$. We denote it by N_0. We have

$$N_0 = \rho^b(b+1)^{n-1}M^{b+1}b!\ \|u\|_{B_{R-(b-1)\rho}}\,.$$

We denote the remaining part $N - N_0$ of N by N_1. We can see that (because $\frac{d}{b} \le 1$)

$$N_1 \le \rho^b \sum_{d=1}^{b-1}(b-d+1)^{n-1}M^{b-d+1}\,\frac{b!}{d!}\ \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\ .$$

Let us study N_1 first. A derivative $\partial^\delta u$ of length d should be associated with ρ^d. We write:

$$N_1 \le \sum_{d=1}^{b-1}(b-d+1)^{n-1}\rho^{b-d}M^{b-d+1}\,\frac{b!}{d!}\,\Big(\rho^d \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\Big).$$

We process the factorial terms $d!$ and $b!$ with the Stirling formula, which we use in the form

$$\exists s, s',\ \forall m \in \mathbb{N}:\quad 0 < s \le \frac{m^m \mathrm{e}^{-m}\sqrt{m}}{m!} \le s' < \infty \tag{1.45}$$

We see that (1.45) implies that there exists a universal constant c_S such that

$$N_1 \le c_S \sum_{d=1}^{b-1}(b-d+1)^{n-1}\rho^{b-d}M^{b-d+1}\mathrm{e}^{d-b}\sqrt{\frac{b}{d}}\,\frac{b^b}{d^d}\,\Big(\rho^d \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\Big).$$

Let us now take advantage of the fact that $\rho \le \frac{R}{2b}$. We find:

$$N_1 \le c_S \sum_{d=1}^{b-1}(b-d+1)^{n-1}M\Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d}\sqrt{\frac{b}{d}}\,\Big(\frac{b}{d}\Big)^d\,\Big(\rho^d \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-(b-1)\rho}}\Big).$$

A priori, the factor $\big(\frac{b}{d}\big)^d$ hampers the estimate. The way to absorb this factor is to couple it with ρ^d inside the seminorm expression. The reason for this possibility is the "small" domain of integration $B_{R-(b-1)\rho}$ for the seminorm (instead of $B_{R-d\rho}$): As a consequence, for each value of d we may introduce the new distance $\rho'_d := (b-1)\rho/d$. Then $B_{R-(b-1)\rho} = B_{R-d\rho'_d}$ and the above inequality becomes

$$N_1 \le c_S \sum_{d=1}^{b-1}(b-d+1)^{n-1}M\Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d}\sqrt{\frac{b}{d}}\,\Big(\frac{b}{b-1}\Big)^d\Big((\rho'_d)^d \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-d\rho'_d}}\Big).$$

Noting that

$$\Big(\frac{b}{b-1}\Big)^d \le \mathrm{e} \quad\text{and}\quad \sqrt{\frac{b}{d}} \le \frac{b}{d} \le b-d+1 \quad\text{for}\quad 1 \le d \le b-1,$$

we simplify the above estimate into

$$N_1 \le c_S\mathrm{e}M \sum_{d=1}^{b-1}(b-d+1)^{n}\Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d}\,\Big((\rho'_d)^d \max_{|\delta|=d}\|\partial^\delta u\|_{B_{R-d\rho'_d}}\Big).$$

Concerning N_0, simpler calculations give

$$N_0 \le c_S M(b+1)^n\Big(\frac{RM}{2\mathrm{e}}\Big)^b\,\|u\|_{B_{R-(b-1)\rho}}.$$

□

Keeping trace of ρ'_d in the bound of N_1 is technically complicated, but not necessary.

Instead we introduce the following weighted seminorms of Sobolev-Morrey type, and in this way, we have proved Lemma 1.6.2 hereafter.

Notation 1.6.1 *(i)* We set $\big[|u|\big]_{0;B_R} = \|u\|_{B_R}$ and for $\ell \in \mathbb{N}$, $\ell > 0$

$$\big[|u|\big]_{\ell;B_R} := \max_{0<\rho\le\frac{R}{2\ell}} \max_{|\delta|=\ell} \rho^\ell \|\partial^\delta u\|_{B_{R-\ell\rho}}.$$

(ii) We set for $\ell \in \mathbb{N}$: $\quad \rho_*^2\big[|f|\big]_{\ell;B_R} := \max_{0<\rho\le\frac{R}{2(\ell+1)}} \max_{|\delta|=\ell} \rho^{2+\ell}\|\partial^\delta f\|_{B_{R-(\ell+1)\rho}}.$

Here ρ_*^2 does not denote a real number, but is a symbolic notation. △

Lemma 1.6.2 (Commutator estimate) *There exists a universal constant c_0 such that the following holds: Let $R > 0$ and let a be an analytic function satisfying $|\partial^\gamma a| \le M^{|\gamma|+1}|\gamma|!$ on B_R for all $\gamma \in \mathbb{N}^n$. Let α and β be any multiindices and $b = |\alpha| + |\beta|$. Then for all $\rho \in (0, \frac{R}{2b}]$ there holds*

$$\rho^b\|a(\mathbf{x})\partial^\beta\partial^\alpha u - \partial^\beta(a(\mathbf{x})\partial^\alpha u)\|_{B_{R-(b-1)\rho}} \le c_0 M \sum_{d=0}^{b-1}(b-d+1)^n\Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d}\big[|u|\big]_{d;B_R}.$$

We are now ready to prove the statement corresponding to Proposition 1.5.1 in the variable coefficient case.

Proposition 1.6.3 *We assume that L is a $N \times N$ second order system with analytic coefficients in $\overline{B}_{R_*}$ and that estimate (1.38) holds for all $\mathbf{u} \in \mathbf{H}_0^2(B_{R_*})$. There exists a constant $A \ge 1$ such that for all $R \in (0, R_*]$ and for all $k \in \mathbb{N}$ with $\mathbf{u} \in \mathbf{H}^{k+2}(B_R)$ there holds*

$$\big[|\mathbf{u}|\big]_{k+2;B_R} \le \sum_{\ell=0}^{k} A^{k+1-\ell}\,\rho_*^2\big[|L\mathbf{u}|\big]_{\ell;B_R} + A^{k+1}\sum_{\ell=0}^{1}\big[|\mathbf{u}|\big]_{\ell;B_R}. \tag{1.46}$$

Proof: Let $\rho \in (0, \frac{R}{2(k+2)})$. We can apply Lemma 1.4.1, therefore estimate (1.42) holds. In the variable coefficient case, we obtain now, instead of (1.44):

$$\max_{|\alpha|=k+2} \rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{B_{R-|\alpha|\rho}} \le A_1'\Big(\max_{|\beta|=k}\rho^{2+|\beta|}\|\partial^\beta L\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + \max_{|\beta|=k}\rho^{2+|\beta|}\|[L,\partial^\beta]\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} + \max_{|\alpha|=k,k+1}\rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{B_{R-|\alpha|\rho}}\Big), \tag{1.47}$$

where $A_1' = A_1 c(n)$, where $c(n) > 0$ depends only on the space dimension n. We apply

Lemma 1.6.2 for $|\beta| = k$ and $|\alpha| \le 2$ (thus $b \le k+2$) and obtain

$$\begin{aligned}
\rho^{2+|\beta|} \|[L, \partial^\beta]\mathbf{u}\|_{B_{R-\rho-|\beta|\rho}} &\le \Big(\frac{R_*}{4}\Big)^{k+2-b} \rho^b \|[L, \partial^\beta]\mathbf{u}\|_{B_{R-(b-1)\rho}} \\
&\le c_0 M \sum_{d=0}^{b-1} (b-d+1)^n \Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d} \big[|\mathbf{u}|\big]_{d;B_R} \\
&\le c_1 \sum_{d=0}^{k+1} (KR)^{k+1-d} \big[|\mathbf{u}|\big]_{d;B_R}
\end{aligned}$$

with positive constants c_1 and K, independent of β and R for $R \le R_*$. Taking the max over ρ in (1.47) gives

$$\begin{aligned}
\big[|\mathbf{u}|\big]_{k+2;B_R} &\le A_1' \Big(\rho_*^2 \big[|L\mathbf{u}|\big]_{k;B_R} + \sum_{d=k}^{k+1} \big[|\mathbf{u}|\big]_{d;B_R} + c_1 \sum_{d=0}^{k+1} (KR)^{k+1-d} \big[|\mathbf{u}|\big]_{d;B_R}\Big) \\
&\le A_2 \Big(\rho_*^2 \big[|L\mathbf{u}|\big]_{k;B_R} + \big[|\mathbf{u}|\big]_{k+1;B_R} + \sum_{d=0}^{k} (KR)^{k-d} \big[|\mathbf{u}|\big]_{d;B_R}\Big). \qquad (1.48)
\end{aligned}$$

Let us prove (1.46) by induction over k. It holds for $k = 0$ if we choose $A \ge A_2$. We then assume that (1.46) holds for $0, \dots, k-1$. In the right hand side of (1.48), we use the estimate of $\big[|\mathbf{u}|\big]_{d;B_R}$ provided by the induction hypothesis and obtain

$$\begin{aligned}
\big[|\mathbf{u}|\big]_{k+2;B_R} \le A_2 \Big(& \rho_*^2 \big[|L\mathbf{u}|\big]_{k;B_R} + \sum_{\ell=0}^{k-1} A^{k-\ell} \rho_*^2 \big[|L\mathbf{u}|\big]_{\ell;B_R} + A^k \sum_{\ell=0}^{1} \big[|\mathbf{u}|\big]_{\ell;B_R} \\
& + \sum_{d=0}^{k} (KR)^{k-d} \Big\{ \sum_{\ell=0}^{d-2} A^{d-1-\ell} \rho_*^2 \big[|L\mathbf{u}|\big]_{\ell;B_R} + A^{d-1} \sum_{\ell=0}^{1} \big[|\mathbf{u}|\big]_{\ell;B_R} \Big\} \Big).
\end{aligned}$$

The remaining task is to gather the coefficients in front of $\rho_*^2 \big[|L\mathbf{u}|\big]_{\ell;B_R}$ and $\big[|\mathbf{u}|\big]_{\ell;B_R}$ and finally, find sufficient conditions on A so that the above inequality yields (1.46) for $d = k+2$.

(i) For $\rho_*^2 \big[|L\mathbf{u}|\big]_{\ell;B_R}$: With $\ell = k$ or $k-1$, it suffices that $A \ge A_2$. With $\ell \le k-2$, the coefficient *divided by* $A^{k+1-\ell}$ is equal to

$$A_2 \Big(A^{-1} + \sum_{d=\ell+2}^{k} (KR)^{k-d} A^{d-1-\ell} A^{-(k+1-\ell)} \Big) = \frac{A_2}{A} \Big(1 + A^{-1} \sum_{d=\ell+2}^{k} \Big(\frac{KR}{A}\Big)^{k-d} \Big). \quad (1.49)$$

We look for conditions on A so that the above expression is less than 1. For this it suffices, for example, to require that $2KR \le A$, $1 \le A$ and $3A_2 \le A$.

(ii) For $\big[|\mathbf{u}|\big]_{\ell;B_R}$ with $\ell = 0$ or 1, we find that the coefficient *divided by* A^{k+1} is here given

by

$$A_2\Big(A^{-1} + \sum_{d=0}^{k} (KR)^{k-d} A^{d-1-\ell} A^{-(k+1-\ell)}\Big)$$

which is less than 1 under the same conditions on A. This ends the proof. □

Remark 1.6.4 Let k be fixed. If the coefficients of L are in $\mathscr{C}^k(\overline{B}_{R_*})$ only, there still exists M such that $|\partial^\gamma a| \le M^{|\gamma|+1}|\gamma|!$ for all γ, $|\gamma| \le k$ and for all coefficients $a = a_{ij}^\alpha$ of L. The estimate (1.46) is then still valid *up to this value of k only.* △

Remark 1.6.5 Let the dimension n be fixed. The constant A in estimate (1.46) continuously depends on R_*, on the constant A_0 in (1.38) and on the analyticity modulus of the coefficients of L on B_{R_*} i.e., the least constant M such that

$$\forall \alpha,\ |\alpha| \le 2,\ \ \forall i,j = 1,\dots,N,\ \ \forall \gamma \in \mathbb{N}^n, \quad |\partial^\gamma a_{ij}^\alpha| \le M^{|\gamma|+1}|\gamma|! \ \text{ on } \ B_{R_*}. \tag{1.50}$$

Thus A can be chosen uniformly for any family $(L^\tau)_\tau$ of elliptic operators such that R_* and A_0 can be taken independently on τ, and such that the analyticity moduli M^τ of the coefficients of L^τ are uniformly bounded in τ. △

1.7 Interior analytic regularity

With the help of Proposition 1.6.3, we have now all material at hands to prove the interior analytic regularity of solutions (the analytic hypoellipticity of elliptic operators).

Theorem 1.7.1 *Let Ω be a domain in $\mathbb{R}^n$ or, more generally, in an analytic manifold of dimension n. Let L be a $N \times N$ system of second order partial differential operators with analytic coefficients, elliptic on Ω. The analytic class on Ω is denoted by $\mathbf{A}(\Omega)$, cf.* (1.16).

(i) For any bounded subdomains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2 \subset \Omega$, there exists a constant A such that for all $k \ge 2$ and $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ such that $L\mathbf{u} \in \mathbf{H}^{k-2}(\Omega_2)$ there holds the a priori estimate

$$\frac{1}{k!}\,|\mathbf{u}|_{k;\,\Omega_1} \le A^{k+1}\Big(\sum_{\ell=0}^{k-2} \frac{1}{\ell!}\,|L\mathbf{u}|_{\ell;\,\Omega_2} + \sum_{\ell=0}^{1} |\mathbf{u}|_{\ell;\,\Omega_2}\Big). \tag{1.51}$$

If $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ satisfies $L\mathbf{u} \in \mathbf{A}(\Omega_2)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega_1)$.

(ii) We assume now that Ω is a compact manifold without boundary. If $\mathbf{u} \in \mathbf{H}^2(\Omega)$ satisfies $L\mathbf{u} \in \mathbf{A}(\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega)$.

Proof: The key point is the proof of the analytic estimate (1.51). Let Ω be a domain in an analytic manifold M and $\overline{\Omega}_1 \subset \Omega_2 \subset \Omega$. For each point $\mathbf{x}_0$ in $\overline{\Omega}_1$ there exists a neighborhood $\mathcal{U}_2(\mathbf{x}_0)$ of $\mathbf{x}_0$ contained in Ω_2, a ball B_{R_*} in $\mathbb{R}^n$ and an analytic map ϕ from $\mathcal{U}_2(\mathbf{x}_0)$ onto B_{R_*}. The equation $L\mathbf{u} = \mathbf{f}$ in $\mathcal{U}_2(\mathbf{x}_0)$ becomes

$$\breve{L}\breve{\mathbf{u}} = \breve{\mathbf{f}} \text{ in } B_{R_*} \quad \text{with} \quad \breve{\mathbf{u}} \circ \phi = \mathbf{u} \text{ and } \breve{\mathbf{f}} \circ \phi = \mathbf{f}.$$

The operator $\breve{L}$ has analytic coefficients in the ball B_{R_*}. By Corollary 1.2.10, estimate (1.38) holds for $\breve{L}$ in a neighborhood of 0. Then Proposition 1.6.3 gives that there exist positive numbers R_0 and A so that the following estimates hold for all $R \le R_0$ and all $k \in \mathbb{N}$, $k \ge 2$,

$$\big[|\breve{\mathbf{u}}|\big]_{k;\,B_R} \le \sum_{\ell=0}^{k-2} A^{k-\ell}\,\rho_*^2\big[|\breve{L}\breve{\mathbf{u}}|\big]_{\ell;\,B_R} + A^k \sum_{\ell=0}^{1} \big[|\breve{\mathbf{u}}|\big]_{\ell;\,B_R}. \tag{1.52}$$

As a consequence of the definitions, we have the inequalities:

$$\begin{cases} (k+1)^{(n-1)/2}\big[|\breve{\mathbf{u}}|\big]_{k;\,B_R} \ge \Big(\dfrac{R}{2k}\Big)^k \,|\breve{\mathbf{u}}|_{k;\,B_{R/2}} \\ \rho_*^2\big[|\breve{L}\breve{\mathbf{u}}|\big]_{\ell;\,B_R} \le \Big(\dfrac{R}{2\ell}\Big)^{2+\ell}\,|\breve{L}\breve{\mathbf{u}}|_{\ell;\,B_R} \\ \displaystyle\sum_{\ell=0,1} \big[|\breve{\mathbf{u}}|\big]_{\ell;\,B_R} \le \max\Big\{\frac{R}{2}, 1\Big\}\,\|\breve{\mathbf{u}}\|_{1;\,B_R}. \end{cases} \tag{1.53}$$

With the help of Stirling's formula, we deduce from (1.52) and the inequalities (1.53) that, with a new constant $\widetilde{A}$ independent of k and $R \le R_0$, we have:

$$\frac{R^k}{k!}\,|\breve{\mathbf{u}}|_{k;\,B_{R/2}} \le \sum_{\ell=0}^{k-2} \widetilde{A}^{k-\ell}\,\frac{R^\ell}{\ell!}\,|\breve{L}\breve{\mathbf{u}}|_{\ell;\,B_R} + \widetilde{A}^k \sum_{\ell=0}^{1} \|\breve{\mathbf{u}}\|_{\ell;\,B_R}.$$

Note that $\widetilde{A}$ and R_0 still depend on the center point $\mathbf{x}_0$ in $\overline{\Omega}_1$. We fix $R = R(\mathbf{x}_0) \le \min\{R_0, R_*\}$ and denote by $\mathcal{U}_1(\mathbf{x}_0)$ the pull-back $\phi^{-1}(B_{R(\mathbf{x}_0)/2})$. Combining the above estimate with estimate (1.22) for the analytic change of variables, we obtain

$$\frac{1}{k!}\,|\mathbf{u}|_{k;\mathcal{U}_1(\mathbf{x}_0)} \le A_{\mathbf{x}_0}^{k+1}\Big(\sum_{\ell=0}^{k-2} \frac{1}{\ell!}\,|L\mathbf{u}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)} + \sum_{\ell=0}^{1} |\mathbf{u}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)}\Big). \tag{1.54}$$

Here $A_{\mathbf{x}_0}$ is a positive number independent of k and $\mathbf{u}$. Extracting from the set of all open sets $\mathcal{U}_1(\mathbf{x}_0)$, $\mathbf{x}_0 \in \overline{\Omega}_1$, a finite covering of the compact set $\overline{\Omega}_1$, we conclude the proof of (1.51).

The remaining parts of Theorem 1.7.1 are now obvious. □

Chapter 2

Estimates up to the boundary

Introduction

In this chapter, we continue the discussion of elliptic problems on smooth domains by introducing and studying elliptic boundary value problems on domains with smooth boundaries. As in the previous chapter, we consider second order $N \times N$ systems of linear partial differential equations, and we complement them by boundary conditions given by systems of linear partial differential equations of order zero and one.

This is again classical material, which we present in such a way that the *results* and *techniques* introduced here can be used – both by applying directly the results where they are applicable and by modeling more complicated techniques on these simpler ones – in later parts of the book treating domains with non-smooth boundaries. We prove the standard main results on local and global a priori estimates and regularity up to the boundary to any order in Sobolev spaces and in spaces of analytic functions. The local versions of these results are valid also near smooth parts of the boundary of non-smooth domains.

Whereas the results presented in this chapter can be considered standard, we put the emphasis on three more technical aspects of the analysis of elliptic boundary value problems:

a) We describe in detail the relation between a boundary value problem with variable coefficients set on a domain with smooth boundary and its associated "tangent" model problem with constant coefficients set on a half-space. Later on, on domains with conical points or edges, we will have to consider several such tangent model problems set on certain model domains invariant under certain symmetry groups. These model problems allow at least partial algebraization via the Fourier transformation corresponding to the symmetries, thereby giving rise to "symbols", simpler operators whose invertibility is the key to ellipticity.

We have seen this technique in a nutshell in the first chapter, where the associated tangent problem was the system with constant coefficients corresponding to the principal part frozen in a point, acting on the whole space $\mathbb{R}^n$ or the periodic space $\mathbb{T}^n$, and the associated symbol was the principal symbol of the system, an operator of multiplication by a matrix function, obtained by Fourier transformation on $\mathbb{R}^n$ or $\mathbb{T}^n$. In the present chapter, the associated tangent problem is a boundary value problem with constant coefficients on the periodic half-space $\mathbb{T}^n_+$ which can be algebraized partially by tangential Fourier transformation, resulting in a family of boundary value problems for systems of ordinary differential equations on the half-axis $\mathbb{R}_+$. We use the invertibility of this family as our definition of ellipticity, leaving aside the well-known discussion of the solutions of these boundary value problems on $\mathbb{R}_+$ which can lead to a completely algebraic formulation of the *Shapiro-Lopatinski* ellipticity condition.

b) We propose a specific framework for the *global* formulation of boundary conditions. Whereas ellipticity is a local condition, to be satisfied in every interior and boundary point of a domain, applications may provide boundary conditions in a global formulation which does not fit into the purely local definition of elliptic boundary conditions often found in textbooks. An important class of such elliptic problems are problems in variational form, which we will discuss in detail in Chapter 3. There one has to take the essential and the natural boundary conditions together to constitute a set of boundary conditions that might satisfy an ellipticity condition. Now it may happen that due to the non-trivial topology of the boundary, the number of equations required to write these conditions in a smooth form exceeds the number of conditions admissible for an elliptic boundary value problem.

Let us look at a very simple example for this phenomenon (this is discussed in more detail in Chapter 4): The equations of linear elasticity for the displacement of an elastic material filling a ball in $\mathbb{R}^3$ and satisfying "non-sliding" boundary conditions. These conditions are expressed by the vanishing of the tangential component of the displacement (essential b.c.) and the normal component of the traction (natural b.c.) on the boundary. Now for this situation of a second order 3×3 system, the number of partial differential equations on the boundary defining the boundary conditions has to be 3, before one can even discuss whether ellipticity conditions are satisfied or not. In our case of a sphere, however, the conditions just described cannot be formulated globally by 3 equations with smooth non-vanishing coefficients; one needs 4 at least. Locally, it is possible to introduce coordinates such that the vanishing of the tangential component can be described by 2 equations; globally it is not.

Our proposed solution for this dilemma is to write the boundary differential operators as the product of systems of differential operators and of projection-valued functions, one system each for the operators of order zero and for those of order one. The total number of components may thus exceed the number N allowed for an elliptic boundary value problem, but we require that the sum of the ranks of the two projection operators is equal to that number N. In our simple example, the projection operators would be the operators of projection onto the tangential and onto the normal components of a vector, thus reducing the total number of components from 4 (or 6 or even 12 for some other quite natural formulations of the same conditions) to the required dimension $N = 3$. The introduction of these projection operators allows us to subsume the elliptic boundary value problems in variational form under the general theory of elliptic boundary value problems.

c) We give a complete proof of the analytic regularity up to the boundary for elliptic boundary value problems with analytic coefficients near analytic parts of the boundary. We present the nested open set technique and Morrey's weighted norm estimates in complete detail and in a form suitable for generalization to the case of non-smooth points on the boundary in later chapters.

Like in the previous chapter, we start with functional analysis (trace spaces of the Sobolev spaces), before introducing elliptic boundary conditions and the technique of parametrices at boundary points, which together with the nested open set method allows us to prove local estimates valid up to the boundary. Then we use a bootstrapping method to get higher order local a priori estimates in their "analytic" version, first in the constant coefficient case, where the understanding of the main arguments is easier, and then in the variable coefficient case, concluding with the proof of Fredholm properties and analytic hypoellipticity.

Since we use the method of local parametrices, our approach for obtaining higher order regularity is founded on similar tools as the calculus of boundary pseudodifferential operators which is also often used to prove regularity of elliptic boundary value problems in the literature [18, 90, 94, 91]. With this technique, it is also possible to obtain results on analytic regularity [92].

Nevertheless, in view of the generalizations to non-smooth domains, we prefer to build on the more classical approach to higher regularity ([4, 5]) and its generalization to the analytic framework ([70, 59]).

Plan of Chapter 2

- **§1** Trace spaces of Sobolev spaces. The model for a domain and its boundary: The periodic half-space.
- **§2** Definition of elliptic boundary value problems for model and general systems: The covering condition and the global writing of boundary conditions with fields of projection operators. Local parametrices. Basic a priori estimates.
- **§3** Higher order regularity of solutions up to the boundary. Fredholm theorem in Sobolev spaces.
- **§4** Basic a priori estimates up to the boundary in nested concentric half-balls, taking the difference of radii into account.
- **§5** Order-independent estimates in nested concentric half-balls in the case of an operator with constant coefficients.
- **§6** Order-independent estimates in nested concentric half-balls in the general case of variable (analytic) coefficients.
- **§7** Analytic regularity up to the boundary for solutions of elliptic boundary value problems with analytic coefficients and analytic data.

§8 Extension of the notion of smooth domain, using geodesic completions and extended boundaries – to handle double boundary points.

Essentials

The regularity results obtained in the previous chapter hold only in the interior of a domain and thence globally on manifolds without boundary; on a domain with boundary they cannot be true if the considered subdomains touch the boundary. The reason is that the solutions of even homogeneous elliptic equations with constant coefficients can grow arbitrarily fast towards the boundary. Therefore there cannot hold any global control of the solution and its derivatives solely by the right hand side of an elliptic partial differential equation. Now some of the irregular behavior of the solution near the boundary will already be eliminated by the choice of function spaces in which we look for solutions, typically Sobolev spaces $\mathsf{H}^1(\Omega)$ or $\mathsf{H}^2(\Omega)$, but in order to gain uniqueness of solutions or even Fredholm properties of the problem, or estimates of Sobolev norms up to the boundary of the solution in terms of the right hand sides, one has to impose further conditions on the boundary behavior of the solutions.

Such *boundary conditions* are typically given in the form of partial differential equations to be satisfied on the boundary, and they are defined by differential operators that will be required to fulfill an ellipticity condition with respect to the second order elliptic system, the *covering* condition, also called *complementing* condition or *Shapiro-Lopatinski* condition.

The first parts of this chapter present the standard definitions of Sobolev spaces on domains and their trace spaces on the boundary. Then we define and discuss ellipticity conditions on the model domain $\mathbb{T}^n_+ = \mathbb{T}^{n-1} \times \mathbb{R}_+$, the *periodic half-space* of dimension $n \geq 2$, before introducing the central notion of an elliptic boundary system near a smooth boundary point of a domain or globally on a smooth bounded domain. On the basis of this definition, one can then use techniques very similar to those applied in the previous chapter and obtain a priori estimates, estimates for higher derivatives, and finally analytic estimates for the solution of an elliptic boundary value problem in terms of the right hand sides in the domain and on the boundary.

We continue to consider second order $N \times N$ systems of linear partial differential operators on domains Ω in $\mathbb{R}^n$ or in manifolds of dimension n. To exclude degeneracies, we treat only the case of dimension $n \geq 2$. Then the boundary $\partial\Omega$ is a smooth manifold of dimension $n-1$, and we have the notion of Sobolev spaces on such a manifold as defined in the previous chapter in Section 1.1.a, see (1.4) and (1.8). For functions in the Sobolev space $\mathsf{H}^k(\Omega)$, we will typically have to consider the space of traces of order 0 and 1, namely the Sobolev spaces $\mathsf{H}^{k-\frac{1}{2}}(\partial\Omega)$ and $\mathsf{H}^{k-\frac{3}{2}}(\partial\Omega)$, respectively.

An important special case are Sobolev spaces on the periodic half-space. Functions u on $\mathbb{T}^n_+$ can be expressed in terms of their tangential Fourier coefficients $\hat{u}(\mathbf{p}')$, $\mathbf{p}' \in \mathbb{Z}^{n-1}$ which are functions of the normal variable t on the half-axis $\mathbb{R}_+$. A useful observation expressing the isotropy of the Sobolev norms is the norm equivalence for $s \geq 0$

$$\|u\|_{s;\mathbb{T}^n_+} \cong \Big(\|\hat{u}(\mathbf{0})\|^2_{s;\mathbb{R}_+} + \sum_{\mathbf{p}' \in \mathbb{Z}^{n-1}\setminus\{0\}} |\mathbf{p}'|^{2s-1} \|\mathfrak{H}_{|\mathbf{p}'|}\hat{u}(\mathbf{p}')\|^2_{s;\mathbb{R}_+} \Big)^{\frac{1}{2}} \tag{2.a}$$

with the notation $(\mathfrak{H}_\rho v)(t) = v(\frac{t}{\rho})$ for $\rho > 0$.

As a first step towards the definition and investigation of elliptic boundary value problems on general smooth domains, we consider suitable *model problems*, namely boundary value problems with constant coefficients on the periodic half-space $\mathbb{T}^n_+$ with boundary Γ, of the following form

$$\begin{cases} L_j \mathbf{u} &= f_j & \text{in } \mathbb{T}^n_+, & j = 1, \dots, N \\ T_k \mathbf{u} &= g_k & \text{on } \Gamma, & k = 1, \dots, N_1 \\ D_k \mathbf{u} &= h_k & \text{on } \Gamma, & k = 1, \dots, N_0. \end{cases} \tag{2.b}$$

Here L_j, T_k and D_k are linear partial differential operators with constant coefficients of order 2, 1 and 0, respectively. We will often write the system (2.b) in more condensed form as

$$\begin{cases} L\mathbf{u} &= \mathbf{f} & \text{in } \mathbb{T}^n_+, \\ T_\Gamma \mathbf{u} &= \mathbf{g} & \text{on } \Gamma \\ D_\Gamma \mathbf{u} &= \mathbf{h} & \text{on } \Gamma. \end{cases} \tag{2.c}$$

or even as $\{L, C_\Gamma\}\mathbf{u} = \{\mathbf{f}, \mathbf{b}\}$. Here $C_\Gamma = (T_\Gamma, D_\Gamma)$ stands for all boundary operators, and this notation includes the operation of restriction to the boundary.

By periodic Fourier transformation in $\mathbb{T}^{n-1}$, the boundary value problem (2.b) becomes a sequence of boundary value problems for a system of ordinary differential equations on the half-axis $\mathbb{R}_+$ for the Fourier coefficients $\hat{\mathbf{u}}(\mathbf{p}')$ of $\mathbf{u}$. If we write $t \mapsto \mathbf{U}(t)$ for such a Fourier coefficient, we obtain with the notation of the previous chapter for the symbols of the differential operators:

$$\begin{cases} L(\mathbf{p}', \mathrm{D}_t)\mathbf{U} &= \mathbf{F} & \text{in } \mathbb{R}_+ \\ T_k(\mathbf{p}', \mathrm{D}_t)\mathbf{U} &= G_k & \text{in } t = 0, & k = 1, \dots, N_1 \\ D_k \mathbf{U} &= H_k & \text{in } t = 0, & k = 1, \dots, N_0 \end{cases} \tag{2.d}$$

Together with these problems for any finite frequency $\mathbf{p}' \in \mathbb{Z}^{n-1}$, we have to consider the corresponding limit frequencies $\boldsymbol{\xi}'$ on the "sphere at infinity" $\mathbb{S}^{n-2}$, namely the systems with the principal parts of the differential operators

$$\begin{cases} L^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} &= \mathbf{F} & \text{in } \mathbb{R}_+ \\ T_k^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} &= G_k & \text{in } t = 0, & k = 1, \dots, N_1 \\ D_k \mathbf{U} &= H_k & \text{in } t = 0, & k = 1, \dots, N_0 \end{cases} \tag{2.e}$$

This latter system now takes the role of a *symbol* of the boundary value problem and is used to define ellipticity.

Definition 2.A *In the case of a boundary value system with constant coefficients (L, C_Γ), let the second order $N \times N$ system L be elliptic. The boundary operator C_Γ is said to* complement *or* cover *the operator L, and the corresponding boundary value problem is called* elliptic, *if for any $\boldsymbol{\xi}' \in \mathbb{R}^{n-1} \setminus \mathbf{0}$ the boundary value problem (2.e) admits, for $\mathbf{F} = \mathbf{0}$ and for any $(\mathbf{G}, \mathbf{H}) \in \mathbb{C}^{N_1+N_0}$, a unique solution $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}_+)$.*

The solutions of the homogeneous differential equation $L^{\mathrm{pr}}\mathbf{U} = \mathbf{0}$ on the half-axis are polynomials times exponential functions that, due to the ellipticity of L, either grow or decay exponentially at infinity. The condition $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}_+)$ in the definition can therefore be replaced by the condition that $\mathbf{U}$ vanishes at $+\infty$, or equivalently by the condition that $\mathbf{U}$ belongs to any Sobolev space on the half-axis or even that it is a tempered distribution.

Another remark is that this definition of ellipticity necessarily requires that the number $N_1 + N_0$ of boundary conditions equals N.

Ellipticity can again be characterized as invertibility modulo lower order terms on $\mathbf{H}^2$, see Proposition 2.2.10 and Corollary 2.2.13:

Theorem 2.B *Let L be a second order elliptic $N \times N$ system with constant coefficients and let C_Γ be a first order constant coefficient $N \times N$ system of boundary operators. Let the target space corresponding to $\mathbf{H}^2(\mathbb{T}^n_+)$ be defined as*

$$\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+) = \mathsf{L}^2(\mathbb{T}^n_+)^N \times \mathsf{H}^{\frac{1}{2}}(\Gamma)^{N_1} \times \mathsf{H}^{\frac{3}{2}}(\Gamma)^{N_0}.$$

(i) The operator $\{L, C_\Gamma\}$ defines an isomorphism between $\mathbf{H}^2(\mathbb{T}^n_+)$ and $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$ if and only if the following is satisfied: The half-axis boundary value problems (2.d) for any $\mathbf{p}' \in \mathbb{Z}^{n-1}$ and (2.e) for any $\boldsymbol{\xi}' \in \mathbb{S}^{n-2}$ admit a unique solution in $\mathbf{L}^2(\mathbb{R}_+)$ for $\mathbf{F} = \mathbf{0}$ and for every $(\mathbf{G}, \mathbf{H}) \in \mathbb{C}^N$, and the complete symbol $L(\mathbf{p}', \tau)$ is invertible for all $\mathbf{p}' \in \mathbb{Z}^{n-1}$ and $\tau \in \mathbb{R}$.

(ii) The boundary operator C_Γ covers L if and only if there exists another system $\{\tilde{L}, \tilde{C}_\Gamma\}$ that has the same principal part as $\{L, C_\Gamma\}$ and defines an isomorphism between $\mathbf{H}^2(\mathbb{T}^n_+)$ and $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$.

Due to the unboundedness of $\mathbb{T}^n_+$, there is no global Fredholm theorem. The corresponding operator between Sobolev spaces on $\mathbb{T}^n_+$ and its boundary will not have finite-dimensional kernel or cokernel, not will its range be closed, in general.

For bounded domains, such Fredholm properties hold, however, and one gets there by using Theorem 2.B for the construction of local parametrices. *Localization* now requires the study of half-balls in the half-space $\mathbb{R}^n_+$ which can also be considered as subsets of the periodic half-space $\mathbb{T}^n_+$. Smooth parts of the boundary $\partial\Omega$ of a domain Ω in $\mathbb{R}^n$ are characterized by the existence, for each boundary point $\mathbf{x}_0$, of a diffeomorphism $\phi_{\mathbf{x}_0}$ between a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ in $\mathbb{R}^n$ and a ball B_R centered at the origin, such that $\phi_{\mathbf{x}_0}(\mathbf{x}_0) = \mathbf{0}$, $\mathcal{U} \cap \Omega$ is mapped to the half-ball $V_R = B_R \cap \mathbb{R}^n_+$ and the boundary part $\mathcal{U} \cap \partial\Omega$ to the flat boundary part B'_R of V_R.

Thus, contrary to the situation in the previous chapter, we need to introduce nonlinear coordinate transformations in order to flatten the boundary locally, even if our domain is in $\mathbb{R}^n$ and not part of a general manifold. This means that one needs a theory of boundary value problems with variable coefficients even if one only wants to treat examples of partial differential operators with constant coefficients. The coordinate transformation $\phi_{\mathbf{x}_0}$ changes a boundary value system $\{L, C_\Gamma\}$ given in a neighborhood of $\mathbf{x}_0 \in \Gamma \subset \partial\Omega$ into another one given on the half-ball V_R with boundary conditions on B'_R. The principal part of the latter, with coefficients frozen at the origin, is the "tangent" object whose symbol plays the role of principal symbol of the original boundary value problem in the boundary point $\mathbf{x}_0$. This procedure leads, in a first step, to the proper definition of ellipticity of a boundary value problem

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ T\mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega, \\ D\mathbf{u} &= \mathbf{h} \quad \text{on } \partial\Omega. \end{cases} \tag{2.f}$$

The associated boundary value problem with constant coefficients on the periodic half-space is

$$\begin{cases} \underline{L}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{f} \quad \text{in } \mathbb{T}^n_+, \\ \underline{T}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma, \\ \underline{D}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{h} \quad \text{on } \Gamma. \end{cases} \tag{2.g}$$

The "tangent" operators $\{\underline{L}_{\mathbf{x}_0}, \underline{T}_{\mathbf{x}_0}, \underline{D}_{\mathbf{x}_0}\}$ are obtained from $\{L, T, D\}$ either, as described above, by substituting the coordinate transformation $\phi_{\mathbf{x}_0}$, then freezing the coefficients in $\mathbf{0}$ and taking the principal part, or, what is equivalent, by taking first the principal part in $\mathbf{x}_0$ and then using the chain rule to transform the partial derivatives:

$$\{\underline{L}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{x}}), \underline{T}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{x}}), \underline{D}_{\mathbf{x}_0}\} = \{L^{\mathsf{pr}}(\mathbf{x}_0; J^\top \mathrm{D}_{\mathbf{x}}), T^{\mathsf{pr}}(\mathbf{x}_0; J^\top \mathrm{D}_{\mathbf{x}}), D(\mathbf{x}_0)\}.$$

Here J is the Jacobi matrix of $\phi_{\mathbf{x}_0}$ in $\mathbf{x}_0$.

Definition 2.C *The boundary operators $C = (T, D)$ are said to cover the system L in the boundary point $\mathbf{x}_0$, and the boundary value problem (2.f) is called elliptic at $\mathbf{x}_0$, if the tangent boundary value problem (2.g) is elliptic in the sense of Definition 2.A.*

It is not hard to see that this definition of ellipticity is independent of the local diffeomorphism $\phi_{\mathbf{x}_0}$ chosen to flatten the boundary.

Now this definition is suitable for *local* descriptions of elliptic boundary value problems, and it is sufficient for proving local a priori estimates, finite regularity and analytic regularity results, and indeed the corresponding global results will be proved by using localization techniques that reduce them ultimately to the form just given.

But in order to describe the *global* form of the class of boundary value problems we are analyzing here, we have to introduce two generalizations. Both have to do with the non-trivial topology of the boundary $\partial\Omega$.

For the first generalization, we allow that the boundary $\partial\Omega$ has several components, called "sides" $\partial_{\mathbf{s}}\Omega$, $\mathbf{s} \in \mathscr{S}$, with different boundary operators given on different sides. This is a notation that will be essential in later chapters where boundary value problems on polygons or mixed boundary value problems are treated. But even for smooth domains, it would be too restrictive to require the same structure for the boundary operators on different connected components of the boundary. Written with this situation in mind, the boundary value problem (2.f) now takes the more detailed form

$$\begin{cases} L\,\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega\,, \\ T_{\mathbf{s}}\,\mathbf{u} &= \mathbf{g}_{\mathbf{s}} \quad \text{on } \partial_{\mathbf{s}}\Omega\,, \quad \mathbf{s} \in \mathscr{S}, \\ D_{\mathbf{s}}\,\mathbf{u} &= \mathbf{h}_{\mathbf{s}} \quad \text{on } \partial_{\mathbf{s}}\Omega\,, \quad \mathbf{s} \in \mathscr{S}, \end{cases} \tag{2.h}$$

The second generalization concerns the number of boundary conditions. As explained in the Introduction, it is made necessary by the fact that quite a few important standard elliptic boundary value problems cannot be written globally with just N boundary operators. Ellipticity, of course, requires that at least locally there must exist coordinate systems such that there are N boundary conditions for our second order $N \times N$ system. But the transformation leading to this form may transform not only the independent variables, but also the components of the vector functions. Just as the flattening of the boundary has to be done locally in coordinate patches covering the boundary, the selection of N linearly independent boundary conditions among the $\hat{N} \geq N$ globally given conditions can be done only locally. In order to describe the global structure which has the right local behavior, we assume that our boundary operators have the form

$$C = (T, D)\,, \quad \text{with} \quad T = \Pi^{\mathsf{T}}\,\hat{T} \quad \text{and} \quad D = \Pi^{\mathsf{D}}\,\hat{D}\,. \tag{2.i}$$

Here $\hat{T}$ and $\hat{D}$ are $\hat{N}_1 \times N$ and $\hat{N}_0 \times N$ systems of partial differential operators of order 1 and 0, respectively, and Π^{T} and Π^{D} are two smooth functions on $\partial\Omega$ with values in the projection operators on $\mathbb{C}^{\hat{N}_1}$ and on $\mathbb{C}^{\hat{N}_0}$, respectively.

The integers $\hat{N}_1$ and $\hat{N}_0$ are smooth functions on $\partial\Omega$, hence constant on each boundary component $\partial_{\mathbf{s}}\Omega$: $\hat{N}_i = \hat{N}_i(\mathbf{s})$ on $\partial_{\mathbf{s}}\Omega$, $\mathbf{s} \in \mathscr{S}$. The ranks N_1 and N_0 of the projector fields Π^{T} and Π^{D} are piecewise constant on $\partial\Omega$, too, by continuity. To emphasize the dependency of the structure of the projectors on $\mathbf{s} \in \mathscr{S}$, we write also $\Pi^{\mathsf{T}}_{\mathbf{s}}$, $\Pi^{\mathsf{D}}_{\mathbf{s}}$.

Any smooth projection-valued function can locally be diagonalized by a matrix-valued function that has the same order of smoothness (see Lemma 2.2.25). This means for our boundary operators (2.i) that after a local smooth transformation of the vector components, there remain only $N_1 + N_0$ boundary conditions for which the Definition 2.C of ellipticity in a point can then be applied, – in particular the ellipticity implies that

$$N_0 + N_1 = N.$$

It can be seen that this notion of ellipticity does not depend on the choice of diagonalization of the projector fields, nor on the projector fields and operators $\hat{T}$ and $\hat{D}$ chosen to write the factorization (2.i), but depends only on the boundary operators $C = (T, D)$

themselves. For the precise details of this global definition of elliptic boundary value problems, see Subsection 2.2.c.

There is one point that is facilitated by the introduction of the projector fields Π^T and Π^D, namely the description of suitable right hand sides in the boundary value problem. The correct space of possible right hand sides $(\mathbf{f},\mathbf{g},\mathbf{h})$ for regularity order k is

$$\mathsf{R}\mathbf{H}^k(\Omega) := \mathsf{H}^{k-2}(\Omega)^N \times \prod_{\mathbf{s}\in\mathscr{S}} \Big(\Pi^T_{\mathbf{s}} \mathsf{H}^{k-\frac32}(\partial_{\mathbf{s}}\Omega)^{\hat N_1(\mathbf{s})} \times \Pi^D_{\mathbf{s}} \mathsf{H}^{k-\frac12}(\partial_{\mathbf{s}}\Omega)^{\hat N_0(\mathbf{s})}\Big)$$

To fix the ideas, consider a simple example similar to the one discussed in the introduction, where this definition may mean that $\mathbf{h}$ is a tangential vector field of regularity $\mathsf{H}^{k-\frac12}$ on a boundary component $\partial_{\mathbf{s}}\Omega$ and $\mathbf{g}$ is a normal field of regularity $\mathsf{H}^{k-\frac32}$, whereas on another component $\partial_{\mathbf{s}'}\Omega$ $\mathbf{g}$ is tangential and $\mathbf{h}$ is normal.

It is clear that for a smooth bounded domain and operators with sufficiently smooth coefficients with the structure just described ("admissible systems"), the operator (L, C) is bounded from $\mathbf{H}^k(\Omega)$ to $\mathsf{R}\mathbf{H}^k(\Omega)$. It is one of the main results of this chapter that under our definition of ellipticity it is also a Fredholm operator between these spaces.

The techniques which we present for proving local and global a priori estimates, finite order and analytic regularity results and Fredholm properties, are extensions of those used in the previous chapter for the boundary-less situation.

The construction of local and then global parametrices gives H^2 estimates up to the boundary, and higher order tangential regularity and the corresponding estimates can then be obtained by using finite differences in tangential direction for the local problem transformed to the half-space. Estimates for higher order derivatives in normal direction use the fact that the boundary is non-characteristic for the elliptic operator L, and therefore the second derivative in normal direction of $\mathbf{u}$ can be expressed by $L\mathbf{u}$ and by derivatives with lower normal order of $\mathbf{u}$, see (2.53).

Analytic estimates are obtained by the nested open set technique, now using concentric half-balls instead of concentric balls. This technique is basically the same as for the interior estimates in the previous chapter, only much more complicated because of the need for anisotropic norm estimates where the number of derivatives in the normal and tangential directions vary independently.

The main results of this chapter can be summarized in the following theorem.

Theorem 2.D *Let Ω be a bounded domain in $\mathbb{R}^n$ and $k \geq 2$ an integer. We assume the following local regular configuration (see also Figure 2.3, page 92): With $\mathcal{U}_1$ and $\mathcal{U}_2$ open sets in $\mathbb{R}^n$ such that $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$, we set*

$$\Omega_1 = \mathcal{U}_1 \cap \Omega, \quad \Omega_2 = \mathcal{U}_2 \cap \Omega \quad \textit{and} \quad \Gamma_2 := \partial\Omega_2 \cap \partial\Omega$$

and assume that Γ_2 is a $\mathscr{C}^k$ submanifold of $\partial\Omega$. Let $\{L, T, D\}$ be an elliptic boundary value system on $\Omega_2 \cup \Gamma_2$ with coefficients of class $\mathscr{C}^{k-2}(\Omega_2\cup\Gamma_2)$ for L, $\mathscr{C}^{k-1}(\Gamma_2)$ for T, and $\mathscr{C}^k(\Gamma_2)$ for D.

(i) *There is a constant* c *such that for any* $\mathbf{f} \in \mathbf{H}^{k-2}(\Omega_2)$, $\mathbf{g} \in \mathbf{H}^{k-\frac{3}{2}}(\Gamma_2)$, $\mathbf{h} \in \mathbf{H}^{k-\frac{1}{2}}(\Gamma_2)$, *and any solution* $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ *of the boundary value problem*

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega_2 \\ T\mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma_2 \\ D\mathbf{u} &= \mathbf{h} \quad \text{on } \Gamma_2 \end{cases} \tag{2.j}$$

there holds $\mathbf{u} \in \mathbf{H}^k(\Omega_1)$, *and there is an estimate*

$$\|\mathbf{u}\|_{k;\,\Omega_1} \leq c\Big(\|\mathbf{f}\|_{k-2;\,\Omega_2} + \|\mathbf{g}\|_{k-\frac{3}{2};\,\Gamma_2} + \|\mathbf{h}\|_{k-\frac{1}{2};\,\Gamma_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\Big). \tag{2.k}$$

(ii) *If* Γ_2 *is an analytic manifold and the coefficients of* $\{L, T, D\}$ *are analytic, then there is a constant* A *independent of* k *such that there holds a more precise a priori estimate*

$$\frac{1}{k!}|\mathbf{u}|_{k;\,\Omega_1} \leq A^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\Big(|\mathbf{f}|_{\ell;\,\Omega_2} + \|\mathbf{g}\|_{\ell+\frac{1}{2};\,\Gamma_2} + \|\mathbf{h}\|_{\ell+\frac{3}{2};\,\Gamma_2}\Big) + \|\mathbf{u}\|_{1;\,\Omega_2}\Big\}. \tag{2.l}$$

(iii) *If the hypotheses are satisfied globally for* $\Omega_1 = \Omega_2 = \Omega$ *and* $\Gamma_2 = \partial\Omega$, *then the system* $\{L, T, D\}$ *defines a Fredholm operator from* $\mathbf{H}^k(\Omega)$ *to* $\mathsf{R}\mathbf{H}^k(\Omega)$.

(iv) *If, in addition to the assumptions of (iii), the boundary* $\partial\Omega$ *and the coefficients of the system are analytic and* $\mathbf{f} \in \mathbf{A}(\Omega)$, $\mathbf{g}_\mathbf{s}, \mathbf{h}_\mathbf{s} \in \mathbf{A}(\partial_\mathbf{s}\Omega)$ *for all* $\mathbf{s} \in \mathscr{S}$, *then any solution* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *belongs to* $\mathbf{A}(\Omega)$.

The elliptic regularity results described by this theorem are valid not only for bounded subdomains Ω of $\mathbb{R}^n$ with smooth boundaries, but in a natural way also for bounded subdomains of smooth manifolds and therefore also for compact smooth manifolds with boundary. This extended class allows to cover a special class of subdomains of $\mathbb{R}^n$ that do not satisfy one of the main conditions of smooth domains in $\mathbb{R}^n$, namely that the domain lies "locally on one side of its boundary".

This seemingly natural condition is satisfied not only for smooth domains, but even for large classes of non-smooth domains, such as Lipschitz domains or domains with continuous boundary. However, in view of examples that we will want to study later on, such as domains with *cracks* or *inward cusps*, we need to consider also domains with multiple boundary points.

Some of these domains Ω with multiple boundary points can be included in the class of smooth domains for which the standard elliptic regularity results hold, by defining the notion of *unfolded boundary* $\partial_*\Omega$. This definition is based on the completion $\overline{\Omega}^*$ of the domain with respect to the metric of its intrinsic geodesic distance. We call Ω an *extended smooth domain* if its geodesic completion $\overline{\Omega}^*$ is a compact smooth manifold with boundary. As a subdomain of this manifold, Ω then will lie locally on one side of its (unfolded) boundary, even if it does not have this property when considered as subdomain of $\mathbb{R}^n$. Details of this construction and examples are discussed in Section 2.8.

2.1 Trace spaces

2.1.a Traces on the boundary of a domain

We recall some well-known facts about the restriction of functions to the boundary, see [76, 57, 42]. Let Ω be a domain in $\mathbb{R}^n$ or in a smooth manifold M of dimension n. In this chapter we assume that $n \geq 2$, in order to avoid degenerate boundaries. Let Ω have a smooth boundary $\partial\Omega$, which means that for each $\mathbf{x}_0 \in \partial\Omega$, there exists a smooth ($\mathscr{C}^\infty$ or analytic) local map ϕ which transforms a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ in M into a ball B centered in 0 in $\mathbb{R}^n$, and so that $\phi(\partial\Omega \cap \mathcal{U})$ is the set $\{(\mathbf{x}', 0) \in B\}$ with $\mathbf{x}' = (x_1, \ldots, x_{n-1})$, and $\phi(\Omega \cap \mathcal{U})$ is the set $\{(\mathbf{x}', x_n) \in B,\ x_n > 0\}$.

The restriction $\gamma_0 : u \mapsto u|_{\partial\Omega}$ to $\partial\Omega$ makes sense for any $u \in \mathscr{C}^\infty(\overline{\Omega})$. Moreover γ_0 can be extended to $\mathsf{H}^m(\Omega)$ for any $m \geq 1$, and, in fact, to $\mathsf{H}^s(\Omega)$ for any $s > 1/2$. The image of the space $\mathsf{H}^m(\Omega)$ by γ_0 can be characterized: This is the Sobolev space $\mathsf{H}^{m-\frac{1}{2}}(\partial\Omega)$ as introduced in Section 1.1.a. It is called the *trace space* of $\mathsf{H}^m(\Omega)$, and γ_0 the *trace operator*. The corresponding norm is denoted by $\|\cdot\|_{m-\frac{1}{2};\partial\Omega}$.

The trace operator is continuous from $\mathsf{H}^m(\Omega)$ onto $\mathsf{H}^{m-\frac{1}{2}}(\partial\Omega)$ and has a continuous right inverse ("lifting") from $\mathsf{H}^{m-\frac{1}{2}}(\partial\Omega)$ into $\mathsf{H}^m(\Omega)$. This property still holds with m replaced by any real number $s > \frac{1}{2}$.

More generally, one can consider traces of normal derivatives. The outer normal unit field (tangent to M) to the boundary of Ω defines a smooth field, denoted by $\mathbf{n}$. The trace of the kth derivative with respect to this vector field, $\gamma_k : u \mapsto \partial_{\mathbf{n}}^k u|_{\partial\Omega}$ defines a continuous operator from $\mathsf{H}^s(\Omega)$ onto $\mathsf{H}^{s-k-\frac{1}{2}}(\partial\Omega)$ for all real $s > k - \frac{1}{2}$. Moreover, the collection of trace operators

$$\boldsymbol{\gamma}_{\partial\Omega}^k := (\gamma_0 \ldots, \gamma_k) : \mathsf{H}^s(\Omega) \longrightarrow \mathsf{H}^{s-\frac{1}{2}}(\partial\Omega) \times \ldots \times \mathsf{H}^{s-k-\frac{1}{2}}(\partial\Omega) \tag{2.1}$$

is bounded and onto, and there exists a continuous simultaneous lifting of the first $k+1$ traces.

These facts extend to the *local* situation, when Ω is any domain, and $\Gamma \subset \partial\Omega$ is a regular part of the boundary of Ω. Then the $k+1$-trace operator $\boldsymbol{\gamma}_\Gamma^k$ on Γ is bounded and onto,

$$\boldsymbol{\gamma}_\Gamma^k : \mathsf{H}^s(\Omega) \longrightarrow \mathsf{H}^{s-\frac{1}{2}}(\Gamma) \times \ldots \times \mathsf{H}^{s-k-\frac{1}{2}}(\Gamma) \tag{2.2}$$

and has a continuous right inverse.

In the degenerate case of dimension $n = 1$, in this situation we still have the existence of restrictions of the corresponding derivatives at the boundary points, due to the Sobolev embedding theorem (1.13).

2.1.b Sobolev spaces on the periodic half-space

The standard model for a domain and its boundary is $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$ with boundary $\mathbb{R}^{n-1}$, but, like in Chapter 1, we will often prefer a periodic model, that is the periodic half-space $\mathbb{T}^n_+$,

$$\mathbb{T}^n_+ = \{\mathbf{x} = (\mathbf{x}', t) : \quad \mathbf{x}' \in \mathbb{T}^{n-1},\ t > 0\}.$$

with its boundary,

$$\Gamma = \{\mathbf{x} = (\mathbf{x}', 0) : \quad \mathbf{x}' \in \mathbb{T}^{n-1}\}.$$

Note that now, for convenience, we denote the coordinate normal to the boundary by t instead of x_n.

The Sobolev spaces on $\mathbb{T}^n_+$ are defined by (1.3) for non-negative integer exponents m, and by (1.4)-(1.5) for positive non-integer exponents s. There holds for all $s \geq 0$

$$\mathsf{H}^s(\mathbb{T}^n_+) = \mathsf{L}^2(\mathbb{T}^{n-1}, \mathsf{H}^s(\mathbb{R}_+)) \cap \mathsf{H}^s(\mathbb{T}^{n-1}, \mathsf{L}^2(\mathbb{R}_+)), \tag{2.3}$$

with equivalence of norms[1].

For $u \in \mathsf{L}^2(\mathbb{T}^n_+)$, let its partial periodic Fourier transform $(\mathscr{F}'_{\mathsf{per}}u)(\mathbf{p}', t)$ be defined as:

$$(\mathscr{F}'_{\mathsf{per}}u)(\mathbf{p}', t) = (2\pi)^{-(n-1)/2} \int_{\mathbb{T}^{n-1}} \mathrm{e}^{-i\,\mathbf{x}'\cdot\mathbf{p}'}\, u(\mathbf{x}', t)\, \mathrm{d}\mathbf{x}', \quad \mathbf{p}' \in \mathbb{Z}^{n-1},\ t \in \mathbb{R}_+. \tag{2.4}$$

Owing to (2.3), this allows a simple characterization of Sobolev spaces on $\mathbb{T}^n_+$: With the partial Fourier coefficients $\hat{u}(\mathbf{p}')$ defined as

$$\hat{u}(\mathbf{p}')(t) := (\mathscr{F}'_{\mathsf{per}}u)(\mathbf{p}', t), \quad \mathbf{p}' \in \mathbb{Z}^{n-1},$$

there holds the equivalence of norms, for any $s \geq 0$, cf. (1.7):

$$\|u\|_{s;\mathbb{T}^n_+} \cong \Big\{ \sum_{\mathbf{p}' \in \mathbb{Z}^{n-1}} \Big(\|\hat{u}(\mathbf{p}')\|^2_{s;\mathbb{R}_+} + |\mathbf{p}'|^{2s} \|\hat{u}(\mathbf{p}')\|^2_{0;\mathbb{R}_+} \Big) \Big\}^{1/2}. \tag{2.5}$$

This is the reason for the introduction of the parameter-dependent H^s semi-norms and norms on $\mathbb{R}_+$: For any real $s \geq 0$, we set for any $\rho \geq 0$

$$\begin{aligned} |v|_{s;\mathbb{R}_+;\rho} &= \Big\{ \sum_{k=0}^{s} \rho^{2(s-k)} |v|^2_{k;\mathbb{R}_+} \Big\}^{1/2} && \text{if } s \in \mathbb{N} \\ |v|_{s;\mathbb{R}_+;\rho} &= \Big\{ \sum_{k=0}^{[s]} \rho^{2(s-k)} |v|^2_{k;\mathbb{R}_+} + |v|^2_{s;\mathbb{R}_+} \Big\}^{1/2} && \text{if } s \notin \mathbb{N}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \|v\|_{s;\mathbb{R}_+;\rho} &= \Big\{ \sum_{\ell=0}^{s} |v|^2_{\ell;\mathbb{R}_+;\rho} \Big\}^{1/2} && \text{if } s \in \mathbb{N} \\ \|v\|_{s;\mathbb{R}_+;\rho} &= \Big\{ \sum_{\ell=0}^{[s]} |v|^2_{\ell;\mathbb{R}_+;\rho} + |v|^2_{s;\mathbb{R}_+;\rho} \Big\}^{1/2} && \text{if } s \notin \mathbb{N}. \end{aligned} \tag{2.7}$$

[1] Equivalence constants depend on s, but are polynomially bounded with respect to s.

We can see that if $m \in \mathbb{N}$, the norm $\|v\|^2_{m;\mathbb{R}_+;\rho}$ coincides with $\sum_{k=0}^{m} \rho^{2(m-k)} \|v\|^2_{k;\mathbb{R}_+}$. We also notice the equivalences, uniform in $\rho > 0$,

$$\|v\|^2_{s;\mathbb{R}_+;\rho} \cong \|v\|^2_{s;\mathbb{R}_+} + \rho^{2s}\|v\|_{0;,\mathbb{R}_+} \cong |v|^2_{s;\mathbb{R}_+} + (1+\rho^2)^s\|v\|_{0;,\mathbb{R}_+}\,, \tag{2.8}$$

in consequence of which (2.5) can be written as

$$\|u\|_{s;\mathbb{T}^n_+} \cong \Big\{ \sum_{\mathbf{p}'\in\mathbb{Z}^{n-1}} \|\hat{u}(\mathbf{p}')\|^2_{s;\mathbb{R}_+;|\mathbf{p}'|} \Big\}^{1/2}. \tag{2.9}$$

The following lemma is the prototype of many similar ones, and the corner stone for homogeneity arguments.

Lemma 2.1.1 *Let $\rho > 0$. Let u belong to $\mathsf{H}^s(\mathbb{R}_+)$ for a fixed real number $s \geq 0$. Let $\mathfrak{H}_\rho$ be the transformation $\mathfrak{H}_\rho : u \mapsto \mathfrak{H}_\rho u$ where*

$$(\mathfrak{H}_\rho u)(t) = u\Big(\frac{t}{\rho}\Big). \tag{2.10}$$

Then $\mathfrak{H}_\rho u$ belongs to $\mathsf{H}^s(\mathbb{R}_+)$ and there holds

$$\|\mathfrak{H}_\rho u\|_{s;\mathbb{R}_+} = \rho^{-s+\frac{1}{2}} |u|_{s;\mathbb{R}_+;\rho}\,. \tag{2.11}$$

This identity (2.11) and the equivalence (2.9) directly yield the equivalence (2.a):

$$\|u\|_{s;\mathbb{T}^n_+} \cong \Big(\|\hat{u}(\mathbf{0})\|^2_{s;\mathbb{R}_+} + \sum_{\mathbf{p}'\in\mathbb{Z}^{n-1}\setminus\{0\}} |\mathbf{p}'|^{2s-1} \|\mathfrak{H}_{|\mathbf{p}'|}\hat{u}(\mathbf{p}')\|^2_{s;\mathbb{R}_+} \Big)^{\frac{1}{2}} \tag{2.a}$$

Remark 2.1.2 This representation allows also a simple proof of the continuity of the trace mapping on the periodic half-space $\mathbb{T}^n_+$ and an explicit construction of a trace lifting. If one wishes, one can then extend these constructions and estimates, by using local coordinates and partitions of unity, from $\mathbb{T}^n_+$ to any bounded domain with a smooth boundary. We will not present this complete proof here, since these results can easily be found in standard literature on Sobolev spaces. But we show the construction for the case of the periodic half-space here, because a similar, more complicated, construction will be needed for traces on infinite cones.

In the one-dimensional setting, one has the obvious "trace estimate"

$$|u(0)| \leq C_s \|u\|_{s;\mathbb{R}_+} \qquad \text{for all } u \in \mathsf{H}^s(\mathbb{R}_+)\,,\ s > \tfrac{1}{2}\,.$$

With (2.a), this gives for $u \in \mathsf{H}^s(\mathbb{T}^n_+)$ and $h(\mathbf{x}') = u(\mathbf{x}', 0)$ the continuity of the trace mapping, where we use $\hat{h}(\mathbf{p}') = \hat{u}(\mathbf{p}', 0) = \mathfrak{H}_{|\mathbf{p}'|}\hat{u}(\mathbf{p}')(0)$:

$$\begin{aligned}\|h\|^2_{s-\frac{1}{2};\mathbb{T}^{n-1}} &\cong |\hat{h}(\mathbf{0})|^2 + \sum_{\mathbf{p}'\in\mathbb{Z}^{n-1}\setminus\{0\}} |\mathbf{p}'|^{2s-1}|\hat{h}(\mathbf{p}')|^2 \\ &\leq C_s^2 \Big(\|\hat{u}(\mathbf{0})\|^2_{s;\mathbb{R}_+} + \sum_{\mathbf{p}'\in\mathbb{Z}^{n-1}\setminus\{0\}} |\mathbf{p}'|^{2s-1} \|\mathfrak{H}_{|\mathbf{p}'|}\hat{u}(\mathbf{p}')\|^2_{s;\mathbb{R}_+} \Big) \cong \|u\|^2_{s;\mathbb{T}^n_+}\,.\end{aligned}$$

For the construction of a trace lifting, we choose a function $\Phi \in \mathsf{H}^s(\mathbb{R}_+)$ satisfying $\Phi(0) = 1$, and for $h \in \mathsf{H}^{s-\frac{1}{2}}(\mathbb{T}^{n-1})$ we define u on $\mathbb{T}^n_+$ by its Fourier coefficients

$$\hat{u}(\mathbf{0})(t) = \Phi(t)\,\hat{h}(\mathbf{0}) \quad \text{and} \quad \hat{u}(\mathbf{p}')(t) = \Phi(t|\mathbf{p}'|)\,\hat{h}(\mathbf{p}') \ \text{ for } \mathbf{p}' \neq \mathbf{0}\,. \tag{2.12}$$

Then h is the boundary trace of u, and $\mathfrak{H}_{|\mathbf{p}'|}\hat{u}(\mathbf{p}')(t) = \Phi(t)\,\hat{h}(\mathbf{p}')$, which by (2.a) immediately gives the continuity of the trace lifting, namely the norm equivalence, valid for any $s > \frac{1}{2}$:

$$\|u\|_{s;\mathbb{T}^n_+} \cong \|\Phi\|_{s;\mathbb{R}_+}\ \|h\|_{s-\frac{1}{2};\mathbb{T}^{n-1}}\ . \qquad \triangle$$

2.2 Complementing boundary conditions

Our model problems for interior points of the domain have constant coefficients and are set on the torus $\mathbb{T}^n$, see §1.2. For regular boundary points, our model domain is the "periodic half-space" $\mathbb{T}^{n-1} \times \mathbb{R}_+$.

2.2.a Model problems on the periodic half-space

Ellipticity is a property that is defined pointwise, and with respect to the principal part of the operator, see (1.27) (a). This was the situation for the partial differential operator in the interior of the domain, and it will be the same here for the operator with boundary conditions.

Thus, in a first step, we introduce the "model boundary value problems", for which we define the fundamental notion of covering boundary condition. Our model boundary value problems are set on the periodic half-space $\mathbb{T}^n_+ = \mathbb{T}^{n-1} \times \mathbb{R}_+$ and its boundary $\Gamma = \mathbb{T}^{n-1} \times \{0\}$, and can be written as

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \mathbb{T}^n_+ \\ C_\Gamma \mathbf{u} = \mathbf{b} & \text{on } \Gamma. \end{cases} \tag{2.13}$$

Here L is a second order $N \times N$ system L with constant coefficients, i.e. the equation $L\mathbf{u} = \mathbf{f}$ corresponds to the system:

$$\sum_{j'=1}^{N} L_{jj'}(\mathrm{D}_\mathbf{x})\,u_{j'} = \sum_{j'=1}^{N} \sum_{|\alpha|\le 2} a^\alpha_{jj'}\,\partial^\alpha_\mathbf{x} u_{j'} = f_j, \quad j = 1, \dots, N,$$

and C_Γ is a system of boundary conditions on Γ written in partial differential form.

We assume that L is elliptic, which means that it satisfies condition (1.27) (a):

$$\forall \boldsymbol{\xi} \in \mathbb{S}^{n-1}, \quad L^{\mathsf{pr}}(\boldsymbol{\xi}) \text{ is invertible,}$$

where $L^{\mathsf{pr}}(\boldsymbol{\xi}) = \big(\sum_{|\alpha|=2} a^\alpha_{jj'}\,(i\boldsymbol{\xi})^\alpha\big)_{jj'}$ is its principal part.

On the periodic half-space $\mathbb{T}^n_+$, the totally homogeneous partial differential equation

$$L^{\mathsf{pr}}\mathbf{u} = \mathbf{0}$$

admits many solutions. This is quite different from the case of the torus $\mathbb{T}^n$, where we have seen, cf. Remark 1.2.4, that for an elliptic system the only solutions of the totally homogeneous problem are constants. On $\mathbb{T}^n_+$ there exist for any $\mathbf{p}' \in \mathbb{Z}^{n-1}$ exponential solutions of the form

$$\mathbb{T}^n_+ \ni (\mathbf{x}', t) \mapsto \mathbf{u}(\mathbf{x}', t) \;=\; e^{i\mathbf{p}'\cdot\mathbf{x}'} e^{i\tau t}\boldsymbol{\eta},$$

where $\tau \in \mathbb{C}$ and $\boldsymbol{\eta} \in \mathbb{C}^N$ are such that

$$\det L^{\mathsf{pr}}(\mathbf{p}', \tau) = 0 \quad \text{and} \qquad L^{\mathsf{pr}}(\mathbf{p}', \tau)\boldsymbol{\eta} = \mathbf{0}.$$

Stability of solutions on $\mathbb{T}^n_+$ can be achieved only if we exclude from the outset all such exponential solutions with $\operatorname{Im}\tau < 0$ which would grow exponentially for $t \to +\infty$. The other ones will be controlled by *boundary conditions* at $t = 0$, whose number should therefore correspond to the dimension of the space of such solutions with $\operatorname{Im}\tau > 0$, the existence of real τ being excluded by the ellipticity of L^{pr}.

Note that the assumption $n \geq 2$ is essential here: In dimension $n = 1$ we would have a finite-dimensional solution space (polynomials of degree 1) of the equation $L^{\mathsf{pr}}\mathbf{u} = 0$ and there is no obvious need for boundary conditions.

For $n \geq 2$, we can determine the form of solutions of the homogeneous equation $L^{\mathsf{pr}}\mathbf{u} = 0$ with the help of the partial periodic Fourier transformation $\mathscr{F}'_{\mathsf{per}}$. We write t for the last coordinate x_n and $L^{\mathsf{pr}} = L^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}'}, \mathrm{D}_t)$, with $\mathrm{D}_{\mathbf{x}'} = (-i\partial_1, \ldots, -i\partial_{n-1})$ and $\mathrm{D}_t = -i\partial_t$, and then there holds for all $\mathbf{p}' \in \mathbb{Z}^{n-1}$ and $t > 0$

$$\mathscr{F}'_{\mathsf{per}}\big(L^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}'}, \mathrm{D}_t)(\mathbf{u})\big)(\mathbf{p}', t) = L^{\mathsf{pr}}(\mathbf{p}', \mathrm{D}_t)(\mathscr{F}'_{\mathsf{per}}\mathbf{u})(\mathbf{p}', t). \tag{2.14}$$

We recall that, according to the convention of definition of symbols,

$$L^{\mathsf{pr}}(\mathbf{p}', \mathrm{D}_t) = \Big(\sum_{|\alpha|=2} a^{\alpha}_{jj'}\, i^{|\alpha|}(\mathbf{p}')^{\alpha'}\mathrm{D}_t^{\alpha_n}\Big)_{1\leq j,j'\leq N} \quad \text{with} \quad \alpha = (\alpha', \alpha_n).$$

Thus the equation $L^{\mathsf{pr}}\mathbf{u} = 0$ is equivalent to the family of ordinary differential systems on $\mathbb{R}_+$:

$$L^{\mathsf{pr}}(\mathbf{p}', \mathrm{D}_t)(\mathscr{F}'_{\mathsf{per}}\mathbf{u})(\mathbf{p}', t) = 0, \quad t > 0, \quad \forall \mathbf{p}' \in \mathbb{T}^{n-1}.$$

We introduce the following spaces of solutions:

Notation 2.2.1 Let L be an elliptic $N \times N$ system with constant coefficients.

(i) For $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$, let $\mathfrak{M}[L^{\mathsf{pr}}; \boldsymbol{\xi}']$ denote the space of solutions $\mathbf{U} = \mathbf{U}(t)$ of the system

$$L^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}(t) = 0 \quad \text{in} \quad \mathbb{R}_+.$$

Likewise, $\mathfrak{M}[L; \boldsymbol{\xi}']$ is the space of solutions of $L(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}(t) = 0$.

(ii) Let $\mathfrak{M}_+[L^{\mathsf{pr}};\boldsymbol{\xi}']$ and $\mathfrak{M}_+[L;\boldsymbol{\xi}']$ denote the subspaces of $\mathfrak{M}[L^{\mathsf{pr}};\boldsymbol{\xi}']$ and $\mathfrak{M}[L;\boldsymbol{\xi}']$ of stable solutions, that is, solutions $\mathbf{U}$ such that $\mathbf{U}(t) \to 0$ as $t \to \infty$. △

Since the system L has constant coefficients, the theory of systems of ordinary differential equations with constant coefficients gives us the structure of the spaces $\mathfrak{M}[L;\boldsymbol{\xi}']$ and $\mathfrak{M}_+[L;\boldsymbol{\xi}']$.

Lemma 2.2.2 *Let L be an elliptic $N \times N$ system with constant coefficients. Then, for all $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$,*

$$\mathfrak{M}[L^{\mathsf{pr}};\boldsymbol{\xi}'] = \operatorname{span}\Big\{\mathbf{U}(t) = \sum_{p=0}^{P} t^p \mathrm{e}^{i\tau t}\boldsymbol{\eta}_p \;:\; \tau \in \mathbb{C} \ \textit{such that}\ \det L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau) = 0,\ \boldsymbol{\eta}_0,\dots,\boldsymbol{\eta}_P \in \mathbb{C}^N \ \textit{sol. of a linear system}\Big\}.$$

The dimension of $\mathfrak{M}[L^{\mathsf{pr}};\boldsymbol{\xi}']$ is $2N$. In particular, if the equation $\det L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau) = 0$ has $2N$ distinct roots, the space $\mathfrak{M}[L^{\mathsf{pr}};\boldsymbol{\xi}']$ is generated by the functions $\mathrm{e}^{i\tau t}\boldsymbol{\eta}$ for each root τ and for the solutions $\boldsymbol{\eta} \in \mathbb{C}^N \setminus \{0\}$ of the linear system $L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau)\boldsymbol{\eta} = 0$.

Since L is elliptic, the roots τ of $\det L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau) = 0$ for $\boldsymbol{\xi}' \neq 0$ are never real. The function $\mathrm{e}^{i\tau t}$ is exponentially decreasing as $t \to +\infty$ if and only if $\operatorname{Im}\tau > 0$. Thus

$$\mathfrak{M}_+[L^{\mathsf{pr}};\boldsymbol{\xi}'] = \operatorname{span}\Big\{\mathbf{U}(t) = \sum_{p=0}^{P} t^p \mathrm{e}^{i\tau t}\boldsymbol{\eta}_p \in \mathfrak{M}[L^{\mathsf{pr}};\boldsymbol{\xi}'] \;:\; \operatorname{Im}\tau > 0\Big\}. \tag{2.15}$$

Again, our general assumption $n \geq 2$ is essential here: For $n = 1$, the only root would be $\tau = 0$, and the definition of the space $\mathfrak{M}_+$ would not make sense.

If the dimension n is at least 3, as a consequence of the ellipticity, the continuity of roots with respect to $\boldsymbol{\xi}' \neq 0$, and the topological fact that $\mathbb{R}^{n-1} \setminus \{0\}$ is connected, the number of roots with positive imaginary part is exactly N, see [85]. For the case when $n = 2$, the general use is to introduce the notion of proper ellipticity:

Definition 2.2.3 *Let L be a $N \times N$ second order system L with constant coefficients.*

- ⋆ *L is called **properly elliptic** if it is elliptic and if for each $\boldsymbol{\xi}' \neq 0$, the equation $\det L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau) = 0$ has N roots with positive imaginary part.*

In order to complement an elliptic system L, the boundary conditions on Γ have to be in one-to-one correspondence with exponentially decreasing solutions of the equation $L^{\mathsf{pr}}(\boldsymbol{\xi}',\mathrm{D}_t)\mathbf{u} = 0$. Thus, at least, we should have N boundary conditions, which motivates the following definition.

Definition 2.2.4 *Let L be a $N \times N$ system of order 2 on $\mathbb{T}^n_+$ with constant coefficients.*

⋆ *An associated set of **boundary operators** on $\Gamma = \partial\mathbb{T}^n_+$ is a set C_Γ of N scalar operators C_k of order 0 or 1 with constant coefficients, composed with the trace operator $\gamma_\Gamma : \mathbf{u} \mapsto \mathbf{u}\big|_\Gamma$,*

$$C_\Gamma = \gamma_\Gamma \circ (C_1, \ldots, C_N)$$

After a possible rearrangement, there exists $0 \le N_1 \le N$ so that $C_1, \ldots, C_{N_1}$ are of order 1 and $C_{N_1+1}, \ldots, C_N$ are of order 0.

⋆ *We alternatively denote by T_k the operators of order 1 and by D_k the operators of order 0, and set*

$$T_\Gamma = \gamma_\Gamma \circ (T_1, \ldots, T_{N_1}), \quad D_\Gamma = \gamma_\Gamma \circ (D_1, \ldots, D_{N_0}), \quad N_0 + N_1 = N\ .$$

The operators T_k and D_k can be written as

$$T_k\mathbf{u} = \sum_{j=1}^{N} \sum_{|\alpha|\le 1} t^\alpha_{kj} \partial^\alpha_{\mathbf{x}} u_j \quad \text{and} \quad D_k\mathbf{u} = \sum_{j=1}^{N} d_{kj} u_j.$$

⋆ *The principal part C^{pr}_Γ of $C_\Gamma = (T_\Gamma, D_\Gamma)$ is defined as $C^{\mathsf{pr}}_\Gamma = (T^{\mathsf{pr}}_\Gamma, D^{\mathsf{pr}}_\Gamma)$ where*

$$T^{\mathsf{pr}}_k\mathbf{u} = \sum_{j=I}^{N} \sum_{|\alpha|=1} t^\alpha_{kj} \partial^\alpha_{\mathbf{x}} u_j \quad \text{and} \quad D^{\mathsf{pr}}_k = D_k.$$

Remark 2.2.5 The extreme cases $N_0 = 0, N_1 = N$ and $N_0 = N, N_1 = 0$ are explicitly allowed here, corresponding to the absence of D_Γ or T_Γ, respectively. If $N_0 = N$ and the operators D_j are linearly independent, then the corresponding boundary value problem is *the Dirichlet problem*. Homogeneous Dirichlet conditions can equivalently be written as $\mathbf{u} = 0$ on Γ. △

Here follows the fundamental notion of covering boundary conditions.

Definition 2.2.6 *Let L be an elliptic $N \times N$ second order system with constant coefficients on $\mathbb{T}^n_+$. Let $C_\Gamma = \{C_1, \ldots, C_N\}$ be a set of N boundary operators on Γ. In accordance with Definition 2.2.4, by $C^{\mathsf{pr}}_k(\boldsymbol{\xi}', \mathrm{D}_t)$ we denote its partially Fourier-transformed principal part, that is*

$$C^{\mathsf{pr}}_k(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} = T^{\mathsf{pr}}_k(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} = \sum_{j=I}^{N} \sum_{|\alpha|=1} t^\alpha_{kj} (i\boldsymbol{\xi})^{\alpha'} \partial^{\alpha_n}_t U_j \quad \text{if} \quad 1 \le k \le N_1$$

and $C^{\mathsf{pr}}_k(\boldsymbol{\xi}', \mathrm{D}_t) = D_{k+1-N_1}$ if $N_1 + 1 \le k \le N$.

⋆ *The set C_Γ is said to **cover** (or **complement**) L on Γ if for each $\boldsymbol{\xi}' \neq 0$ in $\mathbb{R}^{n-1}$ the operator*

$$\begin{array}{rccc} C^{\mathsf{pr}}_\Gamma(\boldsymbol{\xi}') := & \mathfrak{M}_+[L^{\mathsf{pr}}; \boldsymbol{\xi}'] & \longrightarrow & \mathbb{C}^N \\ & \mathbf{U} & \longmapsto & \left\{C^{\mathsf{pr}}_1(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}, \ldots, C^{\mathsf{pr}}_N(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}\right\}\big|_{t=0} \end{array} \tag{2.16}$$

is an isomorphism.

Remark 2.2.7 Let L be an elliptic $N\times N$ second order system with constant coefficients.

(i) In the book [53], the definition of the covering condition for a set C_Γ of N boundary operators is that $C^{\mathsf{pr}}_\Gamma(\boldsymbol{\xi}')$ is injective for all non-zero $\boldsymbol{\xi}'$. This apparently weaker condition implies that L is properly elliptic when $n=2$ (see [53, Lemma 2.2.3]), and is equivalent to ours.

(ii) If instead of requiring in the very definition of C_Γ that $N_0+N_1=N$, we leave the number N_0+N_1 of boundary conditions as a parameter, and define a relaxed covering condition with the image space $\mathbb{C}^{N_0+N_1}$ instead of $\mathbb{C}^N$ in (2.16), then the condition $N_0+N_1=N$ becomes a consequence of this relaxed covering condition: The relaxed covering condition implies that

$$N_0+N_1=\dim\mathfrak{M}_+[L^{\mathsf{pr}};\boldsymbol{\xi}']=\dim\mathfrak{M}_+[L^{\mathsf{pr}};-\boldsymbol{\xi}'].$$

But since L^{pr} is homogeneous of degree 2, $\mathfrak{M}_+[L^{\mathsf{pr}};-\boldsymbol{\xi}']$ is isomorphic to the space $\mathfrak{M}_-[L^{\mathsf{pr}};\boldsymbol{\xi}']$ of growing solutions. As a consequence of the ellipticity of L,

$$\dim\mathfrak{M}_+[L^{\mathsf{pr}};\boldsymbol{\xi}']+\dim\mathfrak{M}_-[L^{\mathsf{pr}};\boldsymbol{\xi}']=2N\,,$$

and $N_0+N_1=N$ follows, as well as the proper ellipticity of L. △

Particular Case 2.2.8 The Dirichlet conditions $D_\Gamma=(D_1,\dots,D_N)$ with independent D_k cover many elliptic operators, but not all of them:

(i) If the system L is *strongly elliptic* (see Definition 3.2.2 in Chapter 3), then it is always covered by its N Dirichlet conditions.

(ii) If L is elliptic only, without being strongly elliptic, it may happen that it is *not covered* by the N Dirichlet conditions: An example is given by the system $L=\mathbf{curl}\,\mathbf{curl}+\nabla\operatorname{div}$ in dimension $n=3$, and, more generally by the Lamé system (cf. Example 1.2.8 *(i)*) with $\lambda=-3\mu$, [63]. △

Particular Case 2.2.9 A situation where the covering condition can be given more explicitly, is when L is properly elliptic and such that for all $\boldsymbol{\xi}'\neq0$ in $\mathbb{R}^{n-1}$, the equation $\det L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau)=0$ has N distinct roots with positive imaginary part, $\tau_\ell(\boldsymbol{\xi}')$, $\ell=1,\dots,N$. Let $\boldsymbol{\eta}_\ell(\boldsymbol{\xi}')\in\mathbb{C}^N$ be non-zero solutions of the equation

$$L^{\mathsf{pr}}(\boldsymbol{\xi}',\tau_\ell(\boldsymbol{\xi}'))\,\boldsymbol{\eta}_\ell(\boldsymbol{\xi}')=0.$$

The space $\mathfrak{M}_+[L^{\mathsf{pr}};\boldsymbol{\xi}']$ of exponentially decreasing solutions has the basis $\mathrm{e}^{i\tau_\ell(\boldsymbol{\xi}')t}\boldsymbol{\eta}_\ell(\boldsymbol{\xi}')$, $\ell=1,\dots,N$. Since

$$C^{\mathsf{pr}}_k(\boldsymbol{\xi}',\mathrm{D}_t)\big(\mathrm{e}^{i\tau_\ell t}\boldsymbol{\eta}_\ell\big)\big|_{t=0}=C^{\mathsf{pr}}_k(\boldsymbol{\xi}',\tau_\ell)\,\boldsymbol{\eta}_\ell$$

we see that the operator (2.16) is an isomorphism if and only if

$$\det\Big(C^{\mathsf{pr}}_k(\boldsymbol{\xi}',\tau_\ell(\boldsymbol{\xi}'))\,\boldsymbol{\eta}_\ell(\boldsymbol{\xi}')\Big)_{1\le k,\ell\le N}\neq0.$$

For the general case of roots with higher multiplicity, there exist in the literature many conditions in algebraic form which express the covering condition given in Def. 2.2.6 which is also known as *Shapiro-Lopatinski condition* [76, 94]. △

Combined with the interior ellipticity condition ($L^{\mathsf{pr}}(\boldsymbol{\xi})$ invertible for all $\boldsymbol{\xi} \in \mathbb{S}^{n-1}$), the covering condition is the counterpart for boundary value problems of the interior ellipticity condition alone for problems without boundary: We have the analogue of Proposition 1.2.2 for boundary value problems with constant coefficients in the periodic half-space,

$$\begin{cases} L\mathbf{u} &= \mathbf{f} & \text{in } \mathbb{T}^n_+ \\ T_k\mathbf{u} &= g_k & \text{on } \Gamma, \quad k = 1, \ldots, N_1 \\ D_k\mathbf{u} &= h_k & \text{on } \Gamma, \quad k = 1, \ldots, N_0. \end{cases} \tag{2.17}$$

Proposition 2.2.10 *Let L be a $N \times N$ system of order 2 with constant coefficients on $\mathbb{T}^n_+$, let $C_\Gamma = \{T_1, \ldots, T_{N_1}, D_1, \ldots, D_{N_0}\}$ be a set of $N_0 + N_1 = N$ boundary operators with constant coefficients on $\Gamma = \partial\mathbb{T}^n_+$, and let L^{pr}, C^{pr}_Γ denote their principal parts. Let us denote by $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$ the target space corresponding to $\mathbf{H}^2(\mathbb{T}^n_+)$, i.e.*

$$\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+) = \mathsf{L}^2(\mathbb{T}^n_+)^N \times \mathsf{H}^{\frac{1}{2}}(\Gamma)^{N_1} \times \mathsf{H}^{\frac{3}{2}}(\Gamma)^{N_0}. \tag{2.18}$$

(i) Let us assume

$$\begin{cases} \text{(a)} & \begin{cases} \forall \boldsymbol{\xi} \in \mathbb{S}^{n-1}, & L^{\mathsf{pr}}(\boldsymbol{\xi}) \text{ is invertible,} \\ \forall \boldsymbol{\xi}' \in \mathbb{S}^{n-2}, & C^{\mathsf{pr}}_\Gamma(\boldsymbol{\xi}') \text{ is an isomorphism} \end{cases} \\ \\ \text{(b)} & \begin{cases} \forall \mathbf{p}' \in \mathbb{Z}^{n-1}, \ \forall \tau \in \mathbb{R}, & L(\mathbf{p}', \tau) \text{ is invertible,} \\ \forall \mathbf{p}' \in \mathbb{Z}^{n-1}, & C_\Gamma(\mathbf{p}') \text{ is an isomorphism.} \end{cases} \end{cases} \tag{2.19}$$

Here $C^{\mathsf{pr}}_\Gamma(\boldsymbol{\xi}')$ is defined in (2.16) and $C_\Gamma(\mathbf{p}')$ is defined similarly, that is:

$$C_\Gamma(\mathbf{p}') : \begin{array}{ccc} \mathfrak{M}_+[L; \mathbf{p}'] & \longrightarrow & \mathbb{C}^N \\ \mathbf{U} & \longmapsto & \{C_1(\mathbf{p}', \mathrm{D}_t)\mathbf{U}, \ldots, C_N(\mathbf{p}', \mathrm{D}_t)\mathbf{U}\}\big|_{t=0}. \end{array}$$

Then for all $(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$, there exists a unique $\mathbf{u} \in \mathbf{H}^2(\mathbb{T}^n_+)$ solution of the model boundary value problem (2.17). In other words, the operator $\mathbb{A} := \{L, C_\Gamma\}$ defines an isomorphism from $\mathbf{H}^2(\mathbb{T}^n_+)$ onto $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$.

(ii) Conversely, if the problem (2.17) is uniquely solvable in $\mathbf{H}^2(\mathbb{T}^n_+)$ for all $(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$, then (2.19) holds.

Remark 2.2.11 This nice characterization of isomorphic boundary value problems with constant coefficients on the periodic half-space comes from [53, Th.2.2.1]. Its principle is similar to that of Proposition 1.2.2 for the case without boundary: Condition (a) involves principal symbols on the unit spheres $\mathbb{S}^{n-1}$ and $\mathbb{S}^{n-2}$, and is the very definition of the ellipticity with covering conditions, whereas condition (b) reproduces similar conditions for the whole symbols, required on the discrete lattice $\mathbb{Z}^{n-1}$. △

Proof: *(i)* **Step 1.** We assume that (2.19) (b) holds. Let $\mathbf{p}' \in \mathbb{Z}^{n-1}$ be fixed. Let us prove that the boundary value problem on $\mathbb{R}^+$

$$(1) \qquad \begin{cases} L(\mathbf{p}', \mathrm{D}_t)\mathbf{U} &= \mathbf{F} & \text{in } \mathbb{R}_+ \\ T_k(\mathbf{p}', \mathrm{D}_t)\mathbf{U} &= G_k & \text{in } t=0, \quad k=1,\ldots,N_1 \\ D_k(\mathbf{p}', \mathrm{D}_t)\mathbf{U} &= H_k & \text{in } t=0, \quad k=1,\ldots,N_0 \end{cases}$$

induces an isomorphism $\mathbf{U} \mapsto (\mathbf{F}, \mathbf{G}, \mathbf{H})$ from $\mathbf{H}^2(\mathbb{R}_+)$ onto $\mathbf{L}^2(\mathbb{R}_+) \times \mathbb{C}^N$.

Let $\mathbf{U} \in \mathbf{H}^2(\mathbb{R}_+)$ associated with the right hand side $(0,0,0)$. Thus $\mathbf{U}$ belongs to $\mathfrak{M}_+[L; \mathbf{p}']$ and $C_\Gamma(\mathbf{p}')\mathbf{U} = 0$. The condition that $C_\Gamma(\mathbf{p}')$ is an isomorphism yields that $\mathbf{U} = 0$.

Let $\mathbf{F} \in \mathbf{L}^2(\mathbb{R}_+)$ and let $\mathbf{F}_0$ be its extension by 0 on $\mathbb{R}_-$. Since by assumption $L(\mathbf{p}', \tau)$ is invertible for all $\tau \in \mathbb{R}$, we can define $\mathbf{U}_0$ by the Fourier formula

$$\mathscr{F}\mathbf{U}_0(\tau) = L(\mathbf{p}', \tau)^{-1}\mathscr{F}\mathbf{F}_0(\tau).$$

Condition (2.19) (b) gives that the principal part $L^{\mathsf{pr}}(\mathbf{p}', \tau)$ is also invertible for any $\tau \neq 0$, and we deduce as in the proof of Proposition 1.2.2 that $\|L(\mathbf{p}', \tau)^{-1}\|_{\mathcal{L}_n}$ is bounded by $C\langle\tau\rangle^{-2}$ for a positive constant C. Therefore we obtain that $\mathbf{U}_0$ belongs to $\mathbf{H}^2(\mathbb{R})$.

Subtracting $\mathbf{U}_0$ from $\mathbf{U}$ in problem (1), we are left to solve problem (1) with $\mathbf{F} = 0$. The latter problem has a solution in $\mathfrak{M}_+[L; \mathbf{p}']$ by the assumption that $C_\Gamma(\mathbf{p}')$ is an isomorphism.

Step 2. We assume that (2.19) (a) holds. The same proof yields that for all $\boldsymbol{\xi}' \in \mathbf{S}^{n-2}$ the boundary value problem on $\mathbb{R}^+$

$$(2) \qquad \begin{cases} L^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} &= \mathbf{F} & \text{in } \mathbb{R}_+ \\ T_k^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} &= G_k & \text{in } t=0, \quad k=1,\ldots,N_1 \\ D_k^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U} &= H_k & \text{in } t=0, \quad k=1,\ldots,N_0. \end{cases}$$

induces an isomorphism $\mathbf{U} \mapsto (\mathbf{F}, \mathbf{G}, \mathbf{H})$ from $\mathbf{H}^2(\mathbb{R}_+)$ onto $\mathbf{L}^2(\mathbb{R}_+) \times \mathbb{C}^N$. Since all operators depend continuously on $\boldsymbol{\xi}'$, there exists a constant $C > 0$ such that for all $\boldsymbol{\xi}' \in \mathbb{S}^{n-2}$, for all $\mathbf{U} \in \mathbf{H}^2(\mathbb{R}_+)$, and $\mathbf{F}$, G_k and H_k given by (2):

$$(3) \qquad \|\mathbf{U}\|_{\mathbf{H}^2(\mathbb{R}_+)} \le C\Big(\|\mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}_+)} + \sum_{k=1}^{N_1}|G_k| + \sum_{k=1}^{N_0}|H_k|\Big).$$

Let us denote by $A^{\mathsf{pr}}(\boldsymbol{\xi}')$ the operator of problem (2). We use now an homogeneity argument to deduce from (3) uniform estimates for all $\boldsymbol{\xi}' \in \mathbb{R}^{n-1} \setminus \{0\}$. Let $\rho > 0$ and recall that $\mathfrak{H}_\rho$ is the transformation $\mathbf{U}(t) \mapsto \mathbf{U}(\frac{t}{\rho})$. For a right-hand side $(\mathbf{F}, \mathbf{G}, \mathbf{H})$, we define the suitable scaling operator $\mathfrak{G}_\rho$ by

$$(4) \qquad \mathfrak{G}_\rho(\mathbf{F}, \mathbf{G}, \mathbf{H}) = (\rho^{-2}\mathfrak{H}_\rho\mathbf{F},\ \rho^{-1}\mathbf{G},\ \mathbf{H}).$$

Then there holds for all $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$ and $\rho > 0$

(5) $$A^{\mathsf{pr}}(\boldsymbol{\xi}') = \mathfrak{G}_\rho^{-1} \circ A^{\mathsf{pr}}\Big(\frac{\boldsymbol{\xi}'}{\rho}\Big) \circ \mathfrak{H}_\rho.$$

Let $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$, $\boldsymbol{\xi}' \neq 0$ and set $\rho = |\boldsymbol{\xi}'|$. Let $(\mathbf{F}, \mathbf{G}, \mathbf{H}) = A^{\mathsf{pr}}(\boldsymbol{\xi}')\mathbf{U}$. We use (3) with $\frac{\boldsymbol{\xi}'}{\rho} \in \mathbb{S}^{n-2}$ together with relation (5) and find

(6) $$\|\mathfrak{H}_\rho \mathbf{U}\|_{\mathbf{H}^2(\mathbb{R}_+)} \le C\Big(\|\rho^{-2}\mathfrak{H}_\rho \mathbf{F}\|_{\mathbf{L}^2(\mathbb{R}_+)} + \sum_{k=1}^{N_1} \rho^{-1}|G_k| + \sum_{k=1}^{N_0} |H_k|\Big).$$

With the help of Lemma 2.1.1, we deduce from (6) after multiplying by $\rho^{3/2}$

(7) $$|\mathbf{U}|_{2;\mathbb{R}_+} + \rho|\mathbf{U}|_{1;\mathbb{R}_+} + \rho^2\|\mathbf{U}\|_{0;\mathbb{R}_+} \le C\Big(\|\mathbf{F}\|_{0;\mathbb{R}_+} + \sum_{k=1}^{N_1} \rho^{\frac12}|G_k| + \sum_{k=1}^{N_0} \rho^{\frac32}|H_k|\Big).$$

Let $\mathbf{p}' \in \mathbb{Z}^{n-1}$ and $\rho = |\mathbf{p}'|$. Let $A(\mathbf{p}')$ denote the operator of problem (1). We set $(\mathbf{F}', \mathbf{G}', \mathbf{H}') = A(\mathbf{p}')\mathbf{U} - A^{\mathsf{pr}}(\mathbf{p}')\mathbf{U}$. We find that $\mathbf{H}' = 0$ and the estimate for $\mathbf{F}'$, $\mathbf{G}'$

(8) $$\|\mathbf{F}'\|_{0;\mathbb{R}_+} + \sum_{k=1}^{N_1} \rho^{\frac12}|G'_k| \le C\big(\|\mathbf{U}\|_{1;\mathbb{R}_+} + \rho\|\mathbf{U}\|_{0;\mathbb{R}_+}\big).$$

We deduce from (7) and (8) that for $|\mathbf{p}'|$ large enough, the a priori estimate (7) also holds for $(\mathbf{F}, \mathbf{G}, \mathbf{H}) = A(\mathbf{p}')\mathbf{U}$ (with C replaced by $2C$). Since $A(\mathbf{p}')$ is invertible for all $\mathbf{p}'$ (step 1) and since there are uniform estimates in any bounded region in $\mathbf{p}'$, we finally find that there exists a constant $C_A > 0$ such that for all $\mathbf{p}' \in \mathbb{Z}^{n-1}$, for all $\mathbf{U} \in \mathbf{H}^2(\mathbb{R}_+)$

$$\|\mathbf{U}\|_{2;\mathbb{R}_+} + |\mathbf{p}'|^2\|\mathbf{U}\|_{0;\mathbb{R}_+} \le C_A\Big(\|L(\mathbf{p}', \mathrm{D}_t)\mathbf{U}\|_{0;\mathbb{R}_+} + (1 + |\mathbf{p}'|^{\frac12})|T_\Gamma(\mathbf{p}', \mathrm{D}_t)\mathbf{U}| + (1 + |\mathbf{p}'|^{\frac32})|D_\Gamma \mathbf{U}|\Big),$$

with the obvious notation $T_\Gamma = \{T_1, \dots, T_{N_1}\}$ and $D_\Gamma = \{D_1, \dots, D_{N-N_1}\}$.

Step 3. Combining the last estimate with the characterizations (1.7) and (2.5) of the Sobolev spaces on $\mathbb{T}^{n-1}$ and $\mathbb{T}^n_+$, we find the estimate

$$\|\mathbf{u}\|_{2;\mathbb{T}^n_+} \le C_A\Big(\|L\mathbf{u}\|_{0;\mathbb{T}^n_+} + \|T_\Gamma \mathbf{u}\|_{\frac12;\mathbb{T}^{n-1}} + \|D_\Gamma \mathbf{u}\|_{\frac32;\mathbb{T}^{n-1}}\Big),$$

together with the existence of a solution.

(ii) See the proof of Theorem 2.2.1 of [53]. □

Definition 2.2.12 *The model boundary value problem*

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \mathbb{T}^n_+ \\ C_\Gamma \mathbf{u} = \mathbf{b} & \text{on } \Gamma \end{cases} \tag{2.20}$$

is called ***elliptic*** *if L is elliptic and covered by C_Γ, see Definition 2.2.6.*

We see that the ellipticity is simply the condition (2.19) (a) on the principal part $(L^{\mathsf{pr}}, C_\Gamma^{\mathsf{pr}})$. Like in the situation without boundary (Corollary 1.2.3) this condition implies the existence of an invertible model problem with the same principal part:

Corollary 2.2.13 *Let L be a $N \times N$ system of order 2 with constant coefficients on $\mathbb{T}^n_+$, and $C_\Gamma = (T_\Gamma, D_\Gamma)$ a set of N boundary conditions on $\Gamma = \partial\mathbb{T}^n_+$. We assume that the boundary value problem (2.20) is elliptic. Then there exists a $N \times N$ system $\tilde{L}$ of order 2 with constant coefficients and a set $\tilde{C}_\Gamma$ of N boundary conditions with the same principal part as L and C_Γ, respectively, and such that*

$$\tilde{\mathbb{A}} := \{\tilde{L}, \tilde{C}_\Gamma\} \text{ is an isomorphism from } \mathbf{H}^2(\mathbb{T}^n_+) \text{ onto } \mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+).$$

Proof: We set, compare with [53, Rem. 2.2.3]:

$$\tilde{L}(\mathrm{D}_{\mathbf{x}}) = L^{\mathsf{pr}}\big(\mathrm{D}_{\mathbf{x}'} + \vec{\tfrac{1}{2}}, \mathrm{D}_t\big) \quad \text{and} \quad \tilde{C}_\Gamma(\mathrm{D}_{\mathbf{x}'}, \mathrm{D}_t) = C_\Gamma^{\mathsf{pr}}\big(\mathrm{D}_{\mathbf{x}'} + \vec{\tfrac{1}{2}}, \mathrm{D}_t\big),$$

where, now, $\vec{\frac{1}{2}}$ is the $(n-1)$-tuple $(\frac{1}{2}, \ldots, \frac{1}{2})$.

Then $\tilde{L}(\mathbf{p}', \tau) = L^{\mathsf{pr}}(\mathbf{p}' + \vec{\frac{1}{2}}, \tau)$ and $\tilde{C}_\Gamma(\mathbf{p}', \mathrm{D}_t) = C^{\mathsf{pr}}(\mathbf{p}' + \vec{\frac{1}{2}}, \mathrm{D}_t)$.

For all $\mathbf{p}' \in \mathbb{Z}^{n-1}$, $\mathbf{p}' + \vec{\frac{1}{2}}$ is not zero. Therefore condition (2.19) (a) for $\mathbb{A} = (L, C_\Gamma)$ yields condition (2.19) (b) for $\tilde{\mathbb{A}} = \{\tilde{L}, \tilde{C}_\Gamma\}$. Proposition 2.2.10 gives the invertibility of $\tilde{\mathbb{A}}$. ☐

Note that, unlike in the situation without boundary, the original elliptic system (L, C_Γ) will *not* be a Fredholm operator from $\mathbf{H}^2(\mathbb{T}^n_+)$ to $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$, in general. The reason is that $\mathbb{T}^n_+$ is unbounded and $\mathbf{H}^1(\mathbb{T}^n_+)$ is not compactly embedded in $\mathbf{L}^2(\mathbb{T}^n_+)$.

2.2.b Local a priori estimates for problems with smooth coefficients

Basically, an *elliptic boundary system* on a domain with smooth boundary is defined by the fact that in each point, after choosing suitable local coordinates, freezing coefficients, and taking the principal part, it looks like the periodic half-space case considered in the preceding section.

In this subsection, we study local systems as obtained in "suitable coordinates", that is, we consider domains, systems and boundary operators as follows:

(i) $\breve{\Omega}$ is a bounded domain in $\mathbb{R}^n_+$, $\breve{\Gamma}$ is a bounded domain in the hyperplane $\{x_n = 0\}$. We assume that $\breve{\Gamma} \subset \partial\breve{\Omega}$, see Figure 2.1.

(ii) $L = (L_{ij})$ is a second order $N \times N$ system with smooth coefficients on $\breve{\Omega} \cup \breve{\Gamma}$.

(iii) $C_{\breve{\Gamma}} = (T_{\breve{\Gamma}}, D_{\breve{\Gamma}})$ is a set of N boundary operators with smooth coefficients on $\breve{\Gamma}$.

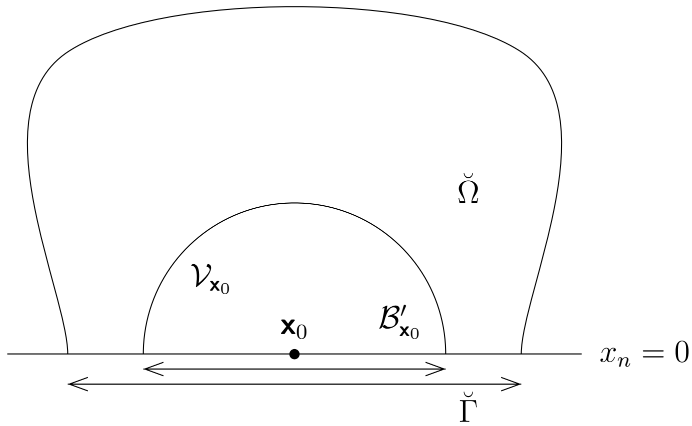


Figure 2.1: Local configurations $(\breve\Omega, \breve\Gamma)$ and $(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$.

The "local" boundary value problem then writes,

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \breve\Omega \\ C_{\breve\Gamma}\mathbf{u} = \mathbf{b} & \text{on } \breve\Gamma. \end{cases} \tag{2.21}$$

Notation 2.2.14 Let $\mathcal{B}$ be a ball centered on the hyperplane $x_n = 0$, let $\mathcal{V} = \mathcal{B} \cap \{x_n > 0\}$ and $\mathcal{B}' = \partial\mathcal{V} \cap \{x_n = 0\}$. Let $k \geq 0$ be an integer.

- ⋆ We set

$$\mathsf{H}^k_0(\mathcal{V}, \mathcal{B}') = \{u \in \mathsf{H}^k(\mathcal{V}), \quad \partial^\ell_r u = 0 \text{ on } \partial\mathcal{V} \setminus \mathcal{B}', \ 0 \leq \ell \leq k-1\}.$$

  Here ∂_r is the radial derivative on the spherical part $\partial\mathcal{V} \setminus \mathcal{B}'$ of $\partial\mathcal{V}$ – thus, it coincides with the normal derivative.

- ⋆ For the natural spaces containing the right hand sides in our boundary value operator $\mathbb{A} = \{L, C_{\breve\Gamma}\}$, we introduce a shorthand notation:

$$\mathsf{R}\mathbf{H}^{k+2}(\mathcal{V}, \mathcal{B}') = \mathsf{H}^k(\mathcal{V})^N \times \mathsf{H}^{k+\frac{1}{2}}(\mathcal{B}')^{N_1} \times \mathsf{H}^{k+\frac{3}{2}}(\mathcal{B}')^{N_0}.$$

  One can think of the symbol R as a functor depending on the differential operators L and $C_{\breve\Gamma}$ via their orders, cf. also (2.18). △

On the model of Proposition 1.2.9, we have the existence of local parametrices near boundary points where ellipticity and covering conditions are satisfied:

Proposition 2.2.15 *Let $\breve\Omega$ and $\breve\Gamma \subset \partial\breve\Omega$ be as above. Let $\mathbf{x}_0 \in \breve\Gamma$. We assume*

a) *L is a second order $N \times N$ system with $\mathscr{C}^0$ coefficients on $\breve\Omega \cup \breve\Gamma$,*

b) *$C_{\breve\Gamma} = (T_{\breve\Gamma}, D_{\breve\Gamma})$ is a set of N boundary operators, where $T_{\breve\Gamma}$ and $D_{\breve\Gamma}$ have $\mathscr{C}^1$ and $\mathscr{C}^2$ coefficients on $\breve\Gamma$ respectively,*

c) *The model boundary value problem*

$$\left\{\begin{array}{rcll} L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})\mathbf{u} &=& \mathbf{f} & \textit{in } \mathbb{T}^n_+ \\ C^{\mathsf{pr}}_{\breve\Gamma}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})\mathbf{u} &=& \mathbf{b} & \textit{on } \Gamma, \end{array}\right.$$

obtained by freezing the coefficients of L and $C_{\breve\Gamma}$ at $\mathbf{x}_0$ and taking the principal parts, is elliptic, cf. Definition 2.2.12.

Then there exist, cf. Figure 2.1,

- ⋆ *a ball $\mathcal{B}_{\mathbf{x}_0}$ of center $\mathbf{x}_0$ defining $\mathcal{V}_{\mathbf{x}_0} = \mathcal{B}_{\mathbf{x}_0} \cap \{x_n > 0\}$ and $\mathcal{B}'_{\mathbf{x}_0} = \partial\mathcal{V}_{\mathbf{x}_0} \cap \{x_n = 0\}$*
- ⋆ *an operator $\mathfrak{E}_{\mathbf{x}_0}$, continuous from $\mathsf{R}\mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$ into $\mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0})$,*

such that for any $\psi', \psi'' \in \mathscr{C}^\infty_0(\mathcal{B}_{\mathbf{x}_0})$ with $\psi''\psi' = \psi'$:

$$\{L, C_{\breve\Gamma}\}\psi''\mathfrak{E}_{\mathbf{x}_0}\,\psi'\,\mathbf{q} = \psi'\mathbf{q} + K\mathbf{q}, \quad \forall \mathbf{q} = (\mathbf{f},\mathbf{g},\mathbf{h}) \in \mathsf{R}\mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0}) \tag{2.22a}$$

$$\psi''\mathfrak{E}_{\mathbf{x}_0}\psi'\{L, C_{\breve\Gamma}\}\,\mathbf{u} = \psi'\mathbf{u} + K'\mathbf{u}, \quad \forall \mathbf{u} \in \mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}) \tag{2.22b}$$

where K and K' are continuous operators

$$K:\ \mathsf{R}\mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0}) \to \mathsf{R}\mathbf{H}^3(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0}) \quad \textit{and} \quad K':\ \mathbf{H}^1(\mathcal{V}_{\mathbf{x}_0}) \to \mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}),$$

and moreover K' is compact from $\mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0})$ into itself.

Proof: Let $\mathbb{A}^{\mathsf{pr}}$ denote the principal part of the boundary value system $\mathbb{A} = \{L, C_{\breve\Gamma}\}$, and

$$\underline{\mathbb{A}}_{\mathbf{x}_0} = \{L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}}), C^{\mathsf{pr}}_{\breve\Gamma}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})\}$$

be this same principal part with coefficients frozen at the point $\mathbf{x}_0$. Since, by assumption, condition (2.19) (a) is satisfied for the boundary value system $\underline{\mathbb{A}}_{\mathbf{x}_0}$, we can apply Corollary 2.2.13: We have an isomorphism $\tilde{\mathbb{A}} = (\tilde L, \tilde C)$ from $\mathbf{H}^2(\mathbb{T}^n_+)$ onto $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$ on the periodic half-space, such that $\tilde{\mathbb{A}}^{\mathsf{pr}} = \underline{\mathbb{A}}_{\mathbf{x}_0}$. Now we use a perturbation argument to link $\tilde{\mathbb{A}}$ to $\mathbb{A}$.

For more versatility, we modify the argument used in the proof of Proposition 1.2.9. We still define the smooth cut-off function ψ on the same way: $\psi \equiv 0$ outside the ball $B_1(\mathbf{0})$ and $\psi \equiv 1$ on $B_{1/2}(\mathbf{0})$. For $R > 0$, we set

(1) $$\mathfrak{A}_R(\mathbf{y};\mathrm{D}_{\mathbf{y}}) := \tilde{\mathbb{A}}(\mathrm{D}_{\mathbf{y}}) + \psi(\mathbf{y})\big(\mathbb{A}^{\mathsf{pr}}(\mathbf{x}_0 + R\mathbf{y};\mathrm{D}_{\mathbf{y}}) - \underline{\mathbb{A}}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{y}})\big), \quad \mathbf{y} \in \mathbb{T}^n_+.$$

We check that the regularity assumptions on the coefficients of L, $T_{\breve\Gamma}$ and $D_{\breve\Gamma}$ imply that the norm of $\psi(\mathbf{y})\big(\mathbb{A}^{\mathsf{pr}}(\mathbf{x}_0 + R\mathbf{y};\mathrm{D}_{\mathbf{y}}) - \underline{\mathbb{A}}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{y}})\big)$ as an operator from $\mathbf{H}^2(\mathbb{T}^n_+)$ into $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$, tends to 0 as $R \to 0$. Thus we can choose $R = R_0$ small enough so that $\mathfrak{A}_{R_0}(\mathbf{y};\mathrm{D}_{\mathbf{y}})$ is an isomorphism from $\mathbf{H}^2(\mathbb{T}^n_+)$ onto $\mathsf{R}\mathbf{H}^2(\mathbb{T}^n_+)$.

For this value of R_0, we consider the change of variables $\mathbf{y} \to \mathbf{x} = \mathbf{x}_0 + R_0\mathbf{y}$ and set

(2) $$(\mathfrak{H}\mathbf{u})(\mathbf{y}) = \mathbf{u}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{x}_0 + R_0\mathbb{T}^n_+, \quad \text{and} \quad \mathfrak{G}(\mathbf{f},\mathbf{g},\mathbf{h}) = (R_0^2\,\mathfrak{H}\mathbf{f}, R_0\,\mathfrak{H}\mathbf{g}, \mathfrak{H}\mathbf{h}).$$

Then the operator $\mathfrak{A}$ defined as

$$\mathfrak{A} = \mathfrak{G}^{-1} \circ \mathfrak{A}_{R_0} \circ \mathfrak{H} \tag{3}$$

is an isomorphism from $\mathbf{H}^2(\mathbf{x}_0 + R_0\mathbb{T}^n_+)$ onto $\mathrm{R}\mathbf{H}^2(\mathbf{x}_0 + R_0\mathbb{T}^n_+)$. Moreover, setting $\mathcal{B}_{\mathbf{x}_0} = B_{R_0/2}(\mathbf{x}_0)$, we find that, by construction, $\mathfrak{A}^{\mathsf{pr}}(\mathbf{x};\mathrm{D}_\mathbf{x}) = \mathbb{A}^{\mathsf{pr}}(\mathbf{x};\mathrm{D}_\mathbf{x})$ for all $\mathbf{x} \in \mathcal{B}_{\mathbf{x}_0}$. We define $\mathfrak{E}_{\mathbf{x}_0}$ like in the proof of Proposition 1.2.9:

For $\quad \mathbf{q} \in \mathrm{R}\mathbf{H}^2_0(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0}) := \mathbf{L}^2(\mathcal{V}_{\mathbf{x}_0}) \times \widetilde{\mathbf{H}}^{\frac12}(\mathcal{B}'_{\mathbf{x}_0}) \times \widetilde{\mathbf{H}}^{\frac32}(\mathcal{B}'_{\mathbf{x}_0})\quad$ we set

$$\mathfrak{E}_{\mathbf{x}_0}\mathbf{q} = \big(\mathfrak{A}^{-1}\mathfrak{P}_0\mathbf{q}\big)\big|_{\mathcal{B}_{\mathbf{x}_0}} \in \mathbf{H}^2(\mathcal{V}_{\mathbf{x}_0}), \tag{4}$$

with the extension operator $\mathfrak{P}_0$, continuous from $\mathrm{R}\mathbf{H}^2_0(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$ into $\mathrm{R}\mathbf{H}^2(\mathbf{x}_0 + R_0\mathbb{T}^n_+)$. The rest of the proof is a consequence of formulas

$$K\mathbf{q} = \psi''(\mathbb{A} - \mathfrak{A})\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{q} + [\mathbb{A}, \psi'']\mathfrak{E}_{\mathbf{x}_0}\psi'\mathbf{q} \tag{5}$$

and

$$K'\mathbf{u} = \psi''\mathfrak{E}_{\mathbf{x}_0}(\mathbb{A} - \mathfrak{A})\psi'\mathbf{u} + \psi''\mathfrak{E}_{\mathbf{x}_0}[\psi', \mathbb{A}]\mathbf{u}. \tag{6}$$

□

Like in Corollary 1.2.10 for interior elliptic estimates, the existence of parametrices allows to prove local a priori estimates up to the boundary.

Corollary 2.2.16 *Let $\breve\Omega$, $\breve\Gamma$ and $\mathbf{x}_0 \in \breve\Gamma$ be as in Proposition 2.2.15. Let the system L and the boundary operators $C_{\breve\Gamma} = \{T_1, \dots, T_{N_1}, D_1, \dots, D_{N_0}\}$ satisfy the assumptions of Proposition 2.2.15. Then there exists a ball $\mathcal{B}^*_{\mathbf{x}_0}$ centered at $\mathbf{x}_0$, defining the half-ball $\mathcal{V}^*_{\mathbf{x}_0}$ and the $n-1$ ball $\mathcal{B}'^*_{\mathbf{x}_0}$ as in Notation 2.2.14, and a constant $A_0 > 0$ such that for all $\mathbf{u} \in \mathbf{H}^2_0(\mathcal{V}^*_{\mathbf{x}_0}, \mathcal{B}'^*_{\mathbf{x}_0})$ there holds*

$$\|\mathbf{u}\|_{2;\mathcal{V}^*_{\mathbf{x}_0}} \le A_0\Big(\|L\mathbf{u}\|_{0;\mathcal{V}^*_{\mathbf{x}_0}} + \sum_{j=1}^{N_1}\|T_j\mathbf{u}\|_{\frac12;\mathcal{B}'^*_{\mathbf{x}_0}} + \sum_{j=1}^{N_0}\|D_j\mathbf{u}\|_{\frac32;\mathcal{B}'^*_{\mathbf{x}_0}} + \|\mathbf{u}\|_{1;\mathcal{V}^*_{\mathbf{x}_0}}\Big). \tag{2.23}$$

2.2.c Elliptic boundary systems in smooth domains

Our boundary value problems will be written in the following condensed general form

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega\,, \\ T\mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega\,, \\ D\mathbf{u} &= \mathbf{h} \quad \text{on } \partial\Omega\,. \end{cases}$$

with a second order $N \times N$ system L and suitable boundary systems T and D of order 1 and 0, respectively. Until now we have defined the ellipticity property for model boundary value problems (2.20) on the periodic half-space. We have seen that a boundary value problem with constant coefficients on the periodic half-space enjoys invertibility

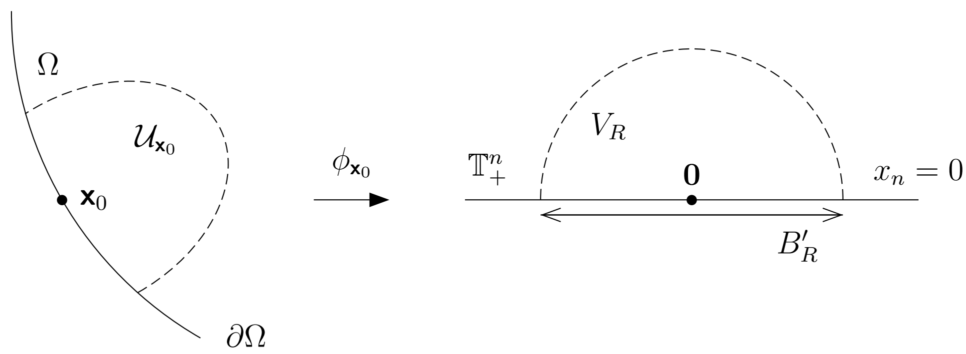


Figure 2.2: Local diffeomorphism $\phi_{\mathbf{x}_0} : \mathcal{U}_{\mathbf{x}_0} \to V_R \cup B'_R$.

properties if its principal part is elliptic, and that local boundary value problems (2.21) have a parametrix around a boundary point $\mathbf{x}_0 = (\mathbf{x}'_0, 0)$ if their principal part frozen at $\mathbf{x}_0$ is elliptic.

For the general case of a smooth domain, besides the ellipticity at each interior point, we "only" have to require the ellipticity property at each boundary point $\mathbf{x}_0$ for an associated boundary value problem

$$\begin{cases} \underline{L}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{f} \quad \text{in } \mathbb{T}^n_+\,, \\ \underline{T}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma\,, \\ \underline{D}_{\mathbf{x}_0}\mathbf{u} &= \mathbf{h} \quad \text{on } \Gamma\,. \end{cases}$$

where the system $\underline{L}_{\mathbf{x}_0}$ and the boundary operators $(\underline{T}_{\mathbf{x}_0}, \underline{D}_{\mathbf{x}_0})$ are obtained by applying a change of variables, freezing the coefficients and taking the principal part. In our case, "suitable coordinates" have to fulfill two requirements: They map a neighborhood of the given boundary point to a neighborhood of the origin of the half-space, and they allow to trivialize the system of boundary conditions so that they take the form considered in the previous subsections, as introduced in Definition 2.2.4.

While the first task is classical and is performed by the use of a diffeomorphism $\phi_{\mathbf{x}_0}$, the second one will be done by a change of basis in the boundary data, in relation with the new, global, definition which we adopt for sets of boundary operators on a smooth boundary.

Let Ω be a domain in $\mathbb{R}^n$. We specify some classical notions and notations. This will also serve as a model for singular domains such as corner domains, edge domains, etc...

Definition & Notation 2.2.17 *Let Ω be domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. Let $\mathbf{x}_0 \in \partial\Omega$. The boundary of Ω is called smooth in $\mathbf{x}_0$ if there exists*

- ⋆ *a neighborhood $\mathcal{U}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ in $\overline{\Omega}$,*
- ⋆ *a number $R > 0$ defining the ball B_R centered at $\mathbf{0}$ with radius R, the half-ball $V_R = B_R \cap \{x_n > 0\}$ and $B'_R = \partial V_R \cap \{x_n = 0\}$,*
- ⋆ *a smooth local diffeomorphism $\phi_{\mathbf{x}_0}$ which sends bijectively $\mathcal{U}_{\mathbf{x}_0}$ onto $V_R \cup B'_R$ and $\mathcal{U}_{\mathbf{x}_0} \cap \partial\Omega$ onto B'_R, and so that $\phi_{\mathbf{x}_0}(\mathbf{x}_0) = \mathbf{0}$, see Figure 2.2.*

Remark 2.2.18 If we require that

$$(\nabla\phi_{\mathbf{x}_0})(\mathbf{x}_0) = \mathbb{I}_n, \tag{2.24}$$

we have to admit more general half-balls V_R of the form $V_R = B_R \cap \{\varphi(\mathbf{x}) > 0\}$ and with $B'_R = \partial V_R \cap \{\varphi(\mathbf{x}) = 0\}$ where φ is a suitable non-zero linear form. Then (2.24) is not a restriction. △

Definition 2.2.19 *Let Ω be an open set in $\mathbb{R}^n$ with boundary $\partial\Omega$.*

- ⋆ *If $\partial\Omega$ is smooth in all of its points $\mathbf{x}_0$, the domain Ω is called **smooth**. By smooth we usually understand $\mathscr{C}^\infty$, but one can also define the class of $\mathscr{C}^k$ domains for $k \geq 2$, by requiring that the local maps $\phi_{\mathbf{x}_0}$ are of regularity $\mathscr{C}^k$.*
- ⋆ *If for all $\mathbf{x}_0 \in \partial\Omega$, the local map $\phi_{\mathbf{x}_0}$ is analytic, Ω is called **analytic**.*
- ⋆ *The connected components of $\partial\Omega$ are called the **sides** of Ω and denoted by $\partial_{\mathbf{s}}\Omega$, with a finite set of indices $\mathbf{s} \in \mathscr{S}$.*

2.2.c (i) Admissible systems of boundary operators

Let L be a $N \times N$ second order system with smooth coefficients on $\overline{\Omega}$. Let $\mathbf{x}_0 \in \partial\Omega$. The principal part $L^{\mathsf{pr}}(\mathbf{x}_0; \mathrm{D}_{\mathbf{x}})$ frozen at $\mathbf{x}_0$ is well defined, cf. Definition 1.2.7. We assume that $\partial\Omega$ is smooth in $\mathbf{x}_0$. The diffeomorphism $\phi_{\mathbf{x}_0}$ transforms L into a similar system which we denote by $\breve{L}_{\mathbf{x}_0}(\breve{\mathbf{x}}; \mathrm{D}_{\breve{\mathbf{x}}})$ in the new coordinates $\breve{\mathbf{x}} = (\mathbf{x}', t)$. We denote by $\underline{L}_{\mathbf{x}_0}$ its principal part frozen at $\mathbf{0}$:

$$\underline{L}_{\mathbf{x}_0}(\mathrm{D}_{\breve{\mathbf{x}}}) = (\breve{L}_{\mathbf{x}_0})^{\mathsf{pr}}(\mathbf{0}; \mathrm{D}_{\breve{\mathbf{x}}}).$$

Note that we have

$$\underline{L}_{\mathbf{x}_0}(\mathrm{D}_{\breve{\mathbf{x}}}) = L^{\mathsf{pr}}(\mathbf{x}_0; J^\top \mathrm{D}_{\breve{\mathbf{x}}}), \tag{2.25}$$

with the Jacobian matrix $J = (\nabla_{\mathbf{x}}\phi_{\mathbf{x}_0})(\mathbf{x}_0)$. Since J is invertible, it is clear that the system L is elliptic at $\mathbf{x}_0$ if and only if $\breve{L}_{\mathbf{x}_0}$ is elliptic at $\mathbf{0}$.

We have now to associate with each point $\mathbf{x}_0$ of the boundary a set $C = (T, D)$ of N boundary operators such that its principal part $\underline{C}_{\mathbf{x}_0}$ frozen at $\mathbf{0}$ in the local coordinates $\breve{\mathbf{x}}$ will satisfy the covering condition with respect to $\underline{L}_{\mathbf{x}_0}$.

The most common use in the literature is to suppose that there exists a global expression of the boundary operators $C = (T, D)$ with a $\mathscr{C}^\infty$ dependency on $\mathbf{x} \in \partial\Omega$. However this representation cannot apply to the classical example of perfect conductor electric boundary conditions in the Maxwell system in $\mathbb{R}^3$ if the domain Ω is homeomorphic to a ball: In this case T represents one operator (the divergence) which is global, and D is the *tangential trace*, which cannot be globally described by two smooth vector fields on the surface of a sphere. Nevertheless, the tangential trace can be correctly described with the help of a function Π^{D} with values in the projection operators of $\mathbb{R}^3$: For this particular situation we can take $\Pi^{D}\mathbf{u} = \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$, with the outer unit normal field $\mathbf{n}$ on $\partial\Omega$. By contrast, the normal trace can be described by a scalar operator $D\mathbf{u} = \mathbf{u} \cdot \mathbf{n}$, but also, by a projection operator $\Pi^{D}\mathbf{u} = (\mathbf{u} \cdot \mathbf{n})\,\mathbf{n}$.

Likewise, we gain generality by the introduction of fields of projection operators to define the first order boundary operators T: Now, we have to compose a larger first order system $\hat{T}$ with a field of projection operators Π^{T}. An example is given by the normal derivative $\partial_{\mathbf{n}}$ for scalar problems ($N = 1$): We can set $Tu = \partial_{\mathbf{n}} u$, but we can also set $Tu = \Pi^{\mathsf{T}} \hat{T} u$ where $\hat{T} = \nabla$ and $\Pi^{\mathsf{T}} \mathbf{g} = (\mathbf{g} \cdot \mathbf{n})\,\mathbf{n}$. For smooth domains, the two representations are equivalent to each other, but as we will see later on, they induce distinct functional analyses in n-dimensional cones for $n \geq 3$.

Definition & Notation 2.2.20 *Let Ω be a smooth domain, with $\partial_{\mathbf{s}}\Omega$, $\mathbf{s} \in \mathscr{S}$, its sides. Let N be a positive integer. We call* ***admissible system of boundary operators*** *of size N and denote by $C = (T, D)$ the structure given, for each $\mathbf{s} \in \mathscr{S}$, by*

- ⋆ *two non-negative integers $N_1 = N_1(\mathbf{s}) \leq N$, and $N_0 = N - N_1$,*
- ⋆ *two integers $\hat{N}_1 = \hat{N}_1(\mathbf{s}) \geq N_1$ and $\hat{N}_0 = \hat{N}_0(\mathbf{s}) \geq N_0$,*
- ⋆ *two smooth functions $\Pi^{\mathsf{T}}_{\mathbf{s}}$ and $\Pi^{\mathsf{D}}_{\mathbf{s}}$ defined on $\partial_{\mathbf{s}}\Omega$ with values in the set of projection operators*

$$\begin{aligned} \Pi^{\mathsf{T}}_{\mathbf{s}} :\ \ & \partial_{\mathbf{s}}\Omega \ni \mathbf{x} \mapsto \Pi^{\mathsf{T}}_{\mathbf{s}}(\mathbf{x}) \in \mathcal{P}(\mathbb{C}^{\hat{N}_1}), \quad \text{with rank } N_1 \\ \Pi^{\mathsf{D}}_{\mathbf{s}} :\ \ & \partial_{\mathbf{s}}\Omega \ni \mathbf{x} \mapsto \Pi^{\mathsf{D}}_{\mathbf{s}}(\mathbf{x}) \in \mathcal{P}(\mathbb{C}^{\hat{N}_0}), \quad \text{with rank } N_0 \end{aligned} \tag{2.26}$$

  Here $\mathcal{P}(E)$ is the set of projection operators $E \to E$.
- ⋆ *two systems of partial differential operators, $\hat{T}_{\mathbf{s}}$ and $\hat{D}_{\mathbf{s}}$ depending smoothly on $\mathbf{x} \in \partial_{\mathbf{s}}\Omega$,*

$$\begin{aligned} &\hat{T}_{\mathbf{s}}(\mathbf{x}) \ \text{ of order } 1 \ \text{ and of size } \ \hat{N}_1 \times N, \quad \forall \mathbf{x} \in \partial_{\mathbf{s}}\Omega \\ &\hat{D}_{\mathbf{s}}(\mathbf{x}) \ \text{ of order } 0 \ \text{ and of size } \ \hat{N}_0 \times N, \quad \forall \mathbf{x} \in \partial_{\mathbf{s}}\Omega \end{aligned} \tag{2.27}$$

  which then define $C = \big(C_{\mathbf{s}}\big)_{\mathbf{s}\in\mathscr{S}}$ by

$$C_{\mathbf{s}} = (T_{\mathbf{s}}, D_{\mathbf{s}}), \quad \text{with} \quad T_{\mathbf{s}} = \Pi^{\mathsf{T}}_{\mathbf{s}}\, \hat{T}_{\mathbf{s}} \quad \text{and} \quad D_{\mathbf{s}} = \Pi^{\mathsf{D}}_{\mathbf{s}}\, \hat{D}_{\mathbf{s}}\,. \tag{2.28}$$

Remark 2.2.21 It is no restriction to assume that $\hat{N}_0 \leq N$, the number of components of $\mathbf{u}$, and $\hat{N}_1 \leq n\,N$, the number of all first order derivatives of the components of $\mathbf{u}$. △

There are two generic particular cases of the previous definitions: The constant coefficient or local cases on one hand, the boundary value problems in variational form on the other hand.

Particular Case 2.2.22 The constant coefficient case (cf. §2.2.a) and the local framework studied in §2.2.b enter trivially in the previous framework with

$$\hat{N}_1 = N_1,\ \hat{N}_0 = N_0, \quad \Pi^{\mathsf{T}} = \mathbb{I}_{N_1},\ \Pi^{\mathsf{D}} = \mathbb{I}_{N_0}, \quad \hat{T} = T,\ \hat{D} = D,$$

where T and D are the $N_1 \times N$ and $N_0 \times N$ systems identified with the sets $(T_1, \cdots, T_{N_1})$ and $(D_1, \cdots, D_{N_0})$, respectively. △

Particular Case 2.2.23 For a boundary value problem in variational form (associated with a sesquilinear form of order 1), there exists a canonical $N \times N$ system B of order 1 on the boundary (the conormal system). Admissible systems of boundary operators are given by

$$D = \Pi^{\mathsf{D}} \quad \text{and} \quad T = \Pi^{\mathsf{T}} B, \quad \text{with} \quad \Pi^{\mathsf{T}} = \mathbb{I}_N - \Pi^{\mathsf{D}}.$$

Here $\hat{N}_0 = \hat{N}_1 = N$, $\hat{D} = \mathbb{I}_N$ and $\hat{T} = B$. This is investigated with more details in the next chapter. △

Example 2.2.24 There are several reasonable choices for the pairs (projection operator, PDE system) determining the boundary operators (2.28).

(i) For the Neumann operator ∂_n associated with the Laplacian, the most standard way is to consider it as a scalar operator, i.e. with $\hat{N}_1 = 1$ and $\hat{T} = T$. Another possibility is to set

$$\hat{N}_1 = n \text{ with } \Pi^{\mathsf{T}} \mathbf{g} = (\mathbf{g} \cdot \mathbf{n})\, \mathbf{n},\ \forall \mathbf{g} \in \mathbb{R}^n, \quad \text{and} \quad \hat{T} = \nabla.$$

(ii) For the three-dimensional elasticity system, more possibilities are natural for boundary conditions on the displacement $\mathbf{u}$ and the traction $\mathbf{t}$.

- ⋆ For the normal component of $\mathbf{u}$, as before, we may choose the scalar formulation $D\mathbf{u} = \mathbf{u} \cdot \mathbf{n}$, or the vector formulation $D\mathbf{u} = \Pi^{\mathsf{D}} \hat{D}$, with $\Pi^{\mathsf{D}} \mathbf{v} = (\mathbf{v} \cdot \mathbf{n})\, \mathbf{n}$ and $\hat{D} = \mathbb{I}_3$. For the tangential component of $\mathbf{u}$, a two-dimension trivialization (i.e., with $\hat{N}_0 = 2$) can be used for certain boundaries, for example a torus. But, in general, one should use $\hat{N}_0 = 3$ with $\Pi^{\mathsf{D}} \mathbf{v} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\, \mathbf{n}$ and $\hat{D} = \mathbb{I}_3$.
- ⋆ Concerning the traction, we have more possibilities. Let $\sigma = \sigma(\mathbf{u})$ be the stress tensor, a symmetric 3×3 matrix. The traction is $\mathbf{t} = \sigma\, \mathbf{n}$, and its normal component can be represented as $T\mathbf{u} = \Pi^{\mathsf{T}} \hat{T} \mathbf{u}$ in three different ways
  - With the trivial representation $T\mathbf{u} = \mathbf{t} \cdot \mathbf{n}$,
  - With the projector $\Pi^{\mathsf{T}} : \mathbf{t} \mapsto (\mathbf{t} \cdot \mathbf{n})\, \mathbf{n}$ in $\mathbb{R}^3$ of rank 1, and with $\hat{T}\mathbf{u} = \mathbf{t}$,
  - With the projector $\Pi^{\mathsf{T}} : \sigma \mapsto (\mathbf{n} \otimes \mathbf{n})\, \sigma\, (\mathbf{n} \otimes \mathbf{n})$ in the space $\mathbb{R}^{3\times 3}$ (i.e., with $\hat{N}_1 = 9$) of rank 1, and with $\hat{T}\mathbf{u} = \sigma$. Here $\mathbf{n} \otimes \mathbf{n}$ is the matrix $\mathbf{n}\,\mathbf{n}^\top$, of coefficients $n_j n_k$.

  The tangential components of the traction can be represented in different ways, too. For instance, the representation with $\hat{N}_1 = 9$ uses the projection operator $\Pi^{\mathsf{T}} : \sigma \mapsto \big(\mathbb{I}_3 - (\mathbf{n} \otimes \mathbf{n})\big)\, \sigma\, (\mathbf{n} \otimes \mathbf{n})$. △

In general, one can think of Π^{T} and Π^{D} as $\hat{N}_1 \times \hat{N}_1$ and $\hat{N}_0 \times \hat{N}_0$ matrix valued $\mathscr{C}^\infty(\partial_{\mathsf{s}}\Omega)$ functions satisfying

$$\forall \mathbf{x} \in \partial_{\mathsf{s}}\Omega \quad \text{and} \quad \Pi^{\mathsf{T}}(\mathbf{x})^2 = \Pi^{\mathsf{T}}(\mathbf{x}), \quad \Pi^{\mathsf{D}}(\mathbf{x})^2 = \Pi^{\mathsf{D}}(\mathbf{x}),$$

respectively. As the example of the projection on the tangential component on the boundary of a ball in $\mathbb{R}^3$ shows, there exists, in general, no global smooth selection of a basis of the image or of the kernel of these projectors. In particular, there is no smooth matrix function diagonalizing Π^D or Π^T globally, and we will therefore by no choice of coordinates whatsoever be able to write our system C globally in components of the form

$$C = \{T_1, \ldots, T_{N_1}, D_1, \ldots, D_{N_0}\} . \tag{2.29}$$

While this is not possible globally, in general, it is possible to do it locally, however.

Lemma 2.2.25 *Let $\mathbf{x} \mapsto \Pi(\mathbf{x})$ be a continuous function with values in the projection operators on $\mathbb{C}^{\widehat{m}}$ for some integer $\widehat{m}$. Then to any $\mathbf{x}_0$ there exists an integer $m \leq \hat{m}$, a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ and a continuous function M with values in the invertible linear operators on $\mathbb{C}^{\widehat{m}}$ such that*

$$\forall \mathbf{x} \in \mathcal{U} : \qquad M(\mathbf{x})\Pi(\mathbf{x})M(\mathbf{x})^{-1} = \pi_m ,$$

where π_m is the projection on the first m components in $\mathbb{C}^{\widehat{m}}$.
If Π is $\mathscr{C}^\infty$ or analytic, then M can be chosen $\mathscr{C}^\infty$ or analytic, too.

Proof: In $\mathbf{x}_0$, $\Pi(\mathbf{x}_0)$ is diagonalizable with eigenvalues 0 and 1: There is an invertible M_0 such that $M_0\Pi(\mathbf{x}_0)M_0^{-1} = \pi_m$. Denoting by $\mathbb{I}_{\widehat{m}}$ the identity on $\mathbb{C}^{\widehat{m}}$, we set

$$M(\mathbf{x}) = M_0 \Big(\Pi(\mathbf{x}_0)\Pi(\mathbf{x}) + (\mathbb{I}_{\widehat{m}} - \Pi(\mathbf{x}_0))(\mathbb{I}_{\widehat{m}} - \Pi(\mathbf{x}))\Big) .$$

We have for all $\mathbf{x}$:

$$M(\mathbf{x})\Pi(\mathbf{x}) = M_0\Pi(\mathbf{x}_0)\Pi(\mathbf{x}) = \pi_m M(\mathbf{x}) .$$

Since $M(\mathbf{x}_0) = M_0$ is invertible, $M(\mathbf{x})$ is invertible for $\mathbf{x}$ in a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$. □

Remark 2.2.26 Note that the Lemma remains valid if we replace everywhere $\mathbb{C}^{\hat{N}_0}$ by $\mathbb{R}^{\hat{N}_0}$. For the question of the existence of a global diagonalization, however, the real and the complex situations are not equivalent. The simplest example that shows this is related to the Möbius strip, a non-trivial two-dimensional real bundle on the unit circle which is trivial over the complex numbers. In our language of projection-valued functions, this is the function

$$\Pi(\theta) = \frac{1}{2}\begin{pmatrix} 1 + \cos\theta & \sin\theta \\ \sin\theta & 1 - \cos\theta \end{pmatrix} ,$$

defined on $\Gamma = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \mid 0 \leq \theta \leq 2\pi\}$. It is not hard to see that the complex-valued matrix function

$$M(\theta) = e^{i\frac{\theta}{2}} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

is continuous and invertible on Γ and diagonalizes $\Pi(\theta)$, and also that there is no real-valued matrix function with these properties. △

2.2.c (ii) Admissible boundary value systems

Definition & Notation 2.2.27 *Let Ω be a smooth domain with sides $\partial_{\mathbf{s}}\Omega$, $\mathbf{s} \in \mathscr{S}$. We call **admissible [boundary value] system** of order 2 on $\overline{\Omega}$ the data of*

- ***a)*** *A second order $N \times N$ system L with smooth coefficients on $\overline{\Omega}$,*
- ***b)*** *An admissible system $C = (T, D)$ of boundary operators of size N on $\partial\Omega$.*

We denote such a system by

$$\mathbb{A} = \{L, T, D\}, \quad \textit{with} \quad T = (T_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}\,,\; D = (D_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}.$$

The boundary value problem associated with the admissible system $\mathbb{A}$ is written as

$$\begin{cases} L\,\mathbf{u} &= \mathbf{f} \quad & \textit{in } \Omega\,, \\ T_{\mathbf{s}}\,\mathbf{u} &= \mathbf{g}_{\mathbf{s}} \quad & \textit{on } \partial_{\mathbf{s}}\Omega\,, \quad \mathbf{s}\in\mathscr{S}, \\ D_{\mathbf{s}}\,\mathbf{u} &= \mathbf{h}_{\mathbf{s}} \quad & \textit{on } \partial_{\mathbf{s}}\Omega\,, \quad \mathbf{s}\in\mathscr{S}, \end{cases} \tag{2.30}$$

where we specify the dependency of the boundary conditions on the connected components $\partial_{\mathbf{s}}\Omega$ of $\partial\Omega$. We may also write it in condensed form

$$\begin{cases} L\,\mathbf{u} &= \mathbf{f} \quad & \textit{in } \Omega\,, \\ T\mathbf{u} &= \mathbf{g} \quad & \textit{on } \partial\Omega\,, \\ D\mathbf{u} &= \mathbf{h} \quad & \textit{on } \partial\Omega\,. \end{cases} \tag{2.31}$$

With our extended definition for admissible systems of boundary operators, we still have to explain the associated spaces for the right hand sides $(\mathbf{f}, \mathbf{g}, \mathbf{h})$, where $\mathbf{g}$ stands for $(\mathbf{g}_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$ and $\mathbf{h}$ for $(\mathbf{h}_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$: If the solution $\mathbf{u}$ is in $\mathsf{H}^k(\Omega)^N := \mathbf{H}^k(\Omega)$, we have

$$\mathbf{f} \in \mathsf{H}^{k-2}(\Omega)^N, \quad \mathbf{g}_{\mathbf{s}} \in \mathsf{H}^{k-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_1(\mathbf{s})}, \quad \mathbf{h}_{\mathbf{s}} \in \mathsf{H}^{k-\frac{1}{2}}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_0(\mathbf{s})}. \tag{2.32}$$

Using the structure (2.28) of the boundary operators $(T_{\mathbf{s}}, D_{\mathbf{s}})$, however, we see that $\mathbf{g}_{\mathbf{s}}(\mathbf{x})$ belongs to the range of $\Pi^T_{\mathbf{s}}(\mathbf{x})$ and $\mathbf{h}_{\mathbf{s}}(\mathbf{x})$ to the range of $\Pi^D_{\mathbf{s}}(\mathbf{x})$ for all $\mathbf{x} \in \partial_{\mathbf{s}}\Omega$. For this reason we introduce the following target spaces:

Notation 2.2.28 Let $k \geq 2$.

- $\star$ We denote by $\mathsf{R}\mathbf{H}^k(\Omega)$ the product of spaces

$$\mathsf{R}\mathbf{H}^k(\Omega) := \mathsf{H}^{k-2}(\Omega)^N \times \prod_{\mathbf{s}\in\mathscr{S}} \Big(\Pi^T_{\mathbf{s}}\mathsf{H}^{k-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_1(\mathbf{s})} \times \Pi^D_{\mathbf{s}}\mathsf{H}^{k-\frac{1}{2}}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_0(\mathbf{s})}\Big)$$

  The norm in the space $\mathsf{R}\mathbf{H}^k(\Omega)$ is defined in the natural way:

$$\|(\mathbf{f},\mathbf{g},\mathbf{h})\|^2_{\mathsf{R}\mathbf{H}^k(\Omega)} = \|\mathbf{f}\|^2_{k-2;\,\Omega} + \|\mathbf{g}\|^2_{k-\frac{3}{2};\,\partial\Omega} + \|\mathbf{h}\|^2_{k-\frac{1}{2};\,\partial\Omega}$$

  where the norms of $\mathbf{g} = (\mathbf{g}_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$ and $\mathbf{h} = (\mathbf{h}_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$ mean the following

$$\|\mathbf{g}\|^2_{k-\frac{3}{2};\,\partial\Omega} = \sum_{\mathbf{s}\in\mathscr{S}} \|\mathbf{g}_{\mathbf{s}}\|^2_{k-\frac{3}{2};\,\partial_{\mathbf{s}}\Omega}\,, \quad \|\mathbf{h}\|^2_{k-\frac{1}{2};\,\partial\Omega} = \sum_{\mathbf{s}\in\mathscr{S}} \|\mathbf{h}_{\mathbf{s}}\|^2_{k-\frac{1}{2};\,\partial_{\mathbf{s}}\Omega}\,. \tag{2.33}$$

⋆ The local version of this definition is, for any open set $\mathcal{U}$:

$$\mathsf{R}\mathbf{H}^k(\mathcal{U}, \partial\Omega \cap \mathcal{U}) := \mathsf{H}^{k-2}(\mathcal{U})^N \times \prod_{\mathbf{s}\in\mathscr{S}} \left(\Pi^T_{\mathbf{s}}\mathsf{H}^{k-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega \cap \mathcal{U})^{\hat{N}_1(\mathbf{s})} \times \Pi^D_{\mathbf{s}}\mathsf{H}^{k-\frac{1}{2}}(\partial_{\mathbf{s}}\Omega \cap \mathcal{U})^{\hat{N}_0(\mathbf{s})}\right).$$

△

Particular Case 2.2.29 If the system of boundary operators is written in the classical form of a global PDE system with the same boundary conditions on all sides of Ω, we can have the following simplifications:

(i) In the Dirichlet case, we can choose $\hat{N}_1(\mathbf{s}) = 0$, $\hat{N}_0(\mathbf{s}) = N$, and the projection operators are trivial: $\Pi^T_{\mathbf{s}} = 0$, $\Pi^D_{\mathbf{s}} = \mathbb{I}_N$, which gives

$$\mathsf{R}\mathbf{H}^k(\Omega) := \mathsf{H}^{k-2}(\Omega)^N \times \mathsf{H}^{k-\frac{1}{2}}(\partial\Omega)^N.$$

(ii) In the Neumann case, on can similarly choose trivial projectors and obtain

$$\mathsf{R}\mathbf{H}^k(\Omega) := \mathsf{H}^{k-2}(\Omega)^N \times \mathsf{H}^{k-\frac{3}{2}}(\partial\Omega)^N.$$

(iii) More generally, the topologically trivial situation corresponds to

$$\mathsf{R}\mathbf{H}^k(\Omega) := \mathsf{H}^{k-2}(\Omega)^N \times \mathsf{H}^{k-\frac{3}{2}}(\partial\Omega)^{N_1} \times \mathsf{H}^{k-\frac{1}{2}}(\partial\Omega)^{N_0}.$$

This illustrates how the representation of boundary operators influences the expression of the spaces for the right hand sides. △

2.2.c (iii) Ellipticity

We will now introduce the notion of ellipticity for an admissible system $\mathbb{A}$ consisting of an interior operator L and of systems $(T_{\mathbf{s}}, D_{\mathbf{s}})$ of boundary operators on each side $\partial_{\mathbf{s}}\Omega$ of Ω. This notion can be viewed as an ellipticity "up to the boundary". It requires the ellipticity on the whole domain $\overline{\Omega}$ as introduced in Definition 1.2.7 and the covering condition on each point of the boundary according to Definition 2.2.6.

We start with the ellipticity of a system $\{L, T, D\}$ at a boundary point $\mathbf{x}_0$. In fact, this is a condition on an associated system $\{\underline{L}_{\mathbf{x}_0}, \underline{T}_{\mathbf{x}_0}, \underline{D}_{\mathbf{x}_0}\}$ of homogeneous operators with constant coefficients, which we qualify as "tangent" to $\{L, T, D\}$, because its definition is based on the diffeomorphism $\phi_{\mathbf{x}_0}$ allowing a local identification of the "manifold" with boundary $(\Omega, \partial\Omega)$ with its tangent space $(\mathbb{R}^n_+, \mathbb{R}^{n-1})$.

Definition 2.2.30 *Let $\mathbb{A} = \{L, T, D\}$ be an admissible system of order 2 on the smooth domain Ω. Let $\mathbf{x}_0 \in \partial\Omega$. Let $J_{\mathbf{x}_0}$ be the Jacobian matrix of the variable change $\phi_{\mathbf{x}_0} : \mathbf{x} \mapsto$*

$\check{\mathbf{x}} = (\mathbf{x}', t)$ *at* $\mathbf{x}_0$*. The model system* $\underline{\mathbb{A}}_{\mathbf{x}_0} = \{\underline{L}_{\mathbf{x}_0}, \underline{T}_{\mathbf{x}_0}, \underline{D}_{\mathbf{x}_0}\}$ ***tangent*** *to* $\mathbb{A} = \{L, T, D\}$ *at the point* $\mathbf{x}_0$ *is defined as follows*

$$\underline{L}_{\mathbf{x}_0}(\mathrm{D}_{\check{\mathbf{x}}}) = L^{\mathsf{pr}}(\mathbf{x}_0; J_{\mathbf{x}_0}^{\top}\mathrm{D}_{\check{\mathbf{x}}}), \tag{2.34}$$

$$\underline{T}_{\mathbf{x}_0}(\mathrm{D}_{\check{\mathbf{x}}}) = \pi_{N_1} M^{\mathsf{T}}(\mathbf{x}_0)\, \hat{T}^{\mathsf{pr}}(\mathbf{x}_0; J_{\mathbf{x}_0}^{\top}\mathrm{D}_{\check{\mathbf{x}}}), \tag{2.35}$$

$$\underline{D}_{\mathbf{x}_0} = \pi_{N_0} M^{\mathsf{D}}(\mathbf{x}_0)\, \hat{D}(\mathbf{x}_0), \tag{2.36}$$

with invertible matrices $M^{\mathsf{T}}(\mathbf{x}_0)$ *and* $M^{\mathsf{D}}(\mathbf{x}_0)$ *trivializing* $\Pi^{\mathsf{T}}(\mathbf{x}_0)$ *and* $\Pi^{\mathsf{D}}(\mathbf{x}_0)$*, respectively, cf. Lemma 2.2.25.*

The ellipticity of the boundary value problem on Ω

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ T\mathbf{u} = \mathbf{g} & \text{on } \partial\Omega, \\ D\mathbf{u} = \mathbf{h} & \text{on } \partial\Omega, \end{cases}$$

at the boundary point $\mathbf{x}_0$ is defined as the ellipticity of its tangent model boundary value problem on the periodic half-space

$$\begin{cases} \underline{L}_{\mathbf{x}_0}\mathbf{u} = \mathbf{f} & \text{in } \mathbb{T}^n_+, \\ \underline{T}_{\mathbf{x}_0}\mathbf{u} = \mathbf{g} & \text{on } \partial\Gamma, \\ \underline{D}_{\mathbf{x}_0}\mathbf{u} = \mathbf{h} & \text{on } \partial\Gamma. \end{cases}$$

Coming back to Definition 2.2.6, this means the following.

Definition 2.2.31 *Let* $\mathbb{A} = \{L, T, D\}$ *be an admissible system of order* 2 *on the smooth domain* Ω*. Let* $\mathbf{x}_0 \in \partial\Omega$*. The system* $\mathbb{A} = \{L, T, D\}$ *is called* ***elliptic*** *at* $\mathbf{x}_0$ *if the following two conditions are satisfied*

a) *The tangent interior operator* $\underline{L}_{\mathbf{x}_0}$ *at* $\mathbf{x}_0$ *is elliptic*

$$\forall \boldsymbol{\xi} \in \mathbb{R}^n,\ \boldsymbol{\xi} \neq 0, \quad \underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}) \text{ is invertible.} \tag{2.37}$$

b) *The system of tangent boundary operators* $\underline{C}_{\mathbf{x}_0} = (\underline{T}_{\mathbf{x}_0}, \underline{D}_{\mathbf{x}_0})$ *covers* $\underline{L}_{\mathbf{x}_0}$*,*

$$\begin{cases} \forall \boldsymbol{\xi}' \in \mathbb{R}^{n-1},\ \boldsymbol{\xi}' \neq 0, \\ \underline{C}_{\mathbf{x}_0}(\boldsymbol{\xi}') \text{ is an isomorphism from } \mathfrak{M}_+[\underline{L}_{\mathbf{x}_0}, \boldsymbol{\xi}'] \text{ onto } \mathbb{C}^N. \end{cases} \tag{2.38}$$

Remark 2.2.32 Like interior ellipticity, the ellipticity at a boundary point is independent of particular definitions:

(i) The definition is independent of the choice of the diffeomorphism $\phi_{\mathbf{x}_0}$.

(ii) The definition is independent of the choice of local diagonalizations of the projectors Π^{D} and Π^{T} and even independent of the choice of these projectors, that is of the splitting of C in (2.28). It only depends on the system $\mathbb{A} = \{L, T, D\}$. △

Proof of *(i)*: Let us consider two suitable diffeomorphisms $\phi^1_{\mathbf{x}_0}$ and $\phi^2_{\mathbf{x}_0}$, corresponding to local coordinates $\breve{\mathbf{x}}^1 = (\mathbf{x}'_1, t_1)$ and $\breve{\mathbf{x}}^2 = (\mathbf{x}'_2, t_2)$, respectively. Then there exist an invertible $(n-1)\times(n-1)$ real matrix A'', a non-zero real number α, and a $1\times(n-1)$ real matrix A'_n such that

$$\begin{pmatrix} A'' & 0 \\ A'_n & \alpha \end{pmatrix} \begin{pmatrix} \mathrm{D}_{\mathbf{x}'_1} \\ \mathrm{D}_{t_1} \end{pmatrix} = \begin{pmatrix} \mathrm{D}_{\mathbf{x}'_2} \\ \mathrm{D}_{t_2} \end{pmatrix} .$$

Let $\underline{L}^1(\mathrm{D}_{\mathbf{x}^1})$ and $\underline{L}^2(\mathrm{D}_{\mathbf{x}^2})$ be the associated tangent operators. The transformation

$$\{t \to \mathbf{U}(t)\} \longmapsto \{t \to \exp(i\langle A'_n, \boldsymbol{\xi}'\rangle)\mathbf{U}(\alpha t)\}$$

induces an isomorphism from $\mathfrak{M}_+[\underline{L}^2, \boldsymbol{\xi}']$ onto $\mathfrak{M}_+[\underline{L}^1, A''(\boldsymbol{\xi}')]$. The equivalence between

$$\underline{C}^1(\boldsymbol{\xi}') : \mathfrak{M}_+[\underline{L}^1, \boldsymbol{\xi}'] \to \mathbb{C}^N \text{ is an isomorphism } \forall \boldsymbol{\xi}' \neq 0$$

and

$$\underline{C}^2(\boldsymbol{\xi}') : \mathfrak{M}_+[\underline{L}^2, \boldsymbol{\xi}'] \to \mathbb{C}^N \text{ is an isomorphism } \forall \boldsymbol{\xi}' \neq 0$$

is now clear. □

The global versions of the definition of ellipticity are the following:

Definition 2.2.33 *Let $\mathbb{A} = \{L, T, D\}$ be an admissible system of order 2 on the smooth domain Ω.*

- *The system $\mathbb{A} = \{L, T, D\}$ is said to be **elliptic on** $\overline{\Omega}$ if*
  - ***a)** For all interior points $\mathbf{x}_0 \in \Omega$, the interior operator L is elliptic at $\mathbf{x}_0$,*
  - ***b)** For all boundary points $\mathbf{x}_0 \in \partial\Omega$, the system $\mathbb{A}$ is elliptic at $\mathbf{x}_0$.*
- *Let $\mathcal{U}$ be open in $\mathbb{R}^n$. The system $\mathbb{A} = \{L, T, D\}$ is said to be **elliptic on** $\mathcal{U} \cap \overline{\Omega}$ if*
  - ***a)** For all interior points $\mathbf{x}_0 \in \mathcal{U} \cap \Omega$, the interior operator L is elliptic at $\mathbf{x}_0$,*
  - ***b)** For all boundary points $\mathbf{x}_0 \in \mathcal{U} \cap \partial\Omega$, the system $\mathbb{A}$ is elliptic at $\mathbf{x}_0$.*
- *If the system $\mathbb{A} = \{L, T, D\}$ is elliptic on $\overline{\Omega}$, the problem (2.30) is said to be an **elliptic boundary value problem**.*

Remark 2.2.34 If the system $\mathbb{A} = \{L, T, D\}$ is elliptic at a point $\mathbf{x}_0$, then there exists a ball B_* centered at $\mathbf{x}_0$ such that $\mathbb{A}$ is elliptic on $B_* \cap \overline{\Omega}$. △

Remark 2.2.35 If Ω is a smooth domain inside a smooth manifold M of dimension n, then we call a system $\mathbb{A} = \{L, T, D\}$ defined on Ω ***elliptic on*** $\overline{\Omega}$ if

a) For all interior points $\mathbf{x}_0 \in \Omega$, the "tangent" interior operator $\underline{L}_{\mathbf{x}_0}$ is elliptic,

b) For all boundary points $\mathbf{x}_0 \in \partial\Omega$, the system $\mathbb{A}$ is elliptic at $\mathbf{x}_0$. △

Before concluding this section with the fundamental Fredholm theorem for elliptic boundary value problems, we introduce localized operators.

Definition & Notation 2.2.36 *Let $\mathbb{A} = \{L, T, D\}$ be an admissible boundary value system of order 2 on Ω. Let $\mathbf{x}_0$ be a point in the boundary $\partial\Omega$. Using the local diffeomorphism $\phi_{\mathbf{x}_0}$ (Def. 2.2.17) and the local trivialization of projection operators (Lemma 2.2.25), we define the* ***localized versions*** *$\breve{L}_{\mathbf{x}_0}$, $\breve{T}_{\mathbf{x}_0}$ and $\breve{D}_{\mathbf{x}_0}$ of L, T and D, respectively, as follows:*

- ⋆ *Let $\phi^*_{\mathbf{x}_0}$ be the change of variables associated with $\phi_{\mathbf{x}_0}$:*
$$(\phi^*_{\mathbf{x}_0}\mathbf{u})(\breve{\mathbf{x}}) = \mathbf{u} \circ \phi^{-1}_{\mathbf{x}_0}(\breve{\mathbf{x}}).$$
- ⋆ *Then the localized version $\breve{L}_{\mathbf{x}_0}$ is the pull-back of L by $\phi_{\mathbf{x}_0}$:*
$$\breve{L}_{\mathbf{x}_0} = \phi^*_{\mathbf{x}_0} \circ L \circ (\phi^*_{\mathbf{x}_0})^{-1}.$$
- ⋆ *Let the neighborhood $\mathcal{U}_{\mathbf{x}_0}$ be chosen small enough so that there exist non-singular $\hat{N}_1 \times \hat{N}_1$ and $\hat{N}_0 \times \hat{N}_0$ matrix valued smooth functions $\mathbf{x} \mapsto M^{\mathsf{T}}(\mathbf{x})$ and $\mathbf{x} \mapsto M^{\mathsf{D}}(\mathbf{x})$ trivializing the projection operators $\Pi^{\mathsf{T}}(\mathbf{x})$ and $\Pi^{\mathsf{D}}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{U}_{\mathbf{x}_0} \cap \partial\Omega$*
$$M^{\mathsf{T}}(\mathbf{x})\Pi^{\mathsf{T}}(\mathbf{x})M^{\mathsf{T}}(\mathbf{x})^{-1} = \pi_{N_1} \quad \text{and} \quad M^{\mathsf{D}}(\mathbf{x})\Pi^{\mathsf{D}}(\mathbf{x})M^{\mathsf{D}}(\mathbf{x})^{-1} = \pi_{N_0} \tag{2.39}$$
where π_{N_j} is the projection on the first N_j components in $\mathbb{C}^{n_j}$, $j = 0, 1$.
- ⋆ *Let $\hat{T}_{\mathbf{x}_0}$ and $\hat{D}_{\mathbf{x}_0}$ be the pull-backs of $\hat{T}$ and $\hat{D}$ by $\phi_{\mathbf{x}_0}$. Then we define the localized versions $\breve{C}_{\mathbf{x}_0} = (\breve{T}_{\mathbf{x}_0}, \breve{D}_{\mathbf{x}_0})$ as*
$$\breve{T}_{\mathbf{x}_0} = \pi_{N_1} M^{\mathsf{T}} \hat{T}_{\mathbf{x}_0} \quad \text{and} \quad \breve{D}_{\mathbf{x}_0} = \pi_{N_0} M^{\mathsf{D}} \hat{D}_{\mathbf{x}_0}.$$

Note that for the principal parts there holds
$$\underline{L}_{\mathbf{x}_0} = (\breve{L}_{\mathbf{x}_0})^{\mathsf{pr}}(\mathbf{0}), \quad \underline{T}_{\mathbf{x}_0} = (\breve{T}_{\mathbf{x}_0})^{\mathsf{pr}}(\mathbf{0}), \quad \underline{D}_{\mathbf{x}_0} = \breve{D}_{\mathbf{x}_0}(\mathbf{0}). \tag{2.40}$$

Particular Case 2.2.37 If the system of boundary operators is written in the classical form of a PDE system with trivial projection operators, we simply have
$$\breve{T}_{\mathbf{x}_0} = \phi^*_{\mathbf{x}_0} \circ T \circ (\phi^*_{\mathbf{x}_0})^{-1} \quad \text{and} \quad \breve{D}_{\mathbf{x}_0} = \phi^*_{\mathbf{x}_0} \circ D \circ (\phi^*_{\mathbf{x}_0})^{-1},$$
that is, $\breve{T}_{\mathbf{x}_0}$ and $\breve{D}_{\mathbf{x}_0}$ are the pull-backs of T and D. △

With notations 2.2.36 at hands, it is now easy to state the relation between local and localized versions of the boundary value problem (2.31), cf. Figure 2.2:

Lemma & Notation 2.2.38 *Let $\mathbb{A} = \{L, T, D\}$ be an admissible system of order 2 on the smooth domain Ω. Let $\mathbf{x}_0 \in \partial\Omega$. Then the* ***local*** *boundary value problem in $\mathcal{U}_{\mathbf{x}_0}$:*
$$\begin{cases} L\,\mathbf{u} = \mathbf{f} & \text{in} \quad \mathcal{U}_{\mathbf{x}_0}, \\ T\,\mathbf{u} = \mathbf{g} & \text{on} \quad \partial\Omega \cap \mathcal{U}_{\mathbf{x}_0}, \\ D\,\mathbf{u} = \mathbf{h} & \text{on} \quad \partial\Omega \cap \mathcal{U}_{\mathbf{x}_0}. \end{cases} \tag{2.41}$$

is equivalent to its ***localized*** *version on the half-ball* V_R

$$\begin{cases} \breve{L}_{\mathbf{x}_0}\,\breve{\mathbf{u}} &= \breve{\mathbf{f}} \quad \text{in} \quad V_R\,, \\ \breve{T}_{\mathbf{x}_0}\,\breve{\mathbf{u}} &= \breve{\mathbf{g}} \quad \text{on} \quad B'_R\,, \\ \breve{D}_{\mathbf{x}_0}\,\breve{\mathbf{u}} &= \breve{\mathbf{h}} \quad \text{on} \quad B'_R\,. \end{cases} \tag{2.42}$$

Here, for $\mathbf{u} \in \mathbf{H}^k(\mathcal{U}_{\mathbf{x}_0})$ *and* $(\mathbf{f},\mathbf{g},\mathbf{h}) \in \mathsf{R}\mathbf{H}^k(\mathcal{U}_{\mathbf{x}_0}, \partial\Omega \cap \mathcal{U}_{\mathbf{x}_0})$*:*

$$\breve{\mathbf{u}} = \mathbf{u} \circ \phi_{\mathbf{x}_0}^{-1}, \quad \breve{\mathbf{f}} = \mathbf{f} \circ \phi_{\mathbf{x}_0}^{-1}, \quad \breve{\mathbf{g}} = (M^T\mathbf{g}) \circ \phi_{\mathbf{x}_0}^{-1}, \quad \breve{\mathbf{h}} = (M^D\mathbf{h}) \circ \phi_{\mathbf{x}_0}^{-1}.$$

We denote by $\mathfrak{U}_{\mathbf{x}_0}$ *and* $\mathfrak{Q}_{\mathbf{x}_0}$ *the transformations*

$$\mathfrak{U}_{\mathbf{x}_0} : \mathbf{u} \mapsto \breve{\mathbf{u}} \quad \text{and} \quad \mathfrak{Q}_{\mathbf{x}_0} : \{\mathbf{f},\mathbf{g},\mathbf{h}\} \mapsto \{\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}\},$$

so that the equivalence between (2.41) *and* (2.42) *can be written in condensed form as*

$$\{L,T,D\}\big|_{\mathcal{U}_{\mathbf{x}_0}} = \mathfrak{Q}_{\mathbf{x}_0}^{-1} \circ \{\breve{L}_{\mathbf{x}_0}, \breve{T}_{\mathbf{x}_0}, \breve{D}_{\mathbf{x}_0}\} \circ \mathfrak{U}_{\mathbf{x}_0}.$$

The boundary conditions in (2.42) can be equivalently written as the system of N independent equations

$$\begin{cases} \breve{T}_{\mathbf{x}_0,k}\,\breve{\mathbf{u}} &= \breve{g}_k \quad \text{on } B'_R, \quad k = 1,\ldots,N_1 \\ \breve{D}_{\mathbf{x}_0,k}\,\breve{\mathbf{u}} &= \breve{h}_k \quad \text{on } B'_R, \quad k = 1,\ldots,N_0, \end{cases}$$

where $\breve{T}_{\mathbf{x}_0,k}$ and $\breve{D}_{\mathbf{x}_0,k}$ are the rows of $\breve{T}_{\mathbf{x}_0}$ and $\breve{D}_{\mathbf{x}_0}$, respectively.

Theorem 2.2.39 *Let* $\mathbb{A} = \{L,T,D\}$ *be an admissible system of order* 2 *on the smooth bounded domain* Ω *with sides* $(\partial_{\mathbf{s}}\Omega)_{\mathbf{s}\in\mathscr{S}}$*. We assume that* $\mathbb{A}$ *is elliptic on* $\overline{\Omega}$*. Then it defines a Fredholm operator* $\mathbf{u} \mapsto (\mathbf{f},\mathbf{g},\mathbf{h})$ *from* $\mathbf{H}^2(\Omega)$ *to* $\mathsf{R}\mathbf{H}^2(\Omega)$ *with, cf. Notation* 2.2.28,

$$\mathsf{R}\mathbf{H}^2(\Omega) = \mathsf{L}^2(\Omega)^N \times \prod_{\mathbf{s}\in\mathscr{S}} \Big(\Pi_{\mathbf{s}}^T \mathsf{H}^{\frac12}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_1(\mathbf{s})} \times \Pi_{\mathbf{s}}^D \mathsf{H}^{\frac32}(\partial_{\mathbf{s}}\Omega)^{\hat{N}_0(\mathbf{s})}\Big)\,.$$

Proof: In the same way as in Theorem 1.2.13, the Fredholm property for elliptic operators on smooth domains will be deduced from the existence of a *regularizer* (or "global parametrix") i.e. a continuous operator $\mathbb{E} : \mathsf{R}\mathbf{H}^2(\Omega) \to \mathbf{H}^2(\Omega)$ such that

(1) $\mathbb{A}\mathbb{E} = \mathbb{I} + K,\ K$ compact in $\mathsf{R}\mathbf{H}^2(\Omega)$, $\mathbb{E}\mathbb{A} = \mathbb{I} + K',\ K'$ compact in $\mathbf{H}^2(\Omega)$.

Let us construct $\mathbb{E}$. We cover $\overline{\Omega}$ with

a) Balls $\mathcal{B}_{\mathbf{x}_0}$, $\mathbf{x}_0 \in \Omega$, for which there exists a parametrix $\mathfrak{E}_{\mathbf{x}_0}$ of the interior operator L (Proposition 1.2.9),

b) Neighborhoods $\mathcal{U}_{\mathbf{x}_0}$, $\mathbf{x}_0 \in \partial\Omega$, for which there exists a parametrix $\mathfrak{E}_{\mathbf{x}_0}$ of the localized operator $\{\breve{L}_{\mathbf{x}_0}, \breve{T}_{\mathbf{x}_0}, \breve{D}_{\mathbf{x}_0}\}$ on $\mathcal{V}_{\mathbf{x}_0} = \phi_{\mathbf{x}_0}$ (This uses Notation 2.2.17 and Lemma 2.2.38 for the localization, and Proposition 2.2.15 for the parametrix).

We cover $\overline{\Omega}$ with a finite subset X of these $\mathcal{B}_{\mathbf{x}_0}$ and $\mathcal{U}_{\mathbf{x}_0}$, we choose a corresponding partition of unity $(\psi'_{\mathbf{x}})_{\mathbf{x}\in X}$ and $\psi''_{\mathbf{x}} \in \mathscr{C}_0^\infty(\mathcal{B}_{\mathbf{x}})$ with $\psi''_{\mathbf{x}}\psi'_{\mathbf{x}} = \psi'_{\mathbf{x}}$. Then we set

a) $\mathbb{E}_{\mathbf{x}_0}(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathfrak{E}_{\mathbf{x}_0}\mathbf{f}$, if $\mathbf{x}_0 \in \Omega$,

b) $\mathbb{E}_{\mathbf{x}_0}(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathfrak{U}_{\mathbf{x}_0}^{-1}\mathfrak{E}_{\mathbf{x}_0}\mathfrak{Q}_{\mathbf{x}_0}(\mathbf{f}, \mathbf{g}, \mathbf{h})$, if $\mathbf{x}_0 \in \partial\Omega$, and using Notation 2.2.38.

Finally we set

$$\mathbb{E} = \sum_{\mathbf{x}\in X} \psi''_{\mathbf{x}}\mathbb{E}_{\mathbf{x}}\psi'_{\mathbf{x}}$$

and have obtained our regularizer, thanks to the properties of the local parametrices. □

Remark 2.2.40 The generalization of the theory of elliptic boundary value problems to smooth manifolds with boundary is natural: There we use local diffeomorphisms $\phi_{\mathbf{x}_0}$ inside the domain too, as indicated in Remark 2.2.35. Thus for elliptic problems on smooth bounded manifolds with boundary, there holds the same Fredholm result as in Theorem 2.2.39. △

2.3 Regularity of solutions up to the boundary

In this section, we obtain higher order Sobolev regularity results by analyzing the mapping properties of the parametrix constructed above in Proposition 2.2.15.

To provide optimal local and global regularity results, we first need to introduce a local version of admissible boundary value systems (compare Definition 2.2.27) with finite regularity.

Definition 2.3.1 *Let $k \geq 2$ be an integer. Let Ω be a domain in $\mathbb{R}^n$ or, more generally, in a $\mathscr{C}^k$ manifold M of dimension n. Let $(\partial_{\mathbf{s}}\Omega)_{\mathbf{s}\in\mathscr{S}}$ be the connected components of $\partial\Omega$. Let Γ be a $\mathscr{C}^k$ subdomain of the boundary of Ω. The boundary value system $\mathbb{A} = \{L, T, D\}$, with $T = (T_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$ and $D = (D_{\mathbf{s}})_{\mathbf{s}\in\mathscr{S}}$, is said to be **admissible** of class $\mathscr{C}^k$ **on** $\Omega \cup \Gamma$ if*

a) *The interior system L is a $N \times N$ system of second order operators with coefficients $a_{ij}^\alpha \in \mathscr{C}^{k-2}(\Omega \cup \Gamma)$,*

b) *For each $\mathbf{s}$ such that $\partial_{\mathbf{s}}\Omega \cap \Gamma \neq \emptyset$, the boundary operators $T_{\mathbf{s}}$ of order 1 and $D_{\mathbf{s}}$ of order 0 have the form (cf. Definition 2.2.20)*

$$T_{\mathbf{s}} = \Pi_{\mathbf{s}}^T\,\hat{T}_{\mathbf{s}}\,, \quad D_{\mathbf{s}} = \Pi_{\mathbf{s}}^D\,\hat{D}_{\mathbf{s}},$$

where the projector field $\Pi_{\mathbf{s}}^T$ and the system $\hat{T}_{\mathbf{s}}$ have $\mathscr{C}^{k-1}(\Gamma \cap \partial_{\mathbf{s}}\Omega)$ coefficients, whereas $\Pi_{\mathbf{s}}^D$ and $\hat{D}_{\mathbf{s}}$ have $\mathscr{C}^k(\Gamma \cap \partial_{\mathbf{s}}\Omega)$ coefficients.

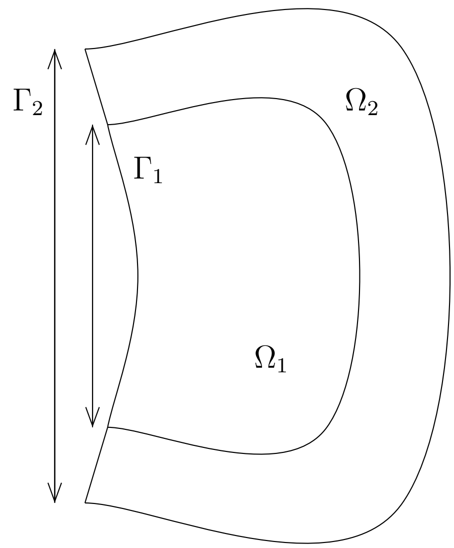


Figure 2.3: Local configurations (Ω_1, Γ_1) and (Ω_2, Γ_2).

*If Γ coincides with $\partial\Omega$, $\mathbb{A} = \{L, T, D\}$ is said to be **admissible** of class $\mathscr{C}^k$ **on** $\overline{\Omega}$.*

Theorem 2.3.2 *Let $k \geq 2$ be an integer. Let Ω be a domain in $\mathbb{R}^n$ or, more generally, in a $\mathscr{C}^k$ manifold M of dimension n. Let $(\partial_{\mathsf{s}}\Omega)_{\mathsf{s}\in\mathscr{S}}$ be the connected components of $\partial\Omega$. Let Γ be a $\mathscr{C}^k$ subdomain of the boundary of Ω. Let $\mathbb{A} = \{L, T, D\}$ be an admissible system of class $\mathscr{C}^k$ on $\Omega\cup\Gamma$, as defined above. We assume that the system $\mathbb{A} = \{L, T, D\}$ is elliptic on $\Omega \cup \Gamma$ (cf. Definition 2.2.31). Then we have the following local and global regularity results and a priori estimates:*

(i) Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be bounded subdomains of M with $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$, and let $\Omega_1 = \mathcal{U}_1 \cap \Omega$ and $\Omega_2 = \mathcal{U}_2 \cap \Omega$. We assume that $\Gamma_2 := \partial\Omega_2 \cap \partial\Omega$ is a subset of Γ. If $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ is a solution of

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega_2 \\ T\mathbf{u} = \mathbf{g} & \text{on } \Gamma_2 \\ D\mathbf{u} = \mathbf{h} & \text{on } \Gamma_2 \end{cases} \tag{2.43}$$

and if $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ belong to the space of local right hand sides $\mathsf{R}\mathbf{H}^k(\Omega_2, \Gamma_2)$, cf. Notation 2.2.28, then $\mathbf{u} \in \mathbf{H}^k(\Omega_1)$, and there is an estimate

$$\|\mathbf{u}\|_{k;\,\Omega_1} \leq c\Big(\|\mathbf{f}\|_{k-2;\,\Omega_2} + \|\mathbf{g}\|_{k-\frac{3}{2};\,\Gamma_2} + \|\mathbf{h}\|_{k-\frac{1}{2};\,\Gamma_2} + \|\mathbf{u}\|_{1;\,\Omega_2}\Big).$$

(ii) We assume moreover that Ω is bounded and $\Gamma = \partial\Omega$. Then the operator $\mathbb{A}$ is Fredholm from $\mathbf{H}^k(\Omega)$ into $\mathsf{R}\mathbf{H}^k(\Omega)$, cf. Notation 2.2.28. Moreover, if the right

hand side $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *belongs to* $\mathsf{R}\mathbf{H}^k(\Omega)$*, then any solution* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *of*

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \\ T\mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega, \\ D\mathbf{u} &= \mathbf{h} \quad \text{on } \partial\Omega, \end{cases} \tag{2.44}$$

belongs to $\mathbf{H}^k(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{k;\,\Omega} \leq c\left(\|\mathbf{f}\|_{k-2;\,\Omega} + \|\mathbf{g}\|_{k-\frac{3}{2};\,\partial\Omega} + \|\mathbf{h}\|_{k-\frac{1}{2};\,\partial\Omega} + \|\mathbf{u}\|_{1;\,\Omega}\right).$$

Like in Chapter 1 for interior estimates, we state the extension of Theorem 2.3.2 to non-integral Sobolev exponents as a Corollary:

Corollary 2.3.3 *Under the conditions of Theorem 2.3.2, let* s *be real,* $2 \leq s \leq k$*.*

(i) For Ω_1*,* Ω_2 *as above, if* $\mathbf{u} \in \mathbf{H}^2(\Omega_2)$ *satisfies* $\mathbb{A}\mathbf{u} \in \mathsf{R}\mathbf{H}^s(\Omega_2, \Gamma_2)$ *(with obvious extension of Notation 2.2.28), then* $\mathbf{u}$ *belongs to* $\mathbf{H}^s(\Omega_1)$ *with the estimate*

$$\|\mathbf{u}\|_{\mathbf{H}^s(\Omega_1)} \leq c(\|\mathbb{A}\mathbf{u}\|_{\mathsf{R}\mathbf{H}^s(\Omega_2,\Gamma_2)} + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega_2)}). \tag{2.45}$$

(ii) Assume that Ω *is bounded and* $\Gamma = \partial\Omega$*. If* $\mathbf{u} \in \mathbf{H}^2(\Omega)$ *satisfies* $\mathbb{A}\mathbf{u} \in \mathsf{R}\mathbf{H}^s(\Omega)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{H}^s(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{\mathbf{H}^s(\Omega)} \leq c(\|\mathbb{A}\mathbf{u}\|_{\mathsf{R}\mathbf{H}^s(\Omega)} + \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}). \tag{2.46}$$

In addition, $\mathbb{A}$ *defines a Fredholm operator* $\mathbb{A}_s : \mathbf{H}^s(\Omega) \mapsto \mathsf{R}\mathbf{H}^s(\Omega)$*. The kernel and the cokernel of* $\mathbb{A}_s$ *do not depend on* s*.*

Like for interior Sobolev estimates (Theorem 1.3.3) the proof of Theorem 2.3.2 relies on refined properties of local parametrices $\mathfrak{E}_{\mathbf{x}_0}$, along the lines of Proposition 2.2.15.

Proposition 2.3.4 *Let* $k \geq 2$ *be an integer. Let us assume that the hypotheses of Proposition 2.2.15 are satisfied, and that, moreover, the system* $\mathbb{A} = (L, C_{\breve{\Gamma}})$ *is admissible of class* $\mathscr{C}^k$ *on* $\breve{\Omega} \cup \breve{\Gamma}$*. Then there exists a ball* $\mathcal{B}_{\mathbf{x}_0}$ *centered at* $\mathbf{x}_0$ *and an operator* $\mathfrak{E}_{\mathbf{x}_0}$ *(the local parametrix) enjoying the same properties as in Proposition 2.2.15, with the extra continuity properties:*

- ⋆ $\mathfrak{E}_{\mathbf{x}_0}$ *is continuous from* $\mathsf{R}\mathbf{H}^k(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$ *into* $\mathbf{H}^k(\mathcal{V}_{\mathbf{x}_0})$*,*
- ⋆ *The operator* K *is continuous from* $\mathsf{R}\mathbf{H}^k(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$ *to* $\mathsf{R}\mathbf{H}^{k+1}(\mathcal{V}_{\mathbf{x}_0}, \mathcal{B}'_{\mathbf{x}_0})$*, and* K' *from* $\mathbf{H}^{k-1}(\mathcal{V}_{\mathbf{x}_0})$ *to* $\mathbf{H}^k(\mathcal{V}_{\mathbf{x}_0})$*.*

Proof: In opposition with the proof of Proposition 1.3.5, for which we proved the regularizing properties of the parametrix by a differential quotient technique, we choose here to present a different proof, well prepared with the help of the new perturbation argument proving Proposition 2.2.15: The statement of this latter proposition can simply be generalized to higher regularity indices, relying on a similar generalization of Proposition 2.2.10. □

2.4 Basic nested a priori estimates

As a preparation for analytic a priori estimates, we come back to a local situation as studied in §2.2.b: We consider a $N \times N$ second order system with smooth coefficients in a neighborhood of $\mathbf{0}$, and a set of N boundary operators $C = \{T_1, \ldots, T_{N_1}, D_1, \ldots, D_{N_0}\}$ with smooth coefficients in a neighborhood of $\mathbf{0}$ in the hyperplane $x_n = 0$.

If we assume that L is elliptic at $\mathbf{0}$ and C covers L at $\mathbf{0}$, according to Corollary 2.2.16 there exists a radius $R_* > 0$ and a constant $A_0 > 0$ such that for all $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$:

$$\|\mathbf{u}\|_{2;V_{R_*}} \leq A_0\Big(\|L\mathbf{u}\|_{0;V_{R_*}} + \sum_{j=1}^{N_1}\|T_j\mathbf{u}\|_{\frac{1}{2};B'_{R_*}} + \sum_{j=1}^{N_0}\|D_j\mathbf{u}\|_{\frac{3}{2};B'_{R_*}} + \|\mathbf{u}\|_{1;V_{R_*}}\Big). \quad (2.47)$$

Here, for $R > 0$, $V_R = \{\mathbf{x} \in \mathbb{R}^n_+ \,;\, |\mathbf{x}| < R\}$ and $B'_R = \partial V_R \cap \{x_n = 0\}$.

Remark 2.4.1 In the following, we will often replace the norms of the right hand sides on the boundary $\sum_{j=1}^{N_1}\|T_j\mathbf{u}\|_{1/2}$ and $\sum_{j=1}^{N_0}\|D_j\mathbf{u}\|_{3/2}$ by the shorthand vector notations $\|T\mathbf{u}\|_{1/2}$ and $\|D\mathbf{u}\|_{3/2}$. This will not lead to confusion with the vector-valued Sobolev norms of the boundary terms $T\mathbf{u}$ and $D\mathbf{u}$ where the operators T and D have the non-diagonalized form (2.28), because these norms before and after diagonalization are equivalent. △

Our first nested open set estimate is the analogue, with boundary terms, of Lemma 1.4.1. Like in Ch.1, we take now estimate (2.47) as our starting point, rather than ellipticity assumptions on L and the boundary operators T and D.

Lemma 2.4.2 *Let L be a $N \times N$ second order system with smooth coefficients. Let $C = \{T_1, \ldots, T_{N_1}, D_1, \ldots, D_{N_0}\}$ be boundary operators with smooth coefficients. We assume that estimate (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$. Let $R \leq R_*$. Then any function $\mathbf{u} \in \mathbf{H}^2(V_R)$ satisfies for any $\rho \in (0, R/2)$*

$$\sum_{|\alpha|\leq 2}\rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{V_{R-|\alpha|\rho}} \leq A_1\Big(\rho^2\|L\mathbf{u}\|_{V_{R-\rho}} + \rho^{\frac{3}{2}}\|T\mathbf{u}\|_{\frac{1}{2};B'_{R-\rho}} + \rho^{\frac{1}{2}}\|D\mathbf{u}\|_{\frac{3}{2};B'_{R-\rho}} + \sum_{|\alpha|\leq 1}\rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{V_{R-|\alpha|\rho}}\Big), \quad (2.48)$$

where the positive constant A_1 is independent of $\mathbf{u}$, R and ρ.

Proof: We apply estimate (2.47) to the function $\chi_{R,\rho}\mathbf{u}$, where $\chi_{R,\rho}$ is the cut-off function defined in (1.40). As a consequence of the estimates (1.41) for the derivatives of $\chi_{R,\rho}$, we note that the operator of multiplication by $\chi_{R,\rho}$ has an operator norm in the Sobolev space $\mathsf{H}^s_{B'_R}$ bounded by $C\rho^{-s}$ for any $s \in [0, 2]$ (in fact for any $s > 0$, but with a constant C

depending on a bound for s). This follows directly from the Leibniz rule if s is an integer, and we used this argument in Ch.1. For non-integral s it follows by interpolation.

We have $D_j(\chi_{R,\rho}\mathbf{u}) = \chi_{R,\rho}D_j\mathbf{u}$ on B'_R, but for the operators of order one T_j there is a non-trivial commutator: $T_j(\chi_{R,\rho}\mathbf{u}) = \chi_{R,\rho}T_j\mathbf{u} + [T_j,\chi_{R,\rho}]\mathbf{u}$. We find then

$$\|\mathbf{u}\|_{2;V_{R-\rho}} \leq A_0\Big(\|L\mathbf{u}\|_{0;V_R} + \|[L,\chi_{R,\rho}]\mathbf{u}\|_{0;V_R} + \sum_{j=1}^{N_0}\|\chi_{R,\rho}D_j\mathbf{u}\|_{\frac32;B'_R} + \sum_{j=1}^{N_1}\|\chi_{R,\rho}T_j\mathbf{u}\|_{\frac12;B'_R} + \sum_{j=1}^{N_1}\|[T_j,\chi_{R,\rho}]\mathbf{u}\|_{\frac12;B'_R} + \|\chi_{R,\rho}\mathbf{u}\|_{1;V_R}\Big)$$

As a result of the continuity of the trace operator $\gamma_0 : \mathsf{H}^1(\mathbb{R}^n_+) \to \mathsf{H}^{\frac12}(\mathbb{R}^{n-1})$, the boundary term $\|[T_j,\chi_{R,\rho}]\mathbf{u}\|_{\frac12;B'_R}$ can be estimated by a domain term :

$$\|[T_j,\chi_{R,\rho}]\mathbf{u}\|_{\frac12;B'_R} \leq c_0\|[T_j,\chi_{R,\rho}]\mathbf{u}\|_{1;V_R}\,.$$

Both expressions $[L,\chi_{R,\rho}]\mathbf{u}$ and $[T_j,\chi_{R,\rho}]\mathbf{u}$ are seen to be composed of terms $\partial^\beta\chi_{R,\rho}\,\partial^\alpha\mathbf{u}$ with $|\alpha|\leq 1$, $|\alpha+\beta|\leq 2$. The estimate of $\|\mathbf{u}\|_{2;V_{R-\rho}}$ continues as

$$\begin{aligned}\|\mathbf{u}\|_{2;V_{R-\rho}} &\leq A'_0\Big(\|L\mathbf{u}\|_{0;V_R} + \|\chi_{R,\rho}D\mathbf{u}\|_{\frac32;B'_R} + \|\chi_{R,\rho}T\mathbf{u}\|_{\frac12;B'_R} + \sum_{|\alpha|\leq1}\sum_{|\beta|\leq 2-|\alpha|}\|\partial^\beta\chi_{R,\rho}\,\partial^\alpha\mathbf{u}\|_{0;V_R}\\ &\overset{(1.41)}{\leq} A''_0\Big(\|L\mathbf{u}\|_{0;\,V_R} + \rho^{-\frac32}\|D\mathbf{u}\|_{\frac32;B'_R} + \rho^{-\frac12}\|T\mathbf{u}\|_{\frac12;B'_R} + \sum_{|\alpha|=1}\rho^{-1}\|\partial^\alpha\mathbf{u}\|_{0;\,V_R} + \rho^{-2}\|\mathbf{u}\|_{0;\,V_R}\Big).\end{aligned}$$

From this it is easy to deduce (2.48) with the same arguments we used before for (1.42). ☐

2.5 Nested a priori estimates for constant coefficients

For estimating higher order derivatives, we can begin by using the same arguments as in the proof of Proposition 1.5.1, if L and the boundary operators D_j and T_j have constant coefficients, but in a first step only for the estimate of *almost tangential derivatives* of $\mathbf{u}$, i. e. partial derivatives of the form $\partial^\beta\partial^\alpha\mathbf{u}$ with $\beta = (\beta',0)$ and $|\alpha|\leq 2$. The operator ∂^β is thus tangential on the boundary B'_R. This class of derivatives can equivalently be written ∂^α with $\alpha = (\alpha',\alpha_n)$ and $\alpha_n \leq 2$. In other words, we are going to prove a series of *anisotropic estimates*, where the number of normal derivatives has a possibly smaller bound than the total number of derivatives. For this purpose, we introduce the following notation:

Notation 2.5.1 For $k, m \in \mathbb{N}$, let the anisotropic seminorm $|u|_{k,m;\mathcal{U}}$ be defined as

$$|u|_{k,\,m;\mathcal{U}} = \max_{\substack{|\alpha|=k\\ \alpha_n\le m}} \|\partial^\alpha u\|_{\mathcal{U}} \,.$$

Note that for $m \geq k$, $|u|_{k,\,m;\mathcal{U}}$ is equivalent to the seminorm $|u|_{k;\mathcal{U}}$ as defined in section 1.1 (1.3).
On the boundary, we use the seminorms

$$|u|_{k,s;\mathcal{U}'} = \max_{|\alpha|=k} \|\partial^\alpha u\|_{s;\mathcal{U}'} \,.$$

Note that these are different from both the norm and the usual seminorm in $\mathsf{H}^{k+s}(\mathcal{U}')$, but this distinction is of no importance for the further applications. △

The proof of the following proposition is based on the estimate (2.47) and its consequence Lemma 2.4.2 only, without any further need of ellipticity and covering conditions.

Proposition 2.5.2 *Let L and the boundary operators D_j, T_j have constant coefficients. We assume that estimate (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$. Then there exists a constant $A \geq 1$ such that for all $R \in (0, R_*)$, for any $\mathbf{u} \in \mathbf{H}^2(V_R)$, all $k \in \mathbb{N}$ and all $\rho \in (0, \frac{R}{k+2}]$ there holds*

$$\begin{aligned}\rho^{k+2}|\mathbf{u}|_{k+2,\,2;\,V_{R-(k+2)\rho}} \leq \sum_{\ell=0}^{k} A^{k+1-\ell} \Big(\rho^{\ell+2}|L\mathbf{u}|_{\ell,\,0;\,V_{R-(\ell+1)\rho}} \\ + \rho^{\ell+\frac{3}{2}}|T\mathbf{u}|_{\ell,\frac{1}{2};B'_{R-(\ell+1)\rho}} + \rho^{\ell+\frac{1}{2}}|D\mathbf{u}|_{\ell,\frac{3}{2};B'_{R-(\ell+1)\rho}}\Big) \\ + A^{k+1}\sum_{|\alpha|\le 1}\rho^{|\alpha|}\|\partial^\alpha \mathbf{u}\|_{V_{R-|\alpha|\rho}} \,. \qquad (2.49)\end{aligned}$$

In particular, if the right hand side is finite, the left hand side is also finite (tangential regularity of the solution up to the boundary).

Proof: All the arguments leading to Proposition 1.5.1 and Lemma 1.3.6 (the nested open set technique and the difference quotients) can be reproduced with the tangential derivatives ∂^β, i.e. with $\beta_n = 0$. The reason for this is that such tangential derivatives commute not only with the operator L, but also with the boundary operators D_j and T_j. □

A new argument has to be employed to obtain an estimate for the remaining derivatives of $\mathbf{u}$: This is the fact that the boundary $x_n = 0$ *is non-characteristic for* L, i. e.

$$L^{\mathsf{pr}}(\boldsymbol{\xi}) = -\sum_{p=1}^{n}\sum_{q=1}^{n} M_{pq}\xi_p\xi_q, \quad \text{where the } N\times N \text{ matrix } M_{nn} \text{ is invertible [2]}. \qquad (2.50)$$

[2] In the notation of Def. 1.2.7, the entries of M_{nn} are the $a_{ij}^{(0,\dots,0,2)}$, $1 \leq i, j \leq N$.

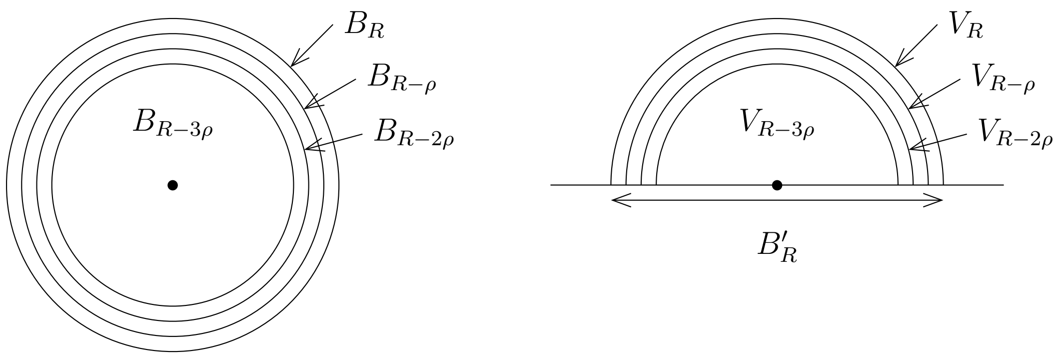


Figure 2.4: Nested neighborhoods (balls and half-balls)

Proposition 2.5.3 *Let L and the boundary operators D_j, T_j have constant coefficients. We assume that estimate (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$ and that (2.50) holds*[3]*. Then there exist two constants*[4] *A, $B \geq 1$ such that for all $k, m \in \mathbb{N}$ with $m \leq k$, for all $R \in (0, R_*]$, for all $\rho \in (0, \frac{R}{k+2}]$ and for all $\mathbf{u} \in \mathbf{H}^2(V_R)$ there holds*

$$\rho^{k+2}|\mathbf{u}|_{k+2,\,m+2;\,V_{R-(k+2)\rho}} \leq \sum_{\ell=0}^{k} A^{k+1-\ell}\Big(\sum_{\nu=0}^{m} B^{m+1-\nu}\,\rho^{\ell+2}|L\mathbf{u}|_{\ell,\,\nu;\,V_{R-(\ell+1)\rho}} + B^{m+1}\rho^{\ell+\frac32}|T\mathbf{u}|_{\ell,\frac12;B'_{R-(\ell+1)\rho}} + B^{m+1}\rho^{\ell+\frac12}|D\mathbf{u}|_{\ell,\frac32;B'_{R-(\ell+1)\rho}}\Big) + A^{k+1}B^{m+1}\sum_{|\alpha|\leq 1}\rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{V_{R-|\alpha|\rho}}. \quad (2.51)$$

In particular, if $L\mathbf{u}$ belongs to $\mathbf{H}^k(V_R)$, then $\mathbf{u}$ belongs to $\mathbf{H}^{k+2}(V_{R-\rho})$ for any $\rho > 0$.

Proof: Let k be fixed. We prove (2.51) by induction over m from 0 to k. For $m = 0$, (2.51) is a direct consequence of (2.49) as soon as $B \geq 1$. Let us assume that (2.51) holds for $m - 1$, that is

$$\rho^{k+2}|\mathbf{u}|_{k+2,\,m+1;\,V_{R-(k+2)\rho}} \leq \sum_{\ell=0}^{k} A^{k+1-\ell}\Big(\sum_{\nu=0}^{m-1} B^{m-\nu}\,\rho^{\ell+2}|L\mathbf{u}|_{\ell,\,\nu;\,V_{R-(\ell+1)\rho}} + B^{m}\rho^{\ell+\frac32}|T\mathbf{u}|_{\ell,\frac12;B'_{R-(\ell+1)\rho}} + B^{m}\rho^{\ell+\frac12}|D\mathbf{u}|_{\ell,\frac32;B'_{R-(\ell+1)\rho}}\Big) + A^{k+1}B^{m}\sum_{|\alpha|\leq 1}\rho^{|\alpha|}\|\partial^\alpha\mathbf{u}\|_{V_{R-|\alpha|\rho}}, \quad (2.52)$$

and let us prove it for m. Condition (2.50) implies that we have the following identity, with

[3] In fact, it is possible to prove that (2.50) is a consequence of estimate (2.47), because estimate (2.47) implies the interior ellipticity of the constant coefficient system L, which, itself, implies (2.50).

[4] The constant A can be chosen as the same as in Proposition 2.5.2.

$N \times N$ matrices N^α:

$$\partial_n^2 \mathbf{u} = M_{nn}^{-1} L\mathbf{u} + \sum_{\substack{|\alpha|\le 2\\ \alpha_n \le 1}} N^\alpha \partial^\alpha \mathbf{u}. \tag{2.53}$$

As a consequence, there exists a positive constant B_0 such that for all $R > 0$ and all $\mathbf{u} \in \mathbf{H}^2(V_R)$ (here *no boundary condition is needed*):

$$|\mathbf{u}|_{2,\,2;\,V_R} \le B_0\big(|L\mathbf{u}|_{0,\,0;\,V_R} + |\mathbf{u}|_{2,\,1;\,V_R}\big). \tag{2.54}$$

Applying (2.54) in $V_{R-(k+2)\rho}$ for $\partial_n^m \partial^\beta \mathbf{u}$ with $\beta = (\beta', 0)$ and $|\beta| = k - m$, we obtain

$$|\mathbf{u}|_{k+2,\,m+2;\,V_{R-(k+2)\rho}} \le B_0\big(|L\mathbf{u}|_{k,\,m;\,V_{R-(k+2)\rho}} + |\mathbf{u}|_{k+2,\,m+1;\,V_{R-(k+2)\rho}}\big).$$

Here again, the commutation of $\partial_n^m \partial^\beta$ with L was used. We multiply the last inequality by ρ^{k+2} and employ (2.52) for the estimation of $|\mathbf{u}|_{k+2,\,m+1;\,V_{R-(k+2)\rho}}$. The result is (2.51) if B was chosen such that $B \ge B_0$. □

Remark 2.5.4 As already mentioned in Remark 1.5.2, we can choose other domains than balls. For example:

$$V_R = (-b_1, b_1) \times \ldots \times (-b_{n-1}, b_{n-1}) \times (0, b_n), \quad \text{with} \quad R = \min_{1\le j\le n} \frac{b_j}{2},$$

and a suitable definition for the nested domains $V_{R-\rho}$ is

$$V_{R-\rho} = (-b_1 + \rho, b_1 - \rho) \times \ldots \times (-b_{n-1} + \rho, b_{n-1} - \rho) \times (0, b_n - \rho).$$

Estimates (2.51) are still valid for such domains. Again, it is also possible to use rectangular domains in polar or spherical coordinates. △

2.6 Nested a priori estimates for variable coefficients

We will now generalize the higher order estimates to the case of operators L, D_j and T_j with variable coefficients. We still consider the local model geometric situation of a half-ball V_R centered at 0, with active boundary $B'_R \subset \mathbb{R}^{n-1}$.

2.6.a Homogeneous boundary conditions

In order not to overly complicate the presentation, we consider first the case of vanishing boundary terms. In the next paragraph we will then prove the complete estimates including the boundary terms.

Like for interior estimates, we need to estimate commutators. Again, weighted semi-norms centered at 0 appear natural, but now *anisotropic semi-norms* have to be considered. We use the model of Notation 1.6.1, and define with Notation 2.5.1:

Notation 2.6.1 *(i)* We set $\big[|u|\big]_{0,0;V_R} = \|u\|_{V_R}$ and for $k \in \mathbb{N},\ k > 0$,

$$\big[|u|\big]_{k,m;V_R} := \max_{0<\rho\le\frac{R}{2k}} \rho^k |u|_{k,m;\,V_{R-k\rho}}.$$

(ii) We set for $k \in \mathbb{N}$: $\quad \rho_*^2\big[|f|\big]_{k,m;V_R} := \max_{0<\rho\le\frac{R}{2(k+1)}} \rho^{2+k} |f|_{k,m;\,V_{R-(k+1)\rho}}.$ △

For the estimate of commutator terms, we need a straightforward extension of Lemma 1.6.2, the proof of which is similar to the very detailed proof of Lemma 1.6.2 given above and is left to the reader:

Lemma 2.6.2 *Let a be an analytic function satisfying $|\partial^\gamma a| \le M^{|\gamma|+1}|\gamma|!$ on B_{R_*} for all $\gamma \in \mathbb{N}^n$. Let α and δ be multiindices such that $|\alpha|+|\delta| \le 2$, and let β be any multiindex. We set*

$$b = |\alpha| + |\beta| + |\delta| \quad \textit{and} \quad m = \alpha_n + \delta_n + \beta_n.$$

Let $R,\ \rho$ be positive numbers, $R \le R_*$ and $\rho \le \frac{R}{2b}$. There exists a constant c_1 independent of $R,\ \rho,\ b$ and m, such that

$$\rho^b \|\partial^\delta\big(a(\mathbf{x})\partial^\beta\partial^\alpha u\big) - \partial^\delta\partial^\beta\big(a(\mathbf{x})\partial^\alpha u\big)\|_{V_{R-(b-1)\rho}} \le c_1 \sum_{d=0}^{b-1} (b-d+1)^n \Big(\frac{RM}{2\mathrm{e}}\Big)^{b-d} \big[|u|\big]_{d,\,m;V_R}. \tag{2.55}$$

The first result is an estimate of almost tangential derivatives, cf. Prop. 2.5.2.

Proposition 2.6.3 *Let L and the boundary operators $T_j,\ D_j$ have analytic coefficients. We assume that estimate (2.47) holds for any $\mathbf{u} \in \mathbf{H}_0^2(V_{R_*}, B'_{R_*})$. Let $\mathbf{u}$ be any function in $\mathbf{H}^2(V_R)$ with $R \le R_*$. We assume that*

$$T_j\mathbf{u}|_{B'_R} = 0,\ \ j = 1,\ldots,N_1 \quad \textit{and} \quad D_j\mathbf{u}|_{B'_R} = 0,\ \ j = 1,\ldots,N_0 = N - N_1.$$

Then there exists a constant $A \ge 1$ independent of $\mathbf{u}$ such that for all $k \in \mathbb{N}$ and for all $R \in (0, R_*]$ there holds

$$\big[|\mathbf{u}|\big]_{k+2,\,2;V_R} \le \sum_{\ell=0}^{k} A^{k+1-\ell}\,\rho_*^2\big[|L\mathbf{u}|\big]_{\ell,\,0;V_R} + A^{k+1}\sum_{\ell=0}^{1}\big[|\mathbf{u}|\big]_{\ell,\ell;V_R}. \tag{2.56}$$

Proof: By induction over k. We know that estimate (2.47) implies (2.48). And estimate (2.48) clearly implies (2.56) for $k = 0$.

We suppose that (2.56) is proven for all $\kappa \le k-1$, i.e. we have for all $d,\ 2 \le d \le k+1$:

$$\big[|\mathbf{u}|\big]_{d,\,2;V_R} \le \sum_{\ell=0}^{d-2} A^{d-1-\ell}\,\rho_*^2\big[|L\mathbf{u}|\big]_{d-2,\,0;V_R} + A^{d-1}\sum_{\ell=0}^{1}\big[|\mathbf{u}|\big]_{\ell,\ell;V_R}. \tag{2.57}$$

Let $\beta = (\beta', 0)$ be a tangential multi-index of length k. Since now the boundary operators do not commute with ∂^β, we cannot apply directly (2.48) to $\partial^\beta \mathbf{u}$. Instead, we use the nested open set technique like in the proof of Lemma 2.4.2, starting by applying the estimate (2.47) to the function $\chi_{R-(k+2)\rho,\rho}\,\partial^\beta \mathbf{u}$. Let us abbreviate $\chi_{R-(k+2)\rho,\rho}$ by χ and $R-(k+1)\rho$ by R'. Noting that $D_j\mathbf{u} = 0$, $[D_j, \chi] \equiv 0$ and $T_j\mathbf{u} = 0$ on B'_R, we obtain

$$
\begin{aligned}
\|\partial^\beta \mathbf{u}\|_{2;V_{R'-\rho}} &\le A_0\Big\{ \|L\partial^\beta \mathbf{u}\|_{0;V_{R'}} + \|[L,\chi]\partial^\beta \mathbf{u}\|_{0;V_{R'}} + \sum_{j=1}^{N_0} \|\chi[D_j,\partial^\beta]\mathbf{u}\|_{\frac32;B'_{R'}} \\
&\qquad + \sum_{j=1}^{N_1}\Big(\|\chi[T_j,\partial^\beta]\mathbf{u}\|_{\frac12;B'_{R'}} + \|[T_j,\chi]\partial^\beta \mathbf{u}\|_{\frac12;B'_{R'}} + \|\chi\partial^\beta \mathbf{u}\|_{1;V_{R'}} \Big)\Big\} \\
&\le A'_0\Big\{ \|L\partial^\beta \mathbf{u}\|_{0;V_{R'}} + \|[L,\chi]\partial^\beta \mathbf{u}\|_{0;V_{R'}} + \sum_{j=1}^{N_0} \|\chi[D_j,\partial^\beta]\mathbf{u}\|_{2;V_{R'}} \\
&\qquad + \sum_{j=1}^{N_1}\Big(\|\chi[T_j,\partial^\beta]\mathbf{u}\|_{1;V_{R'}} + \|[T_j,\chi]\partial^\beta \mathbf{u}\|_{1;V_{R'}} + \|\chi\partial^\beta \mathbf{u}\|_{1;V_{R'}} \Big)\Big\}.
\end{aligned}
\tag{2.58}
$$

Here we have used, as in the proof of Lemma 2.4.2, the continuity of the trace operator from $\mathrm{H}^1(\mathbb{R}^n_+)$ into $\mathrm{H}^{\frac12}(\mathbb{R}^{n-1})$ and from $\mathrm{H}^2(\mathbb{R}^n_+)$ into $\mathrm{H}^{\frac32}(\mathbb{R}^{n-1})$. Recalling (1.41), we deduce

$$
\begin{aligned}
\|\partial^\beta \mathbf{u}\|_{2;V_{R'-\rho}} &\le A''_0\Big\{ \|\partial^\beta L\mathbf{u}\|_{0;V_{R'}} + \|[L,\partial^\beta]\mathbf{u}\|_{0;V_{R'}} + \sum_{\ell=0}^{1} \rho^{\ell-2}\|\partial^\beta \mathbf{u}\|_{\ell;V_{R'}} \\
&\qquad + \sum_{j=1}^{N_0}\sum_{\ell=0}^{2} \rho^{\ell-2}\|[D_j,\partial^\beta]\mathbf{u}\|_{\ell;V_{R'}} + \sum_{j=1}^{N_1}\sum_{\ell=0}^{1} \rho^{\ell-2}\|[T_j,\partial^\beta]\mathbf{u}\|_{\ell;V_{R'}} \Big\}.
\end{aligned}
\tag{2.59}
$$

We multiply this estimate by ρ^{k+2} and bound the three commutator terms: Lemma 2.6.2 used successively with $b = k+1$, k and $k-1$, and with $m = 2$ yields

$$
\begin{aligned}
\rho^{k+2}\|[L,\partial^\beta]\mathbf{u}\|_{V_{R'}} + \sum_{j=1}^{N_0}\sum_{\ell=0}^{2} \rho^{k+\ell}\|[D_j,\partial^\beta]\mathbf{u}\|_{\ell;V_{R'}} + \sum_{j=1}^{N_1}\sum_{\ell=0}^{1} \rho^{k+\ell}\|[T_j,\partial^\beta]\mathbf{u}\|_{\ell;V_{R'}} \\
\le c_2 \sum_{\ell=0}^{2}\sum_{d=0}^{k+1-\ell} (KR)^{k+2-\ell-d}\ \big[|\mathbf{u}|\big]_{d,2;V_R}
\end{aligned}
$$

with positive constants c_2 and K, independent of $\mathbf{u}$, R and β. Thus we have obtained

$$\rho^{k+2}\|\partial^\beta \mathbf{u}\|_{2;V_{R'-\rho}} \le A_0''\Big\{\rho^{k+2}\|\partial^\beta L\mathbf{u}\|_{0;V_{R'}} + \sum_{\ell=0}^{1}\rho^{k+\ell}\|\partial^\beta \mathbf{u}\|_{\ell;V_{R'}} + c_2\sum_{\ell=0}^{2}\sum_{d=0}^{k+1-\ell}(KR)^{k+2-\ell-d}\,\big[|\mathbf{u}|\big]_{d,2;V_R}\Big\}. \quad (2.60)$$

Taking the max over $\rho \in (0, \frac{R}{k+2}]$ finally gives the analogue of the interior estimate (1.48):

$$\big[|\mathbf{u}|\big]_{k+2,2;V_R} \le A_2\Big(\rho_*^2\big[|L\mathbf{u}|\big]_{k,0;V_R} + \big[|\mathbf{u}|\big]_{k+1,2;V_R} + \sum_{d=0}^{k}(KR)^{k-d}\,\big[|\mathbf{u}|\big]_{d,2;V_R}\Big). \quad (2.61)$$

From the inequalities (2.57) and (2.61) one deduces (2.56) for k as in the concluding arguments of the proof of Proposition 1.6.3. □

If the boundary B'_R is non-characteristic for L, we are now able to bound any derivative of $\mathbf{u}$, and obtain the final "analytic" a priori estimates:

Proposition 2.6.4 *Let L and the boundary operators T_j, D_j have analytic coefficients. We assume that (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$ and that (2.50) holds for the principal symbol $L^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi})$ of L for all $\mathbf{x} \in B'_{R_*}$. Then there exist two constants A, $B \ge 1$ such that for all $k,m \in \mathbb{N}$ with $m \le k$, for all $R \in (0, R_*]$, and for all $\mathbf{u} \in \mathbf{H}^2(V_R)$ satisfying the boundary conditions $T_j\mathbf{u} = 0$ and $D_j\mathbf{u} = 0$ on B'_R, there holds*

$$\big[|\mathbf{u}|\big]_{k+2,\,m+2;\,V_R} \le \sum_{\ell=0}^{k}\sum_{\nu=0}^{\min\{\ell,m\}} A^{k+1-\ell}B^{m+1-\nu}\,\rho_*^2\big[|L\mathbf{u}|\big]_{\ell,\,\nu;\,V_R} + A^{k+1}B^{m+1}\sum_{\ell\le 1}\big[|\mathbf{u}|\big]_{\ell,\,\ell;\,V_R}. \quad (2.62)$$

Proof: We prove (2.62) by induction over k and m for $(k,m) \in \mathbb{N}^2$ with $m \le k$. For $m = 0$ and any k, (2.62) is a direct consequence of (2.56) as soon as $B \ge 1$. Let us fix (k,m) and let us assume that (2.62) holds for any (κ,μ) satisfying either $\kappa = k$ and $\mu \le m-1$, or $\kappa \le k-1$ and $\mu \le \min\{\kappa, m\}$:

$$\big[|\mathbf{u}|\big]_{\kappa+2,\,\mu+2;\,V_R} \le \sum_{\ell=0}^{\kappa}\sum_{\nu=0}^{\mu} A^{\kappa+1-\ell}B^{\mu+1-\nu}\,\rho_*^2\big[|L\mathbf{u}|\big]_{\ell,\,\nu;\,V_R} + A^{\kappa+1}B^{\mu+1}\sum_{\ell\le 1}\big[|\mathbf{u}|\big]_{\ell,\,\ell;\,V_R}. \quad (2.63)$$

By assumption, condition (2.50) for $L^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi})$ is valid for all $\mathbf{x} \in B'_R$. This implies the following variable-coefficient version of (2.53)

$$\partial_n^2\mathbf{u} = M_{nn}^{-1}(\mathbf{x})L\mathbf{u} + \sum_{\substack{|\alpha|\le 2\\ \alpha_n\le 1}} N^\alpha(\mathbf{x})\partial^\alpha\mathbf{u}, \quad (2.64)$$

where the matrices M_{nn}^{-1} and N^{α} have smooth coefficients. Therefore estimate (2.54) still holds. Applying (2.54) in $V_{R-(k+2)\rho}$ to $\partial_n^m \partial^\beta \mathbf{u}$ with $\beta = (\beta', 0)$ and $|\beta| = k - m$, we obtain

$$\left|\mathbf{u}\right|_{k+2,\,m+2;\,V_{R-(k+2)\rho}} \le B_0 \Big\{ \left|L\mathbf{u}\right|_{k,\,m;\,V_{R-(k+2)\rho}} + \left|\mathbf{u}\right|_{k+2,\,m+1;\,V_{R-(k+2)\rho}} + \left|[L, \partial_n^m \partial^\beta]\mathbf{u}\right|_{0,\,0;\,V_{R-(k+2)\rho}} \Big\}.$$

We multiply the above inequality by ρ^{k+2}, use Lemma 2.6.2 to bound the commutator and take the max over ρ:

$$\begin{aligned}
\big[|\mathbf{u}|\big]_{k+2,\,m+2;\,V_R} &\le B_0 \Big\{ \rho_*^2 \big[|L\mathbf{u}|\big]_{k,\,m;\,V_R} + \big[|\mathbf{u}|\big]_{k+2,\,m+1;\,V_R} \\
&\qquad + c_2 \sum_{d=0}^{k+1} (KR)^{k+2-d} \big[|\mathbf{u}|\big]_{d,\,\min\{m+2,d\};\,V_R} \Big\}. \\
&\le B_0 \Big\{ \rho_*^2 \big[|L\mathbf{u}|\big]_{k,\,m;\,V_R} + c_2 (KR)^{k+1} \big(KR \big[|\mathbf{u}|\big]_{0,\,0;\,V_R} + \big[|\mathbf{u}|\big]_{1,\,1;\,V_R} \big) \\
&\qquad + \big[|\mathbf{u}|\big]_{k+2,\,m+1;\,V_R} + c_2 \sum_{\kappa=0}^{k-1} (KR)^{k-\kappa} \big[|\mathbf{u}|\big]_{\kappa+2,\,\min\{m,\kappa\}+2;\,V_R} \Big\}.
\end{aligned}$$

By the induction hypothesis, we can use (2.63) with $(\kappa, \mu) = (k, m-1)$ for the estimation of $\big[|\mathbf{u}|\big]_{k+2,\,m+1;\,V_R}$ and with $(\kappa, \min\{m, \kappa\})$, $\kappa = k-1, \ldots, 0$, for $\big[|\mathbf{u}|\big]_{\kappa+2,\,\min\{m,\kappa\}+2;\,V_R}$.

We finally obtain (2.62), provided that A and B are such that

$$\frac{B_0}{B} + B_0 c_2 \sum_{j=1}^{k+1} \Big(\frac{KR}{A}\Big)^j \le 1.$$

For this we can choose any A and B such that $B \ge 2B_0$, $A \ge 2KRB_0c_2$. □

2.6.b Inhomogeneous boundary conditions

The procedure followed in the previous paragraph can be applied in the same way even if no homogeneous boundary conditions are satisfied. In this section, we will state the corresponding results and indicate how the proofs have to be modified to cover the general case.

We start by introducing weighted seminorms on the boundary in analogy to Notation 2.6.1.

Notation 2.6.5 Using Notation 2.5.1, we set for $k \in \mathbb{N}$:

$$\rho_*^{\frac{1}{2}} \big[|g|\big]_{k,\frac{3}{2};B'_R} := \max_{0<\rho\le\frac{R}{2(k+1)}} \rho^{\frac{1}{2}+k} |g|_{k,\frac{3}{2};B'_{R-(k+1)\rho}},$$

$$\rho_*^{\frac{3}{2}} \big[|h|\big]_{k,\frac{1}{2};B'_R} := \max_{0<\rho\le\frac{R}{2(k+1)}} \rho^{\frac{3}{2}+k} |h|_{k,\frac{1}{2};B'_{R-(k+1)\rho}}.$$

△

For the almost tangential derivatives, we find the analogue of Proposition 2.6.3.

Proposition 2.6.6 *Let L and the boundary operators T_j, D_j have analytic coefficients. We assume that estimate (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$. Then there exists a constant $A \ge 1$ such that for all $k \in \mathbb{N}$, for all $R \in (0, R_*]$, and for all $\mathbf{u} \in \mathbf{H}^2(V_R)$ there holds*

$$\big[|\mathbf{u}|\big]_{k+2,\,2;V_R} \le \sum_{\ell=0}^{k} A^{k+1-\ell}\Big(\rho_*^2\big[|L\mathbf{u}|\big]_{\ell,\,0;V_R} + \rho_*^{\frac{3}{2}}\big[|T\mathbf{u}|\big]_{\ell,\frac{1}{2};B'_R} + \rho_*^{\frac{1}{2}}\big[|D\mathbf{u}|\big]_{\ell,\frac{3}{2};B'_R}\Big) + A^{k+1}\sum_{\ell=0}^{1}\big[|\mathbf{u}|\big]_{\ell,\ell;V_R}. \quad (2.65)$$

Proof: The proof of Proposition 2.6.3 goes through in the present situation if we add boundary terms in some of the formulas. Thus, equation (2.57) reads now

$$\big[|\mathbf{u}|\big]_{d,\,2;V_R} \le \sum_{\ell=0}^{d-2} A^{d-1-\ell}\Big(\rho_*^2\big[|L\mathbf{u}|\big]_{d-2,\,0;V_R} + \rho_*^{\frac{3}{2}}\big[|T\mathbf{u}|\big]_{d-2,\frac{1}{2};B'_R} + \rho_*^{\frac{1}{2}}\big[|D\mathbf{u}|\big]_{d-2,\frac{3}{2};B'_R}\Big) + A^{d-1}\sum_{\ell=0}^{1}\big[|\mathbf{u}|\big]_{\ell,\ell;V_R}. \quad (2.66)$$

In eq. (2.58), we have to add the terms $\|\chi\partial^{\beta'}D_j\mathbf{u}\|_{\frac{3}{2};B'_{R'}}$ in the first sum and $\|\chi\partial^{\beta'}T_j\mathbf{u}\|_{\frac{1}{2};B'_{R'}}$ in the second sum on the right hand sides.

In eq. (2.59), the corresponding extra terms on the right hand side are $\rho^{-\frac{3}{2}}\|D_j\mathbf{u}\|_{k,\frac{3}{2};B'_{R'}}$ and $\rho^{-\frac{1}{2}}\|T_j\mathbf{u}\|_{k,\frac{1}{2};B'_{R'}}$, respectively.

In equation (2.60), the extra boundary terms on the right hand side are

$$\rho^{k+\frac{3}{2}}\|T\mathbf{u}\|_{k,\frac{1}{2};B'_{R'}} + \rho^{k+\frac{1}{2}}\|D\mathbf{u}\|_{k,\frac{3}{2};B'_{R'}},$$

and, finally, in equation (2.61), we have to add

$$\rho_*^{\frac{3}{2}}\big[|T\mathbf{u}|\big]_{k,\frac{1}{2};B'_R} + \rho_*^{\frac{1}{2}}\big[|D\mathbf{u}|\big]_{k,\frac{3}{2};B'_R}$$

in the parenthesis on the right hand side. □

The main result on the analytic a priori estimates is the analogue of Proposition 2.6.4.

Proposition 2.6.7 *Let L and the boundary operators T_j, D_j have analytic coefficients. We assume that (2.47) holds for any $\mathbf{u} \in \mathbf{H}^2_0(V_{R_*}, B'_{R_*})$ and that (2.50) holds for the principal symbol $L^{\mathsf{pr}}(\mathbf{x};\boldsymbol{\xi})$ of L for all $\mathbf{x} \in B'_{R_*}$. Then there exist two constants A, $B \geq 1$ such that for all $k, m \in \mathbb{N}$ with $m \leq k$, for all $R \in (0, R_*]$, and for all $\mathbf{u} \in \mathbf{H}^2(V_R)$ there holds*

$$\big[|\mathbf{u}|\big]_{k+2,\,m+2;\,V_R} \leq \sum_{\ell=0}^{k} A^{k+1-\ell}\Big\{ \sum_{\nu=0}^{\min\{\ell,m\}} B^{m+1-\nu}\,\rho_*^2\big[|L\mathbf{u}|\big]_{\ell,\,\nu;\,V_R} + B^{m+1}\Big(\rho_*^{\frac{3}{2}}\big[|T\mathbf{u}|\big]_{\ell,\,\frac{1}{2};\,B'_R} + \rho_*^{\frac{1}{2}}\big[|D\mathbf{u}|\big]_{\ell,\,\frac{3}{2};\,B'_R}\Big)\Big\} + A^{k+1}B^{m+1}\sum_{\ell\leq 1}\big[|\mathbf{u}|\big]_{\ell,\,\ell;\,V_R}. \tag{2.67}$$

Proof: The proof of Proposition 2.6.4 can be repeated in the case of inhomogeneous boundary conditions almost without change. The only place where boundary terms appear is in the induction hypothesis, equation (2.63), where one has to add

$$B^{\mu+1}\sum_{\ell=0}^{\kappa} A^{\kappa+1-\ell}\Big(\rho_*^{\frac{3}{2}}\big[|T\mathbf{u}|\big]_{\ell,\,\frac{1}{2};\,B'_R} + \rho_*^{\frac{1}{2}}\big[|D\mathbf{u}|\big]_{\ell,\,\frac{3}{2};\,B'_R}\Big)$$

on the right hand side. □

Remark 2.6.8 For later reference, we note that the constants A and B in estimates (2.62) and (2.67) depend continuously on R_*, on the constant A_0 in (2.47), on the maximum of the coefficients of the matrix $M_{nn}^{-1}(\mathbf{x})$ in (2.64) and on the analyticity modulus of the coefficients of L and D_j, T_j on V_{R_*} and B'_{R_*}, compare Remark 1.6.5 and (1.50). △

2.7 Analytic regularity up to the boundary

Our preparation for the proof of the analytic version of Theorem 2.3.2 is now complete. Let us recall that $\mathbf{A}(\Omega)$ denotes the class of N-component analytic functions on $\overline{\Omega}$. The analytic shift theorem states that the solutions of an elliptic boundary problem with analytic data are analytic. Analytic data means analytic domain, coefficients and right hand sides. For our general class of "mixed" boundary value problems $\mathbb{A} = \{L, T, D\}$ with $T = (T_{\mathsf{s}})_{\mathsf{s}\in\mathscr{S}}$ and $D = (D_{\mathsf{s}})_{\mathsf{s}\in\mathscr{S}}$, we understand by analytic boundary data $\mathbf{g} = (\mathbf{g}_{\mathsf{s}})_{\mathsf{s}\in\mathscr{S}}$ and $\mathbf{h} = (\mathbf{h}_{\mathsf{s}})_{\mathsf{s}\in\mathscr{S}}$ vector functions such that all components of $\mathbf{g}_{\mathsf{s}}$ and $\mathbf{h}_{\mathsf{s}}$ are analytic.

Theorem 2.7.1 *Let Ω be a domain in $\mathbb{R}^n$ or, more generally, in an analytic manifold M of dimension n. Let $(\partial_{\mathsf{s}}\Omega)_{\mathsf{s}\in\mathscr{S}}$ be the connected components of $\partial\Omega$. Let Γ be an analytic part of the boundary of Ω. Let L be a $N \times N$ system of second order operators with* *analytic coefficients over $\Omega\cup\Gamma$. Let $C=\{T,D\}$ be an admissible set of boundary operators, cf. Notation 2.2.20, on Γ, with analytic coefficients. We assume that the boundary value system $\mathbb{A}=\{L,T,D\}$ is elliptic on $\Omega\cup\Gamma$ (cf. Definition 2.2.31).*

(i) Let two bounded subdomains $\Omega_1=\mathcal{U}_1\cap\Omega$ and $\Omega_2=\mathcal{U}_2\cap\Omega$ be given with $\mathcal{U}_1$ and $\mathcal{U}_2$ open in M and $\overline{\mathcal{U}}_1\subset\mathcal{U}_2$. We assume that $\Gamma_2:=\partial\Omega_2\cap\partial\Omega$ is contained in Γ. Then there exists a constant A such that any solution $\mathbf{u}\in\mathbf{H}^2(\Omega_2)$ of (2.43) satisfies for all $k\in\mathbb{N}$, $k\geq 2$ the improved a priori estimates ("finite analytic estimates")

$$\frac{1}{k!}|\mathbf{u}|_{k;\Omega_1}\leq A^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\Big(|\mathbf{f}|_{\ell;\Omega_2}+\|\mathbf{g}\|_{\ell+\frac12;\Gamma_2}+\|\mathbf{h}\|_{\ell+\frac32;\Gamma_2}\Big)+\|\mathbf{u}\|_{1;\Omega_2}\Big\}. \tag{2.68}$$

If in addition the right hand sides $\mathbf{f}$, $\mathbf{g}=(\mathbf{g}_\mathbf{s})_{\mathbf{s}\in\mathscr{S}}$ and $\mathbf{h}=(\mathbf{h}_\mathbf{s})_{\mathbf{s}\in\mathscr{S}}$ are such that $\mathbf{f}\in\mathbf{A}(\Omega_2)$ and all components of $\mathbf{g}_\mathbf{s}$ and $\mathbf{h}_\mathbf{s}$ are in $\mathsf{A}(\Gamma_2\cap\partial_\mathbf{s}\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega_1)$.

(ii) If Ω is bounded and $\Gamma=\partial\Omega$ and if the right hand side is such that $\mathbf{f}\in\mathbf{A}(\Omega)$ and all components of $\mathbf{g}_\mathbf{s}$ and $\mathbf{h}_\mathbf{s}$ are in $\mathsf{A}(\partial_\mathbf{s}\Omega)$, then any solution $\mathbf{u}\in\mathbf{H}^2(\Omega)$ of problem (2.44) belongs to $\mathbf{A}(\Omega)$.

Proof: The key point is the proof of the analytic estimate (2.68). Let Ω be a domain in an analytic manifold M, and $\Omega_m=\mathcal{U}_m\cap\Omega$, $m=1,2$ with $\overline{\mathcal{U}}_1\subset\mathcal{U}_2$. Let $\mathbf{x}_0$ be a fixed point in $\overline{\Omega}_1$.

• If $\mathbf{x}_0$ belongs to Ω, it is an interior point: There exist two neighborhoods $\mathcal{U}_1(\mathbf{x}_0)$ and $\mathcal{U}_2(\mathbf{x}_0)$ of $\mathbf{x}_0$, such that $\overline{\mathcal{U}}_1(\mathbf{x}_0)\subset\mathcal{U}_2(\mathbf{x}_0)\subset\Omega_2$ and a positive constant $A_{\mathbf{x}_0}$ so that there holds the local estimate (1.54), shown at the end of the proof of Theorem 1.3.3:

$$\frac{1}{k!}\,|\mathbf{u}|_{k;\mathcal{U}_1(\mathbf{x}_0)}\leq A_{\mathbf{x}_0}^{k+1}\Big(\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\,|L\mathbf{u}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)}+\sum_{\ell=0}^{1}|\mathbf{u}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)}\Big). \tag{2.69}$$

• If $\mathbf{x}_0$ belongs to $\partial\Omega$, then by assumption, $\mathbf{x}_0$ is in Γ which is an analytic part of the boundary $\partial\Omega$. There exist a neighborhood $\mathcal{U}_2(\mathbf{x}_0)$ of $\mathbf{x}_0$ in the manifold M, $\mathcal{U}_2(\mathbf{x}_0)\subset\mathcal{U}_2$, a ball B_{R_*} in $\mathbb{R}^n$ and an analytic map ϕ from $\mathcal{U}_2(\mathbf{x}_0)$ onto B_{R_*} such that $\phi(\mathcal{U}_2(\mathbf{x}_0)\cap\Omega)$ is the half-ball V_{R_*} and $\phi(\mathcal{U}_2(\mathbf{x}_0)\cap\partial\Omega)$ is the $n-1$ dimensional ball B'_{R_*}. We can assume that the projector-valued functions Π^T and Π^D are diagonalizable on $\mathcal{U}_2(\mathbf{x}_0)$ (see (2.39)). The boundary value problem (2.43) is transformed into

$$\breve{L}\breve{\mathbf{u}}=\breve{\mathbf{f}}\text{ in }V_{R_*}\quad\text{with}\quad\breve{\mathbf{u}}\circ\phi=\mathbf{u}\text{ and }\breve{\mathbf{f}}\circ\phi=\mathbf{f},$$

with boundary conditions $\breve{T}_j\breve{\mathbf{u}}=\breve{g}_j$ and $\breve{D}_j\breve{\mathbf{u}}=\breve{h}_j$ on B'_{R_*}. The operators $\breve{L},\breve{T}_j,\breve{D}_j$ have analytic coefficients and form an elliptic system.

By Corollary 2.2.16, estimate (2.47) holds in V_R. Moreover, condition (2.50) at 0 is a consequence of the ellipticity of $\breve{L}$. We can then choose a sufficiently small positive number

R_0 and apply Proposition 2.6.7 which implies that here exist positive numbers A and B so that the following estimate holds for all $R \le R_0$ and all $k \in \mathbb{N}$, $k \ge 2$

$$[|\breve{\mathbf{u}}|]_{k,k;\,V_R} \le A^{k+1}B^{k+1}\Big\{\sum_{\ell=0}^{k} A^{-\ell}\big(\sum_{\nu=0}^{\ell} B^{-\nu}\rho_*^2[|\breve{L}\breve{\mathbf{u}}|]_{\ell,\,\ell;\,V_R} + \rho_*^{\frac{3}{2}}[|\breve{T}\breve{\mathbf{u}}|]_{\ell,\,\frac12;\,B'_R} + \rho_*^{\frac12}[|\breve{D}\breve{\mathbf{u}}|]_{\ell,\,\frac32;\,B'_R}\big) + \sum_{\ell\le 1}[|\breve{\mathbf{u}}|]_{\ell,\,\ell;V_R}\Big\}.$$

Simplifying, we can write this with a constant $\widetilde{A}$ as

$$[|\breve{\mathbf{u}}|]_{k,k;\,V_R} \le \widetilde{A}^{k+1}\Big\{\sum_{\ell=0}^{k}\big(\rho_*^2[|\breve{L}\breve{\mathbf{u}}|]_{\ell,\,\ell;\,V_R} + \rho_*^{\frac32}[|\breve{T}\breve{\mathbf{u}}|]_{\ell,\,\frac12;\,B'_R} + \rho_*^{\frac12}[|\breve{D}\breve{\mathbf{u}}|]_{\ell,\,\frac32;\,B'_R}\big) + \sum_{\ell\le 1}[|\breve{\mathbf{u}}|]_{\ell,\,\ell;V_R}\Big\}. \quad (2.70)$$

Similarly as the inequalities (1.53) for the case of the ball, we have estimates for the half-ball linking the weighted semi-norms on V_R and the semi-norms on V_R and $V_{R/2}$. We also have the analogous estimates on the boundary between the weighted seminorms defined in Notation 2.6.5 and the standard Sobolev norms, namely:

$$\begin{aligned} \rho_*^{\frac12}[|\breve{g}|]_{\ell,\,\frac32;\,B'_R} &\le \big(\tfrac{R}{2(\ell+1)}\big)^{\frac12+\ell}\|\breve{g}\|_{\ell+\frac32;\,B'_R}\,; \\ \rho_*^{\frac32}[|\breve{h}|]_{,\,\frac12;\,B'_R} &\le \big(\tfrac{R}{2(\ell+1)}\big)^{\frac32+\ell}\|\breve{h}\|_{\ell+\frac12;\,B'_R}\,. \end{aligned}$$

Therefore, with the help of the Stirling formula, we deduce from (2.70) that, with a new constant $\widetilde{A}$ independent of k and of $R \le R_0$, we have:

$$\frac{R^k}{k!}\,|\breve{\mathbf{u}}|_{k;\,V_{R/2}} \le \widetilde{A}^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{R^\ell}{\ell!}\,\Big(|\breve{\mathbf{f}}|_{\ell;\,V_R} + \|\breve{\mathbf{g}}\|_{\ell+\frac12;B'_R} + \|\breve{\mathbf{h}}\|_{\ell+\frac32;B'_R}\Big) + \sum_{\ell=0}^{1}\|\breve{\mathbf{u}}\|_{\ell;\,V_R}\Big\}.$$

Note that for this kind of estimate, the precise powers $\ell+1/2$ or $\ell+3/2$ and the question whether we used Sobolev seminorms or norms on the boundary, do not play any role.

The constants $\widetilde{A}$ and R_0 may still depend on the point $\mathbf{x}_0$ in $\overline{\Omega}_1$. We fix $R = R(\mathbf{x}_0) \le \min\{R_0, R_*\}$ and denote by $\mathcal{U}_1(\mathbf{x}_0)$ the pull-back $\phi^{-1}(B_{R(\mathbf{x}_0)/2})$. Combining the above estimate with the estimate (1.22) on the analytic change of variables, which holds by interpolation correspondingly also for the boundary Sobolev norms in the trace spaces, we obtain

$$\frac{1}{k!}\,|\mathbf{u}|_{k;\mathcal{U}_1(\mathbf{x}_0)\cap\Omega} \le A_{\mathbf{x}_0}^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\ \Big(|\mathbf{f}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)\cap\Omega} + \|\mathbf{g}\|_{\ell+\frac{1}{2};\mathcal{U}_2(\mathbf{x}_0)\cap\partial\Omega} + \|\mathbf{h}\|_{\ell+\frac{3}{2};\mathcal{U}_2(\mathbf{x}_0)\cap\partial\Omega}\Big) + \sum_{\ell=0}^{1}|\mathbf{u}|_{\ell;\mathcal{U}_2(\mathbf{x}_0)\cap\Omega}\Big\}. \tag{2.71}$$

Here $A_{\mathbf{x}_0}$ is a positive number independent of k and $\mathbf{u}$.

Letting finally $\mathbf{x}_0$ vary over $\overline{\Omega}_1$, we can extract a finite covering of the compact set $\overline{\Omega}_1$ by open sets $\mathcal{U}_1(\mathbf{x}_0)$. The proof of (2.68) then follows by combining a finite number of estimates (2.69) and (2.71).

The remaining statements of Theorem 2.7.1 are obvious consequences of (2.71). □

2.8 Extended smooth domains

Until now, we have considered domains Ω that are smooth in the usual sense. By including also subdomains of smooth manifolds, we have, in fact, covered a rather large class of examples that goes substantially beyond the class of smooth bounded subdomains of $\mathbb{R}^n$. There exists, however, a class of domains in $\mathbb{R}^n$ or in a fixed smooth manifold M (the sphere $\mathbb{S}^{n-1}$ being an important example later on) for which all the elliptic regularity results of this chapter are valid in an obvious way, but which are not adequately described by this standard notion of smoothness. They are not even Lipschitz domains. To recognize their smoothness, one has to embed their boundary in a manifold different from the one that contains Ω as a subdomain. Let us give an example in $\mathbb{R}^2$:

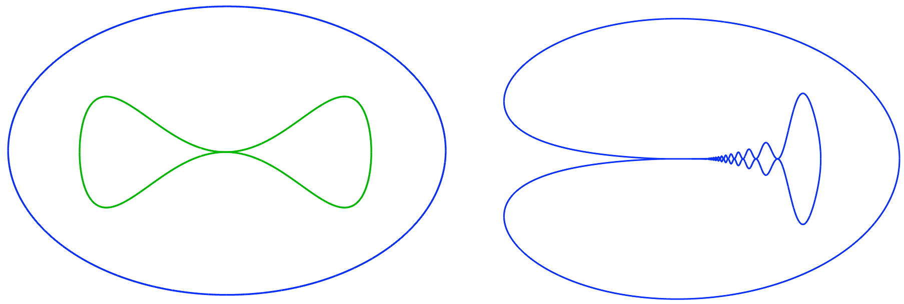
Let Ω be the domain between the curve Γ in $\mathbb{R}^2$ described by the equation

$$y^2 = x^4 - x^6$$

and the circle of radius two (see Figure 2.5(a)). The (analytic) curve Γ has a double point at the origin, and therefore Ω is not a smooth domain in the usual sense, because it does not lie locally on one side of its boundary. In order to embed Ω as a smooth domain into a smooth manifold, we have to count the origin as a double boundary point, thus moving out of $\mathbb{R}^2$. It is clear, however, that in doing so, we recover all the regularity results, including analytic regularity, presented in this chapter.

In the specific example, we can explicitly exhibit a manifold that contains Ω as a smooth subdomain, namely one that is a double cover of a neighborhood of the origin and smooth in a neighborhood of Ω, for instance the Riemann surface of the function $\sqrt{1-2z^2}$. In more general cases, a more intrinsic construction, involving only an extended definition of the boundary of Ω, is desirable, and one can do it as follows:

We assume that Ω is a bounded domain (i.e. an open and connected set) in a complete Riemannian manifold M. We equip Ω with the metric of the intrinsic *geodesic distance*



(a) (b)

Figure 2.5: Extended smooth domains: analytic (a) and $\mathscr{C}^\infty$ (b)

d defined by

$$d(\mathbf{x}, \mathbf{y}) = \inf\{ \text{ length of } \gamma \mid \gamma \in C_{\mathbf{x},\mathbf{y}}\} \tag{2.72}$$

where $C_{\mathbf{x},\mathbf{y}}$ is the set of all curves of finite length connecting $\mathbf{x}$ and $\mathbf{y}$ inside Ω. In the case of $M = \mathbb{R}^n$, one can take the infimum over polygonal lines.

This metric space (Ω, d) has a *completion*, defined in the usual way as the set of equivalence classes of Cauchy sequences.

Notation 2.8.1 Let Ω be a domain in $\mathbb{R}^n$ or in a smooth manifold. We denote

⋆ by $\overline{\Omega}^*$ its ***geodesic completion***, defined by Cauchy sequences for the intrinsic geodesic distance d in Ω (2.72),

⋆ by $\partial_*\Omega$ its ***unfolded boundary***, i.e.

$$\partial_*\Omega = \overline{\Omega}^* \setminus \Omega$$

△

We will now discuss some of the properties of the geodesic completion. As this discussion belongs to the field of elementary topology, we leave most of the details to the reader.

Since the intrinsic geodesic distance d dominates the ordinary distance in M, the Cauchy sequences defining $\overline{\Omega}^*$ converge also in M, and thus there is a natural projection $\pi : \overline{\Omega}^* \to \overline{\Omega}$ which is the identity inside Ω and which is continuous (Lipschitz continuous with Lipschitz constant 1) on $\overline{\Omega}^*$. In the general case, there hold the inclusions

$$\Omega \subset \pi\overline{\Omega}^* \subset \overline{\Omega}\,. \tag{2.73}$$

If Ω is a Lipschitz domain, then the geodesic distance in Ω and the ordinary distance are equivalent, hence in this case the geodesic completion coincides with the closure and the unfolded boundary coincides with the ordinary boundary.

On the other hand, it is easy to construct examples, even in $\mathbb{R}^2$, showing that the following possibilities exist:

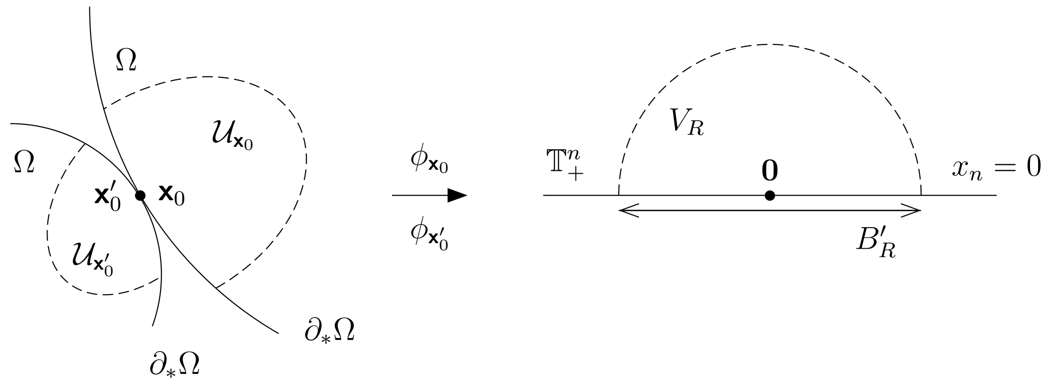


Figure 2.6: Local diffeomorphisms $\phi_{\mathbf{x}_0}$, $\phi_{\mathbf{x}'_0}$ for points $\mathbf{x}_0$, $\mathbf{x}'_0$ such that $\pi(\mathbf{x}_0) = \pi(\mathbf{x}'_0)$.

- ⋆ A bounded domain Ω which is unbounded with respect to the geodesic distance d,
- ⋆ A bounded domain Ω for which $\overline{\Omega}^*$ is not compact,
- ⋆ A bounded domain Ω for which both inclusions in (2.73) are strict, so that $\pi(\partial_*\Omega)$ is not closed,
- ⋆ A bounded domain Ω and a boundary point $\mathbf{x} \in \partial\Omega$ with infinite multiplicity: The set of preimages $\pi^{-1}(\mathbf{x})$ is infinite.

All of these are pathologies that we do not want to consider for our smooth domains. We will therefore make the general assumption that $\overline{\Omega}^*$ is compact. This does not yet imply that all boundary points are of finite multiplicity, but as soon as some regularity assumption is added, finite multiplicity will follow.

The definition of smoothness that we shall adopt basically states that $\overline{\Omega}^*$ is a smooth manifold with boundary.

Definition 2.8.2 *We say that the subdomain Ω of the smooth manifold M is* ***smooth in the extended sense*** *if $\overline{\Omega}^*$ is compact and for any point $\mathbf{x}_0$ in the unfolded boundary $\partial_*\Omega$ there exists, cf. Figure 2.6,*

- ⋆ *a neighborhood $\mathcal{U}_{\mathbf{x}_0}$ of $\mathbf{x}_0$ in $\overline{\Omega}^*$,*
- ⋆ *a number $R > 0$ defining the ball B_R centered at $\mathbf{0}$ with radius R, the half-ball $V_R = B_R \cap \{x_n > 0\}$ and $B'_R = \partial V_R \cap \{x_n = 0\}$,*
- ⋆ *a "smooth local diffeomorphism" $\phi_{\mathbf{x}_0}$ which sends bijectively $\mathcal{U}_{\mathbf{x}_0}$ onto $V_R \cup B'_R$ and $\mathcal{U}_{\mathbf{x}_0} \cap \partial\Omega$ onto B'_R, and so that $\phi_{\mathbf{x}_0}(\mathbf{x}_0) = \mathbf{0}$.*

Here "smooth" means that all derivatives of the inverse $\phi_{\mathbf{x}_0}^{-1}$ have continuous extensions from the open set V_R to $V_R \cup B'_R$, and "local diffeomorphism" means that the Jacobian matrix $\nabla\phi_{\mathbf{x}_0}^{-1}(\breve{\mathbf{x}})$ is invertible for any point $\breve{\mathbf{x}} \in V_R \cup B'_R$.

• *If moreover $\phi_{\mathbf{x}_0}^{-1}$ can be chosen analytic on $V_R \cup B'_R$ for all $\mathbf{x}_0 \in \partial_*\Omega$, we speak of analytic regularity of Ω. Note that in this situation, the neighborhood $\mathcal{U}_{\mathbf{x}_0} \cap \partial_*\Omega$ of $\mathbf{x}_0$ in the unfolded boundary is an analytic submanifold of $\mathbb{R}^n$ or M.*

Note that prior to this definition, the completion $\overline{\Omega}^*$ did not have a differentiable structure, only its interior Ω did. This is the reason why we have to use the inverse diffeomorphism in Definition 2.8.2, because $\phi_{\mathbf{x}_0}^{-1}$ is defined on the set $V_R \cup B'_R$ which does have a differentiable structure.

This notion of smoothness still allows examples of domains that have an infinite number of isolated multiple points on the boundary, in addition to whole subcurves of the boundary consisting of multiple points (see Figure 2.5(b)). In the smooth case, however, the multiplicity of a point on the boundary is at most two. This can be seen as follows:

Suppose that $\mathbf{x}_j$, $j = 1, 2, 3$, are distinct points in $\pi^{-1}(\mathbf{x})$. Let ρ_0 be their minimal mutual geodesic distance. Then using the local diffeomorphisms $\phi_{\mathbf{x}_j}$ with the half-balls V_{R_j} and choosing some ρ such that

$$0 < \rho < \min\{\rho_0/2, R_1, R_2, R_3\},$$

we obtain that the three open sets $\phi_{\mathbf{x}_j}^{-1}(V_\rho)$ of $\mathbb{R}^n$ (or M) are disjoint, they have a common boundary point $\mathbf{x}$, and each one has smooth boundary in the neighborhood of $\mathbf{x}$. This is clearly impossible.

For our extended smooth domains, *all the elliptic regularity results shown in this chapter remain valid*. One can see this on one hand by revisiting the techniques used in the proofs, and noticing in particular that our definition of Sobolev spaces on general domains in Chapter 1 was already given in an intrinsic way well adapted to the definition of the class of extended smooth domains. See in particular the definition (1.5) of fractional Sobolev-Slobodeckii norms. Thus tools like trace theorems and trace liftings will be available in an obvious way.

On the other hand, a more global argument can be made that uses the fact that our domains that are smooth in the extended sense can also be considered as domains that are smooth in the standard sense if one embeds them into a larger manifold. One can construct the *double* $\widetilde{\Omega}$ of the manifold with boundary $\overline{\Omega}^*$ and give it the structure of a smooth or analytic manifold (see for example [94, §5.12] for a presentation of the boundaryless double of a manifold with boundary). Then $\widetilde{\Omega}$ is a smooth compact manifold without boundary, Ω is a smooth subdomain whose closure in $\widetilde{\Omega}$ is $\overline{\Omega}^*$ and whose boundary is the unfolded boundary $\partial_*\Omega$. This implies immediately the validity of the regularity results of this chapter.

The problem with the abstract point of view needed for this global argument is that it implies the construction of another manifold, even if the original domain was a subdomain of $\mathbb{R}^n$ or of a standard manifold like $\mathbb{S}^{n-1}$. It is in particular this latter situation that will be considered later on in Part II where smooth domains G in $\mathbb{S}^{n-1}$ will appear as bases of smooth cones. It is then preferable, as we are proposing here, to stay inside the sphere $\mathbb{S}^{n-1}$ and simply introduce a generalized notion of boundary in the form of the unfolded boundary ∂_*G, instead of introducing an abstract manifold providing a double cover of some parts of $\mathbb{S}^{n-1}$.

In any case, with the definition of the differentiable structure on the manifold $\overline{\Omega}^*$, it is now clear what the space $\mathscr{C}^\infty(\overline{\Omega}^*)$ of smooth functions on $\overline{\Omega}^*$ means. Let us end this discussion with the remark that the Sobolev imbedding theorem implies that with our definition of the Sobolev spaces, for any domain Ω smooth in the extended sense there holds

$$\mathscr{C}^\infty(\overline{\Omega}^*) = \mathsf{H}^\infty(\Omega) = \bigcap_{m\geq 0} \mathsf{H}^m(\Omega) \,. \tag{2.74}$$

Chapter 3

Variational formulations

Introduction

Many stationary or periodic physical systems can be modeled by elliptic equations in *variational form*. Such a form provides a *unique solution* to the problem, in general, and, consequently, a natural way of discretizing these equations by a *Galerkin formulation*, too. Variational forms are defined on *variational spaces* which, in the situation of second order equations, are subspaces of $\mathbf{H}^1(\Omega)$. Boundary conditions arise from two sources: The definition of the variational space may contain boundary conditions ("essential boundary conditions"), and further conditions may appear from integration by parts ("natural boundary conditions").

Under a simple condition on the variational form (the *coercivity*) the set of these boundary conditions is complementing the equations inside the domain. In this chapter, we recall these notions and link them to the interior estimates and estimates up to the boundary which are proved in the previous chapter. The main issue is how to go from the variational regularity, which is simply $\mathbf{H}^1$, to the basic local $\mathbf{H}^2$ regularity which is the starting point of our analysis in Chapters 1 and 2. For the same reason (the discrepancy between standard basic regularity and variational regularity) the results of [4, 5] cannot be applied to variational solutions right away.

Plan of Chapter 3

§1 Sesquilinear forms and variational spaces, essential and natural boundary conditions.

§2 Coercivity and strong coercivity. Ellipticity and covering condition for boundary value problems issued from a coercive variational formulation.

§3 Solvability of coercive variational problems.

§4 Regularity of variational solutions: The $\mathbf{H}^2$ regularity for $\mathbf{L}^2$ interior right hand side allows to deduce from Chapter 2 higher Sobolev and analytic regularity for solutions of coercive variational problems with sufficiently smooth data. Lower Sobolev regularity ($\mathbf{H}^s$ with $1 < s < 2$) is proved for less regular data.

§5 Extension to Robin type boundary conditions.

Essentials

The boundary value problems which were considered in Chapter 2 are determined by the data of an interior system L inside the domain and operators (T, D) on its boundary. The variational problems which we consider now are a subset of these problems: They are determined by

- an integro-differential *sesquilinear form* a of order 1 with smooth coefficients

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{|\alpha|\le 1}\sum_{|\gamma|\le 1}\int_\Omega a_{ij}^{\alpha\gamma}(\mathbf{x})\,\partial_{\mathbf{x}}^{\alpha}u_j(\mathbf{x})\,\partial_{\mathbf{x}}^{\gamma}\overline{v}_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{3.a}$$

  defined for $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{H}^1(\Omega) = \mathrm{H}^1(\Omega)^N$,

- a subspace $\mathbf{V}$ of $\mathbf{H}^1(\Omega)$ determined by *essential boundary conditions*

$$\mathbf{V} = \{\mathbf{u}\in\mathbf{H}^1(\Omega) : \ \Pi^{\mathsf{D}}\gamma_0\mathbf{u} = 0 \ \text{ on } \ \partial\Omega\}\,, \tag{3.b}$$

  where Π^{D} is a smooth field of orthogonal projection operators $\mathbb{C}^N \to \mathbb{C}^N$.

For a right hand side $\mathbf{q}$ given in the dual space $\mathbf{V}'$ of $\mathbf{V}$, we denote by $\langle\mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}}$ the extension of the duality pairing $(\mathbf{q},\mathbf{v}) \mapsto \int_\Omega \mathbf{q}\cdot\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x}$ of $\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ to $\mathbf{V}'\times\mathbf{V}$. The associated variational problem is written as

$$\textit{Find}\quad \mathbf{u}\in\mathbf{V}\quad \textit{such that}\quad \forall\mathbf{v}\in\mathbf{V},\quad \mathsf{a}(\mathbf{u},\mathbf{v}) = \big\langle\mathbf{q},\mathbf{v}\big\rangle_{\overline{\Omega}}\,. \tag{3.c}$$

From the determination of a and $\mathbf{V}$, we deduce the following expressions for interior and boundary operators

- the interior operator

$$L = \big(L_{ij}\big)_{1\le i,j\le N} \quad\text{with}\quad L_{ij} = \sum_{|\alpha|\le 1}\sum_{|\gamma|\le 1}(-1)^{|\gamma|}\partial_{\mathbf{x}}^{\gamma}a_{ij}^{\alpha\gamma}(\mathbf{x})\,\partial_{\mathbf{x}}^{\alpha}, \quad \mathbf{x}\in\overline{\Omega}. \tag{3.d}$$

- the operators on the boundary

$$D\mathbf{u} = \Pi^{\mathsf{D}}\gamma_0\mathbf{u} \quad\text{and}\quad T\mathbf{u} = \Pi^{\mathsf{T}}B\mathbf{u}. \tag{3.e}$$

  with the trace operator γ_0, the complementing projector

$$\Pi^{\mathsf{T}} = \mathbb{I} - \Pi^{\mathsf{D}}, \tag{3.f}$$

and the associate $N \times N$ *conormal system* at the boundary, B,

$$B = \big(B_{ij}\big)_{1\le i,j\le N} \quad \text{with} \quad B_{ij} = \sum_{|\alpha|\le 1}\sum_{|\gamma|=1} \mathbf{n}^\gamma(\mathbf{x}) a_{ij}^{\alpha\gamma}(\mathbf{x})\, \partial_{\mathbf{x}}^\alpha, \quad \mathbf{x}\in\partial\Omega. \tag{3.g}$$

Here for any multiindex $\gamma = (\gamma_1,\ldots,\gamma_n)$ of length 1, $\mathbf{n}^\gamma(\mathbf{x})$ is the component $\ell = \ell(\gamma)$ of the unit outward normal $\mathbf{n}(\mathbf{x})$ to $\partial\Omega$ at the point $\mathbf{x}$, where ℓ is the unique index such that $\gamma_\ell = 1$.

The operators (T, D) enter our general framework in Chapter 2, see (2.i): Here the super-dimensions $\hat{N}_0$ and $\hat{N}_1$ are both set to N, $\hat{D} = \mathbb{I}$, Π^{T} is defined by (3.f) and $\hat{T} = B$. The relation with the general boundary value problems considered in the previous chapter is given in the following lemma.

Lemma 3.A *In the framework above, let* $\mathbf{f}$ *be given in* $\mathbf{L}^2(\Omega)$ *and* $\mathbf{g} = (g_1,\ldots,g_N)$ *be given in* $\mathbf{H}^{-\frac12}(\partial\Omega)$. *Let* $\langle \mathbf{f},\mathbf{v}\rangle_\Omega$ *denote the hermitian product in* $\mathbf{L}^2(\Omega)$ *and* $\langle \mathbf{g},\mathbf{w}\rangle_{\partial\Omega}$ *the duality pairing in* $\mathbf{H}^{-\frac12}(\partial\Omega)\times\mathbf{H}^{\frac12}(\partial\Omega)$. *The expression*

$$\big\langle \mathbf{q},\mathbf{v}\big\rangle_{\overline{\Omega}} = \big\langle \mathbf{f},\mathbf{v}\big\rangle_{\Omega} + \big\langle \mathbf{g}, \Pi^{\mathsf{T}}\gamma_0\mathbf{v}\big\rangle_{\partial\Omega}, \quad \forall\mathbf{v}\in\mathbf{V}, \tag{3.h}$$

defines an element $\mathbf{q}$ *of* $\mathbf{V}'$. *If* $\mathbf{u}$ *is a solution of the variational problem* (3.c)*, then* $\mathbf{u}$ *solves the boundary value problem*

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \textit{in } \Omega,\\ \Pi^{\mathsf{T}}B\mathbf{u} = \Pi^{\mathsf{T}}\mathbf{g} & \textit{on } \partial\Omega,\\ \Pi^{\mathsf{D}}\gamma_0\mathbf{u} = 0 & \textit{on } \partial\Omega. \end{cases} \tag{3.i}$$

As particular cases we have

- the Dirichlet problem when $\Pi^{\mathsf{D}} = \mathbb{I}$ (thus $\Pi^{\mathsf{T}} = 0$ and $\mathbf{V} = \mathbf{H}^1_0(\Omega)$);
- the Neumann problem when $\Pi^{\mathsf{D}} = 0$ (thus $\Pi^{\mathsf{T}} = \mathbb{I}$ and $\mathbf{V} = \mathbf{H}^1(\Omega)$).

More generally, the boundary conditions $\Pi^{\mathsf{D}}\gamma_0\mathbf{u} = 0$ and $\Pi^{\mathsf{T}}B\mathbf{u} = \Pi^{\mathsf{T}}\mathbf{g}$ are called ***essential*** and ***natural*** boundary conditions, respectively.

The form a is said ***coercive*** on $\mathbf{V}$ if there exist positive constants c and C such that

$$\forall\mathbf{u}\in\mathbf{V}, \quad \operatorname{Re}\, \mathsf{a}(\mathbf{u},\mathbf{u}) \ge c\|\mathbf{u}\|^2_{1;\Omega} - C\|\mathbf{u}\|^2_{0;\Omega}. \tag{3.j}$$

The form a is said ***strongly coercive*** if C can be set to 0 in (3.j).

If a is coercive on $\mathbf{V}$, then problem (3.c) is solvable by the Fredholm alternative. The connection with elliptic boundary value problem is simple to state (*cf.* Theorem 3.2.6):

Theorem 3.B *If* a *is* $\mathbf{V}$*-coercive, the system* L (3.d) *is elliptic and is covered by the system of boundary operators* (T, D) *determined by* (3.e)–(3.g).

In order to bridge the $\mathbf{H}^1$ variational regularity with the basic $\mathbf{H}^2$ regularity required as an assumption in Theorem 2.D, we prove the following theorem (*cf.* Theorem 3.4.1)

Theorem 3.C *Let Ω be a smooth bounded domain and a be a coercive form on $\mathbf{V}$ with coefficients in $\mathscr{C}^1(\overline{\Omega})$. Let $\mathbf{u} \in \mathbf{V}$ be a solution of problem (3.c) for a right hand side $\mathbf{q}$ defined by (3.h) with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then $\mathbf{u}$ belongs to $\mathbf{H}^2(\Omega)$ with the estimate*

$$\|\mathbf{u}\|_{2;\,\Omega} \leq C\Big(\|\mathbf{f}\|_{0;\,\Omega} + \|\mathbf{g}\|_{\frac{1}{2};\,\partial\Omega} + \|\mathbf{u}\|_{1;\,\Omega}\Big). \tag{3.k}$$

Therefore, solutions of a coercive variational problem satisfy the same local and global regularity properties than $\mathbf{H}^2$ solutions of elliptic second order systems with covering boundary conditions. In particular, if Ω has an analytic boundary, the form a and the projection operators Π^D have analytic coefficients, then any variational solution $\mathbf{u}$ of problem (3.c) for a right hand side defined by (3.h) with $\mathbf{f} \in \mathbf{A}(\Omega)$ and $\mathbf{g} \in \mathbf{A}(\partial\Omega)$, belongs to $\mathbf{A}(\Omega)$.

The question of lower regularity of variational solutions (in $\mathbf{H}^s$ for $1 < s < 2$) makes sense. For this, we introduce a couple of notations: For any $s > \frac{1}{2}$, let $\mathbf{H}^s_{\mathsf{V}}(\Omega)$ denote the space

$$\mathbf{H}^s_{\mathsf{V}}(\Omega) = \{\mathbf{u} \in \mathbf{H}^s(\Omega),\ \Pi^D\gamma_0\mathbf{u} = 0 \text{ on } \partial\Omega\}, \tag{3.l}$$

and for $s \geq 1,\ s \neq \frac{3}{2}$, we define $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ differently depending on whether $s < \frac{3}{2}$ or not:

- If $s \in [1, \frac{3}{2})$, $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ is the dual space of $\mathbf{H}^{2-s}_{\mathsf{V}}(\Omega)$.
- If $s > \frac{3}{2}$, $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ is the space of $\mathbf{q} \in \mathbf{V}'$ for which there exists a couple $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{s-2}(\Omega) \times \prod_{\mathbf{s}\in\mathscr{S}} \Pi^T_{\mathbf{s}} \mathsf{H}^{s-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega)^N$ such that the representation (3.h) holds (with obvious extensions of the duality pairings $\langle\cdot,\cdot\rangle_\Omega$ and $\langle\cdot,\cdot\rangle_{\partial\Omega}$).

We note that the operator $\mathbf{u} \mapsto \big(\mathbf{v} \mapsto \mathsf{a}(\mathbf{u},\mathbf{v})\big)$ is continuous from $\mathbf{H}^s_{\mathsf{V}}(\Omega)$ into $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ for all $s \geq 1$, with $s \neq \frac{3}{2}$. The elliptic shift result for $1 < s < 2$ takes the form (see Theorem 3.4.5):

Theorem 3.D *Let Ω be a smooth bounded domain and a be a coercive form on $\mathbf{V}$ with coefficients in $\mathscr{C}^1(\overline{\Omega})$. Let $s \in (1,2)$, $s \neq \frac{3}{2}$. Let $\mathbf{u} \in \mathbf{V}$ be a solution of problem (3.c) for a right hand side $\mathbf{q} \in \mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{H}^s(\Omega)$ with the estimate*

$$\|\mathbf{u}\|_{s;\,\Omega} \leq c\Big(\|\mathbf{q}\|_{\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)} + \|\mathbf{u}\|_{1;\,\Omega}\Big). \tag{3.m}$$

The main results of this chapter can be extended to variational problems based on more general sesquilinear form $\tilde{\mathsf{a}}$ containing a lower order term on the boundary

$$\tilde{\mathsf{a}}(\mathbf{u},\mathbf{v}) = \mathsf{a}(\mathbf{u},\mathbf{v}) + \int_{\partial\Omega} (Z\gamma_0\mathbf{u})\cdot\gamma_0\overline{\mathbf{v}}\,\mathrm{d}\sigma, \tag{3.n}$$

with a sesquilinear form a given by (3.a) and a smooth function Z defined on $\partial\Omega$ with values in $N \times N$ matrices. The new variational problem is

$$\textit{Find} \quad \mathbf{u} \in \mathbf{V} \quad \textit{such that} \quad \forall \mathbf{v} \in \mathbf{V}, \quad \tilde{\mathsf{a}}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}}\,. \tag{3.o}$$

With a variational space $\mathbf{V}$ given by (3.b), we have the equivalence (Lemma 3.5.1)

$$\tilde{\mathsf{a}} \text{ coercive on } \mathbf{V} \quad \Longleftrightarrow \quad \mathsf{a} \text{ coercive on } \mathbf{V}.$$

If $\mathbf{q}$ has the representation (3.h), then solutions $\mathbf{u}$ of problem (3.o) are solutions of the elliptic boundary value problem (of Robin type)

$$\begin{cases} L\mathbf{u} &= \mathbf{f} & \text{in } \Omega, \\ \Pi^T B\mathbf{u} + \Pi^T Z\gamma_0\mathbf{u} &= \Pi^T\mathbf{g} & \text{on } \partial\Omega, \\ \Pi^D\gamma_0\mathbf{u} &= 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.p}$$

Since problem (3.p) is a lower order perturbation of (3.i), we can easily show that all regularity results proved for solutions of problem (3.c) still hold for solutions of (3.o).

3.1 Variational spaces and forms

Let Ω be a bounded domain in $\mathbb{R}^n$ (or in a smooth manifold of dimension n). Whereas a big part of the material presented in this chapter is valid under rather weak smoothness assumptions, we will leave the discussion of some classes of non-smooth domains to later chapters and assume here that the boundary $\partial\Omega$ is smooth.

3.1.a Sesquilinear forms

Let $N \geq 1$ be the dimension of the system. We consider a sesquilinear form a of order 1 defined for $\mathbf{u}$ and $\mathbf{v}$ in $\mathsf{H}^1(\Omega)^N = \mathbf{H}^1(\Omega)$ by

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{|\alpha|\leq 1} \sum_{|\gamma|\leq 1} \int_\Omega a_{ij}^{\alpha\gamma}(\mathbf{x})\, \partial_{\mathbf{x}}^\alpha u_j(\mathbf{x})\, \partial_{\mathbf{x}}^\gamma \overline{v}_i(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{3.1}$$

with complex valued coefficients $a_{ij}^{\alpha\gamma}$, smooth up to the boundary of Ω. As soon as the coefficients are bounded on Ω, the form a is continuous over $\mathbf{H}^1(\Omega)$:

$$|\mathsf{a}(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_{1;\,\Omega}\, \|\mathbf{v}\|_{1;\,\Omega}\,. \tag{3.2}$$

The second order $N \times N$ system L associated with a is obtained by integration by part against any test function $\mathbf{v}$ in $\mathscr{C}_0^\infty(\Omega)^N$:

$$L = \big(L_{ij}\big)_{1\leq i,j\leq N} \quad \text{with} \quad L_{ij} = \sum_{|\alpha|\leq 1} \sum_{|\gamma|\leq 1} (-1)^{|\gamma|} \partial_{\mathbf{x}}^\gamma a_{ij}^{\alpha\gamma}(\mathbf{x})\, \partial_{\mathbf{x}}^\alpha, \quad \mathbf{x} \in \overline{\Omega}. \tag{3.3}$$

Here $\partial^\gamma_{\mathbf{x}} a^{\alpha\gamma}_{ij} \partial^\alpha_{\mathbf{x}}$ is the second order operator $u \mapsto \partial^\gamma_{\mathbf{x}}(a^{\alpha\gamma}_{ij}(\mathbf{x})\partial^\alpha_{\mathbf{x}} u)$. The system L has smooth coefficients and there holds

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \big\langle L\mathbf{u}\ ,\ \mathbf{v}\big\rangle_\Omega, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\ \mathbf{v} \in \mathbf{H}^1_0(\Omega). \tag{3.4}$$

Here $\langle\ ,\ \rangle_\Omega$ is the duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}^1_0(\Omega)$.

For any multiindex $\gamma = (\gamma_1, \ldots, \gamma_n)$ of length 1 there exists ℓ such that $\gamma_k = \delta_{k\ell}$: then we define $\mathbf{n}^\gamma(\mathbf{x})$ as the component ℓ of the unit outward normal $\mathbf{n}(\mathbf{x})$ to $\partial\Omega$ at the point $\mathbf{x}$. The associate $N \times N$ *conormal system* at the boundary B, is defined as

$$B = \big(B_{ij}\big)_{1\le i,j\le N} \quad \text{with} \quad B_{ij} = \sum_{|\alpha|\le 1}\sum_{|\gamma|=1} \mathbf{n}^\gamma(\mathbf{x}) a^{\alpha\gamma}_{ij}(\mathbf{x})\, \partial^\alpha_{\mathbf{x}}, \quad \mathbf{x} \in \partial\Omega. \tag{3.5}$$

Integrating by parts for $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$, we find

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \int_\Omega L\mathbf{u}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} B\mathbf{u}\cdot\bar{\mathbf{v}}\,\mathrm{d}\sigma. \tag{3.6}$$

Let $\mathbf{H}^1(\Omega; L)$ denote the maximal domain of L

$$\mathbf{H}^1(\Omega; L) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : L\mathbf{u} \in \mathbf{L}^2(\Omega)\}. \tag{3.7}$$

Formula (3.6) allows to extend the continuous mapping:

$$\mathbf{H}^2(\Omega) \ni \mathbf{u} \longmapsto B\mathbf{u} \in \mathbf{H}^{\frac12}(\partial\Omega)$$

to a continuous mapping

$$B : \mathbf{H}^1(\Omega; L) \longrightarrow \mathbf{H}^{-\frac12}(\partial\Omega). \tag{3.8}$$

Here $\mathbf{H}^{-\frac12}(\partial\Omega)$ is the dual space of $\mathbf{H}^{\frac12}(\partial\Omega)$. We denote by $\langle\ ,\ \rangle_{\partial\Omega}$ their duality pairing. The continuity (3.8) is easily seen as follows: Let $\gamma_0^- : \mathrm{H}^{\frac12}(\partial\Omega) \to \mathrm{H}^1(\Omega)$ be a continuous right inverse ("lifting") of the trace mapping γ_0. Then we can write (3.6) as

$$\big\langle B\mathbf{u}, \mathbf{g}\big\rangle_{\partial\Omega} = \mathsf{a}(\mathbf{u}, \gamma_0^-\mathbf{g}) - \int_\Omega L\mathbf{u}\cdot\overline{\gamma_0^-\mathbf{g}}\,\mathrm{d}\mathbf{x}$$

and we see that the right hand side defines a continuous sesquilinear form on $\mathbf{H}^1(\Omega; L) \times \mathbf{H}^{\frac12}(\partial\Omega)$. Thus we obtain

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \int_\Omega L\mathbf{u}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \big\langle B\mathbf{u}, \gamma_0\mathbf{v}\big\rangle_{\partial\Omega}, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega; L),\ \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{3.9}$$

Definition 3.1.1 *Let* a *be a sesquilinear form of order* 1 *according to* (3.1).

*(i) A **variational space*** $\mathbf{V} = \mathbf{V}(\Omega)$ *is a closed subspace of* $\mathbf{H}^1(\Omega)$ *which contains* $\mathbf{H}^1_0(\Omega)$.

(ii) *The variational problem associated with the sesquilinear form* a *and the space* $\mathbf{V}$ *is formulated as*

$$\text{Find} \quad \mathbf{u} \in \mathbf{V} \quad \text{such that} \quad \forall \mathbf{v} \in \mathbf{V}, \quad \mathsf{a}(\mathbf{u}, \mathbf{v}) = \big\langle \mathbf{q}, \mathbf{v} \big\rangle_{\overline{\Omega}}, \tag{3.10}$$

for $\mathbf{q}$ *given in the dual space* $\mathbf{V}'$ *of* $\mathbf{V}$. *Here* $\langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}}$ *denotes the extension of the duality pairing* $(\mathbf{q}, \mathbf{v}) \mapsto \int_\Omega \mathbf{q} \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x}$ *of* $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ *to* $\mathbf{V}' \times \mathbf{V}$.

(iii) *For* $\mathbf{V} = \mathbf{H}^1_0(\Omega)$, (3.10) *defines the* ***Dirichlet problem*** *for the system* L.

(iv) *For* $\mathbf{V} = \mathbf{H}^1(\Omega)$, (3.10) *defines the* ***Neumann problem*** *associated with the system* L *by the sesquilinear form* a.

We have the classical distributional expressions of Dirichlet and Neumann problems:

Lemma 3.1.2 *(i) If* $\mathbf{V} = \mathbf{H}^1_0(\Omega)$*, for any* $\mathbf{q} \in \mathbf{H}^{-1}(\Omega)$ *the problem* (3.10) *can be written as*

$$\begin{cases} L\mathbf{u} = \mathbf{q} & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

(ii) If $\mathbf{V} = \mathbf{H}^1(\Omega)$*, for a right hand side* $\mathbf{q}$ *given by the* $\mathbf{L}^2$ *scalar product against an element* $\mathbf{f} \in \mathbf{L}^2(\Omega)$ *:* $\langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}}$*, the problem* (3.10) *can be written as*

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ B\mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Proof: *(i)* is quite obvious: Let $\mathbf{u}$ be a solution of problem (3.10) with $\mathbf{V} = \mathbf{H}^1_0(\Omega)$. The boundary condition $\mathbf{u} = 0$ is provided by the space $\mathbf{H}^1_0(\Omega)$, and the equation $L\mathbf{u} = \mathbf{q}$ in Ω is an equality in $\mathbf{H}^{-1}(\Omega)$ which is a direct consequence of (3.4): There holds

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \big\langle L\mathbf{u}\,,\, \mathbf{v} \big\rangle_\Omega = \big\langle \mathbf{q}, \mathbf{v} \big\rangle_{\overline{\Omega}}, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega).$$

(ii) Let $\mathbf{u}$ be a solution of problem (3.10) with $\mathbf{V} = \mathbf{H}^1(\Omega)$ with $\mathbf{q}$ given by $\langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}}$, with $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Formulation (3.10) gives, in particular

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega).$$

With (3.4), this gives the equality $L\mathbf{u} = \mathbf{f}$ in $\mathbf{L}^2(\Omega)$. Thus $\mathbf{u}$ belongs to $\mathbf{H}^1(\Omega; L)$ and (3.9) yields with (3.10) that

$$\int_\Omega L\mathbf{u} \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \big\langle B\mathbf{u}, \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Since $L\mathbf{u} = \mathbf{f}$, we simply obtain

$$\big\langle B\mathbf{u}, \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

which means that $B\mathbf{u} = 0$ as an element of $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. □

Remark 3.1.3 The Dirichlet problem depends on L only and not on the sesquilinear form a such that relation (3.3) holds. In contrast, Neumann conditions depend on a and not only on L. We give examples of this in the next chapter, Section 4.8. △

3.1.b Essential and natural boundary conditions

In many applications, the variational space $\mathbf{V}$ intermediate between $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ is determined by pointwise restrictions imposed on the boundary traces. Such *essential boundary conditions* can often be described by the vanishing of some vector components of the trace, or equivalently by the inclusion of the trace in some subspace of $\mathbb{C}^N$, where this subspace can vary from point to point. Typical examples of such conditions are the vanishing of the tangential or of the normal components of a vector field if $N = n$.

From a mathematical point of view, this type of essential boundary conditions could be described by the introduction of some vector bundle on the boundary, the condition being that the trace of the solution is a section of this bundle. In order to make this idea precise, one would have to cope with the fact that the variational solutions in $\mathbf{H}^1(\Omega)$ do not have well-defined pointwise traces, so that one would have to study distributional sections of such vector bundles.

We choose to introduce a less abstract, but not less general, variant of this idea, based on the description of the essential boundary conditions by *fields of projection operators*. This variant turns out to work well with the definition of natural boundary conditions by integration by parts and, most importantly, it will let us subsume the class of our elliptic problems in variational form under the general class of elliptic boundary value problems the regularity of which we studied in Chapters 1 and 2. The only additional (but not really restrictive) requirement that we now make about the projectors is that they have to be *orthogonal* and, of course, that the first order boundary operators T are defined by the natural boundary conditions corresponding to the essential conditions defined by the zero order boundary operators D.

We assume therefore that we are given a smooth mapping Π^D from the boundary $\partial\Omega$ to the space of orthogonal projection operators $\mathbb{C}^N \to \mathbb{C}^N$:

$$\Pi^D : \partial\Omega \ni \mathbf{x} \longmapsto \Pi^D(\mathbf{x}), \quad \Pi^D(\mathbf{x}) \circ \Pi^D(\mathbf{x}) = \Pi^D(\mathbf{x}), \quad \Pi^D(\mathbf{x})^* = \Pi^D(\mathbf{x}). \tag{3.13}$$

The rank of the operator Π^D may vary with the index $\mathbf{s}$ describing the connected components $(\partial_{\mathbf{s}}\Omega)_{\mathbf{s}\in\mathscr{S}}$ of $\partial\Omega$. We may write $\Pi^D_{\mathbf{s}}$ for the restriction of Π^D to $\partial_{\mathbf{s}}\Omega$. In components we have $\Pi^D(\mathbf{x}) = \big(\pi_{ij}(\mathbf{x})\big)_{i,j=1,\ldots,N}$ with smooth ($\mathscr{C}^\infty$ or analytic) functions π_{ij} on $\partial\Omega$.

We see that the multiplication by $\Pi^D(\cdot)$ defines a projector $\Pi^D : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega)$ for all $s \in \mathbb{R}$ with the orthogonality property

$$\forall \mathbf{G} \in \mathbf{H}^{-s}(\partial\Omega), \quad \mathbf{g} \in \mathbf{H}^s(\partial\Omega): \quad \big\langle \Pi^D\mathbf{G}, \mathbf{g} \big\rangle_{\partial\Omega} = \big\langle \mathbf{G}, \Pi^D\mathbf{g} \big\rangle_{\partial\Omega} = \big\langle \Pi^D\mathbf{G}, \Pi^D\mathbf{g} \big\rangle_{\partial\Omega}.$$

Therefore, with the complementing projector $\Pi^T := \mathbb{I} - \Pi^D$, we have

$$\big\langle B\mathbf{u}, \gamma_0\mathbf{v} \big\rangle_{\partial\Omega} = \big\langle \Pi^T B\mathbf{u}, \Pi^T\gamma_0\mathbf{v} \big\rangle_{\partial\Omega} + \big\langle \Pi^D B\mathbf{u}, \Pi^D\gamma_0\mathbf{v} \big\rangle_{\partial\Omega}, \tag{3.14}$$

for any $\mathbf{u} \in \mathbf{H}^1(\Omega; L)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$.

Definition 3.1.4 *Let Π^D be a smooth field of orthogonal projection operators $\mathbb{C}^N \to \mathbb{C}^N$. The associated variational space $\mathbf{V}$ is the subspace of $\mathbf{H}^1(\Omega)$ defined as*

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \ \Pi^D \gamma_0 \mathbf{u} = 0 \ \text{ on } \ \partial\Omega\}. \tag{3.15}$$

Note that if $\Pi^D \equiv \mathbb{I}$, the space $\mathbf{V}$ coincides with $\mathbf{H}^1_0(\Omega)$, corresponding to the Dirichlet problem, whereas if $\Pi^D \equiv 0$, $\mathbf{V}$ coincides with $\mathbf{H}^1(\Omega)$, defining the Neumann problem. More generally than given in Lemma 3.1.2, the distributional formulation of problem (3.10) writes as follows:

Lemma 3.1.5 *With the variational space (3.15) and $\Pi^T := \mathbb{I} - \Pi^D$, for a right hand side $\mathbf{q}$ given by the $\mathbf{L}^2$ scalar product against an element $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the variational problem (3.10) can be written as:*

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \ \Omega \\ \Pi^T B\mathbf{u} &= 0 \quad \text{on } \ \partial\Omega \\ \Pi^D \gamma_0 \mathbf{u} &= 0 \quad \text{on } \ \partial\Omega. \end{cases} \tag{3.16}$$

Definition & Notation 3.1.6 *The conditions $\Pi^D \gamma_0 \mathbf{u} = 0$ are called **essential** boundary conditions whereas the conditions $\Pi^T B\mathbf{u} = 0$ are called **natural** boundary conditions. We denote D and T the boundary operators*

$$D\mathbf{u} = \Pi^D \gamma_0 \mathbf{u} \quad \text{and} \quad T\mathbf{u} = \Pi^T B\mathbf{u}. \tag{3.17}$$

Proof of Lemma 3.1.5: Let $\mathbf{u}$ be a solution of problem (3.10). The interior equation $L\mathbf{u} = \mathbf{f}$ is obtained like in the Neumann case. Combining (3.14) with (3.9), we obtain

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega L\mathbf{u} \cdot \overline{\mathbf{v}} \, \mathrm{d}\mathbf{x} + \big\langle \Pi^T B\mathbf{u}, \Pi^T \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} + \big\langle \Pi^D B\mathbf{u}, \Pi^D \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Using formulation (3.10) in $\mathbf{V}$, this becomes

$$\int_\Omega L\mathbf{u} \cdot \overline{\mathbf{v}} \, \mathrm{d}\mathbf{x} + \big\langle \Pi^T B\mathbf{u}, \Pi^T \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \ \text{such that} \ \Pi^D \mathbf{v} = 0,$$

and since $L\mathbf{u} = \mathbf{f}$

$$\text{(1)} \qquad \big\langle \Pi^T B\mathbf{u}, \Pi^T \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \ \text{such that} \ \Pi^D \gamma_0 \mathbf{v} = 0.$$

Let us prove that, in fact, the above equality holds for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Let $\mathbf{v} \in \mathbf{H}^1(\Omega)$, and consider $\mathbf{g} = \Pi^T \gamma_0 \mathbf{v} \in \mathbf{H}^{\frac12}(\partial\Omega)$. Let $\mathbf{v}' \in \mathbf{H}^1(\Omega)$ be a lifting of the traces $\mathbf{g}$. Thus $\Pi^D \gamma_0 \mathbf{v}' = 0$ and $\Pi^T \gamma_0 \mathbf{v}' = \Pi^T \gamma_0 \mathbf{v}$. Therefore $\langle \Pi^T B\mathbf{u}, \Pi^T \gamma_0 \mathbf{v}' \rangle_{\partial\Omega} = 0$ implies that $\langle \Pi^T B\mathbf{u}, \Pi^T \gamma_0 \mathbf{v} \rangle_{\partial\Omega} = 0$.

Since Π^T is a self-adjoint projection operator we deduce

$$\langle \Pi^T B\mathbf{u}, \gamma_0\mathbf{v}\rangle_{\partial\Omega} = 0, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

which yields that $\Pi^T B\mathbf{u} = 0$ as an element of $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. □

In components, the boundary conditions in (3.16) read:

$$\left\{ \begin{array}{rcll} \sum_{j=1}^{N} \big(\delta_{ij} - \pi_{ij}(\mathbf{x})\big)(Bu)_j(\mathbf{x}) &=& 0 & i = 1,\dots,N \\ \sum_{j=1}^{N} \pi_{ij}(\mathbf{x})u_j(\mathbf{x}) &=& 0 & i = 1,\dots,N\,. \end{array} \right. \tag{3.18}$$

This boundary system has $2N$ equations, which are not independent, however. It has the form of the general boundary systems considered in Chapter 2 with $\hat{N}_1 = \hat{N}_0 = N$, see Definition 2.2.20. By using a local diagonalization of the projectors as in Subsection 2.2.c and in particular Lemma 2.2.25, we see that they are equivalent to a system of N independent equations like those in Definition 2.2.4. Let us recall this in more detail.

We note that the continuity of $\mathbf{x} \mapsto \Pi^D(\mathbf{x})$ implies that the rank of $\Pi^D(\mathbf{x})$ is constant on each connected component $\partial_\mathsf{s}\Omega$ of $\partial\Omega$. Therefore for any $\mathbf{x}_0 \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ such that on $\mathcal{U}$, we can find smooth bases in $\operatorname{im}\Pi^D$ and $\ker\Pi^D$: There exists $N_0 \in \mathbb{N}$, $N_0 \le N$ and $d_{ij} \in \mathscr{C}^\infty(\mathcal{U})$, $i,j = 1,\dots,N$ such that the vectors

$$\begin{array}{ll} \Big\{\big(d_{ij}(\mathbf{x})\big)_{j=1,\dots,N} : \ i = 1,\dots,N_0\Big\} & \text{span} \ \ \operatorname{im}\Pi^D(\mathbf{x}) \\ \Big\{\big(d_{ij}(\mathbf{x})\big)_{j=1,\dots,N} : \ i = N_0+1,\dots,N\Big\} & \text{span} \ \ \ker\Pi^D(\mathbf{x}) = \operatorname{im}\Pi^T. \end{array}$$

Then we define

$$t_{ij}(\mathbf{x}) = \sum_{k=1}^{N} d_{N_0+i,k}(\mathbf{x})B_{kj} \quad i = 1,\dots,N_1 = N - N_0, \quad j = 1,\dots,N.$$

Thus the boundary system (3.18) becomes, for $\mathbf{x} \in \mathcal{U}$, equivalent to

$$\left\{ \begin{array}{rcll} \sum_{j=1}^{N} t_{ij}(\mathbf{x})u_j(\mathbf{x}) &=& 0 & i = 1,\dots,N_1 \\ \sum_{j=1}^{N} d_{ij}(\mathbf{x})u_j(\mathbf{x}) &=& 0 & i = 1,\dots,N_0. \end{array} \right. \tag{3.19}$$

Thus, locally, we find the standard form of our boundary system (2.16) in Chapter 2.

A typical example which shows that the transformation between (3.18) and (3.19) cannot be done globally on $\partial\Omega$ is the following: Let $\Omega \subset \mathbb{R}^n$ and $N = n$. Let

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \int_\Omega \partial_{x_i}u_j\,\partial_{x_i}\overline{v}_j\,\mathrm{d}\mathbf{x}.$$

With the unit outward normal vector $\mathbf{n}$ and the normal derivative $\partial_{\mathbf{n}} = \mathbf{n} \cdot \nabla$, the standard Green formula is

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega -\Delta \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \partial_{\mathbf{n}} \mathbf{u} \cdot \overline{\mathbf{v}} \, \mathrm{d}\sigma.$$

We define the normal component $\mathbf{u}_{\mathbf{n}} = (\mathbf{u} \cdot \mathbf{n})\, \mathbf{n}$ and the tangential component $\mathbf{u}_{\mathbf{t}} = \mathbf{u} - \mathbf{u}_{\mathbf{n}}$ of a vector $\mathbf{u}$. Now let

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{u}_{\mathbf{t}} = 0 \ \text{ on } \ \partial\Omega\}.$$

This is a natural choice in electrostatics, for example. Then the boundary value problem (3.16) becomes

$$\begin{cases} -\Delta \mathbf{u} & = & \mathbf{f} & \text{in } \ \Omega \\ \partial_{\mathbf{n}} \mathbf{u}_{\mathbf{n}} & = & 0 & \text{on } \ \partial\Omega \\ \mathbf{u}_{\mathbf{t}} & = & 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.20}$$

If Ω is a ball in $\mathbb{R}^3$, then it is well known that one cannot find a basis of the tangent space smoothly on all of $\partial\Omega$. Therefore one cannot, in general, write the boundary system in (3.20) with only $N = n$ linearly independent boundary operators with smooth coefficients.

Remark 3.1.7 $\ker \Pi^D(\cdot)$ and $\operatorname{im} \Pi^D(\cdot)$ define vector bundles on $\partial\Omega$, subbundles of the bundle $\partial\Omega \times \mathbb{C}^N$. But while the latter is trivial, the former two are only locally trivial, in general. △

3.2 Coercivity and ellipticity

The solvability of the variational problem (3.10) is ensured by the following classical condition on the sesquilinear form a, in relation with the variational space $\mathbf{V}$:

Definition 3.2.1 *The form a is said **coercive** on $\mathbf{V}$ (or $\mathbf{V}$-coercive for short) if there exist positive constants c and C such that*

$$\forall \mathbf{u} \in \mathbf{V}, \quad \operatorname{Re} \, \mathsf{a}(\mathbf{u}, \mathbf{u}) \geq c \|\mathbf{u}\|^2_{1;\Omega} - C \|\mathbf{u}\|^2_{0;\Omega}. \tag{3.21}$$

*The form a is said **strongly coercive** if C can be set to 0 in (3.21).*

Before stating in the next section the precise consequences of the $\mathbf{V}$-coercivity of a on the solvability of problem (3.10), let us mention several results giving criteria of $\mathbf{V}$-coercivity.

Definition 3.2.2 *Let a be a sesquilinear form, as given by formula (3.1).*

(i) The form a *is said* ***strongly elliptic*** *on* Ω *if there exists* $c > 0$ *such that for all* $\mathbf{x} \in \Omega$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ *and* $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathbb{C}^N$ *there holds*

$$\operatorname{Re} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{|\alpha|=1} \sum_{|\gamma|=1} a_{ij}^{\alpha\gamma}(\mathbf{x})\, \boldsymbol{\xi}^{\alpha+\gamma}\, \eta_j\, \overline{\eta}_i \quad \geq \quad c\, |\boldsymbol{\xi}|^2 \sum_{i=1}^{N} |\eta_i|^2. \tag{3.22}$$

(ii) Using the principal symbols $L_{ij}^{\mathsf{pr}}(\mathbf{x}; \boldsymbol{\xi})$ *of the terms of the associate* $N \times N$ *system* L (3.3), *the above condition can be written as*

$$\operatorname{Re} \sum_{i=1}^{N} \sum_{j=1}^{N} L_{ij}^{\mathsf{pr}}(\mathbf{x}; \boldsymbol{\xi})\, \eta_j\, \overline{\eta}_i \quad \geq \quad c\, |\boldsymbol{\xi}|^2 \sum_{i=1}^{N} |\eta_i|^2 \tag{3.23}$$

and is the definition of the ***strong ellipticity*** *of the system* L.

It is clear that condition (3.23) is stronger than the standard ellipticity of the system L which only requires the invertibility of the matrix $\big(L_{ij}^{\mathsf{pr}}(\mathbf{x}; \boldsymbol{\xi})\big)$ for non-zero $\boldsymbol{\xi}$. This condition guarantees that the Gårding inequality (3.21) holds, see [36] or [76, Ch.3, Th.7.3]:

Lemma 3.2.3 *If the sesquilinear form* a *is strongly elliptic on* Ω*, then it is coercive on* $\mathbf{H}_0^1(\Omega)$.

Thus, the strong ellipticity yields the $\mathbf{V}$-coercivity for the Dirichlet problem, but does not ensure the coercivity for other boundary conditions. The stronger condition of "formal positivity" given below implies the $\mathbf{V}$-coercivity of a on any variational space $\mathbf{V}$ contained in $\mathbf{H}^1(\Omega)$:

Definition 3.2.4 *The form* a*, given by* (3.1)*, is said* ***formally positive*** *if there exists a constant* $c > 0$ *such that for all* $\mathbf{x} \in \Omega$ *and* $\psi_i^\alpha \in \mathbb{C}$, $i = 1 \dots, N$, $|\alpha| = 1$, *there holds the estimate*

$$\operatorname{Re} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{|\alpha|=1} \sum_{|\gamma|=1} a_{ij}^{\alpha\gamma}(\mathbf{x})\, \psi_j^\alpha\, \overline{\psi}_i^\gamma \quad \geq \quad c \sum_{i=1}^{N} \sum_{|\alpha|=1} |\psi_i^\alpha|^2. \tag{3.24}$$

Taking $\psi_i^\alpha = \boldsymbol{\xi}^\alpha \eta_i$ we see immediately that the formal positivity of a implies its strong ellipticity. There holds, see [76, Ch.3, Th.7.5]:

Lemma 3.2.5 *If the sesquilinear form* a *is formally positive on* Ω*, then it is coercive on* $\mathbf{H}^1(\Omega)$.

In another direction, an important fact is that the $\mathbf{V}$-coercivity of a implies the ellipticity of the boundary value problem (3.16):

Theorem 3.2.6 *Let* a *be a* $\mathbf{V}$*-coercive sesquilinear form with smooth coefficients on* $\overline{\Omega}$*, with a space* $\mathbf{V}$ *defined by the essential boundary conditions* $\Pi^{\mathsf{D}}\mathbf{u} = 0$ *on* $\partial\Omega$*, cf.* (3.15).

Then the system $\mathbb{A} = \{L, T = \Pi^{\mathsf{T}} B, D = \Pi^{\mathsf{D}} \gamma_0\}$ defining the boundary value problem (3.16), is elliptic on $\overline{\Omega}$ (cf. Definition 2.2.31).

Proof: We have to consider separately interior and boundary points.

a) Let $\mathbf{x}_0 \in \Omega$. We have to prove that the principal part $L^{\mathsf{pr}}(\mathbf{x}_0; \mathrm{D}_{\mathbf{x}})$ of L frozen at $\mathbf{x}_0$, satisfies

(1) $$L^{\mathsf{pr}}(\mathbf{x}_0; \boldsymbol{\xi}) \quad \text{invertible} \quad \forall \boldsymbol{\xi} \neq 0.$$

By homogeneity it suffices to prove (1) for $|\boldsymbol{\xi}|$ large enough.

We start with a scaled Poincaré inequality: Let us consider the Poincaré inequality (1.10) for $m = 1$ on the unit ball B_1:

$$\|u\|_{0;\,B_1} \le c_{1,B_1} \, |u|_{1;\,\Omega}\,, \quad \forall u \in \mathrm{H}^1_0(B_1).$$

Using the dilation $\mathbf{x} \longrightarrow R\mathbf{x}$ we deduce immediately that there holds on the ball $B_R = B_R(\mathbf{x}_0)$ of radius R

$$\|u\|_{0;\,B_R} \le c_{1,B_1} R |u|_{1;\,B_R}\,, \quad \forall u \in \mathrm{H}^1_0(B_R)\,. \tag{3.25}$$

Combining this with the coercivity estimate (3.21), we find that

(2) $$\exists R_0 > 0,\ \exists c_0 > 0, \quad \mathrm{Re}\ \mathsf{a}(\mathbf{u}, \mathbf{u}) \ge c_0 |\mathbf{u}|^2_{1;\,B_R} \quad \forall R \le R_0,\ \forall \mathbf{u} \in \mathbf{H}^1_0(B_R).$$

Let $\mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}$ denote the principal part of the form a frozen in $\mathbf{x}_0$:

$$\mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{|\alpha|=1} \sum_{|\gamma|=1} \int_\Omega a^{\alpha\gamma}_{ij}(\mathbf{x}_0)\, \partial^\alpha u_j(\mathbf{x})\, \partial^\gamma \overline{v}_i(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Using the regularity of the coefficients $a^{\alpha\gamma}_{ij}$, we find that there exists a positive constant C_1 such that

$$|\mathsf{a}(\mathbf{u}, \mathbf{u}) - \mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u}, \mathbf{u})| \le C_1 \big(R \|\mathbf{u}\|^2_{1;\,B_R} + \|\mathbf{u}\|_{1;\,B_R} \|\mathbf{u}\|_{0;\,B_R} \big) \quad \forall \mathbf{u} \in \mathbf{H}^1_0(B_R).$$

The scaled Poincaré inequality (3.25) then yields another positive constant C_1' such that

$$|\mathsf{a}(\mathbf{u}, \mathbf{u}) - \mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u}, \mathbf{u})| \le C_1' R \|\mathbf{u}\|^2_{1;\,B_R}\,.$$

Combining with the improved coercivity estimate (2), we find that it is valid for $\mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}$:

(3) $$\exists R_1 > 0,\ \exists c_1 > 0, \quad \mathrm{Re}\ \mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u}, \mathbf{u}) \ge c_1 |\mathbf{u}|^2_{1;\,B_R} \quad \forall R \le R_1,\ \forall \mathbf{u} \in \mathbf{H}^1_0(B_R).$$

Since, now both members are homogeneous with respect to dilation, we deduce that (3) is valid without condition on R: In particular

(4) $$\exists c_1 > 0, \quad \mathrm{Re}\ \mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u}, \mathbf{u}) \ge c_1 |\mathbf{u}|^2_{1;\,B_1} \quad \forall \mathbf{u} \in \mathbf{H}^1_0(B_1).$$

At this point we can note that c_1 does not depend on $\mathbf{x}_0$. Therefore, by continuity of coefficients, (4) is still valid for $\mathbf{x}_0$ in the boundary of Ω.

Let us prove (1) for $|\boldsymbol{\xi}|$ large enough. Let $\psi \in \mathscr{C}_0^\infty(B_1)$, $\psi \equiv 1$ on $B_{1/2}$. We use (4) with the special functions

$$\text{(5)} \qquad \mathbf{u}(\mathbf{x}) = \psi(\mathbf{x})\mathrm{e}^{i\mathbf{x}\cdot\boldsymbol{\xi}}\boldsymbol{\eta}, \quad \boldsymbol{\eta} \in \mathbb{C}^N.$$

Examining both members of (4), we can prove

$$\text{(6)} \qquad \operatorname{Re} \mathsf{a}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u},\mathbf{u}) \le \|\psi\|^2_{0;\,B_1} \operatorname{Re} \langle L^{\mathsf{pr}}(\mathbf{x}_0;\boldsymbol{\xi})\boldsymbol{\eta},\boldsymbol{\eta}\rangle + C_2|\boldsymbol{\xi}||\boldsymbol{\eta}|^2$$

and

$$\text{(7)} \qquad |\mathbf{u}|^2_{1;\,B_1} \ge \|\psi\|^2_{0;\,B_1} |\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 - c_2|\boldsymbol{\xi}||\boldsymbol{\eta}|^2,$$

for two positive constants C_2 and c_2 independent of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Combining (6) and (7) with (4) we find that there exists $\rho > 0$ such that

$$\text{(8)} \qquad \operatorname{Re} \langle L^{\mathsf{pr}}(\mathbf{x}_0;\boldsymbol{\xi})\boldsymbol{\eta},\boldsymbol{\eta}\rangle \ge \tfrac{1}{2}\, c_1|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 \quad \forall\boldsymbol{\xi},\ |\boldsymbol{\xi}| \ge \rho,\ \forall\boldsymbol{\eta} \in \mathbb{C}^N.$$

This obviously implies the invertibility of $L^{\mathsf{pr}}(\mathbf{x}_0;\boldsymbol{\xi})$. Note that we even prove that L is strongly elliptic, i.e. that (3.23) holds.

b) Let $\mathbf{x}_0 \in \partial\Omega$. We have two conditions to verify: The ellipticity of the interior operator and the covering condition for the boundary operators. All these conditions have to be checked on the "tangent" operators $\underline{L}_{\mathbf{x}_0}$, $\underline{T}_{\mathbf{x}_0}$ and $\underline{D}_{\mathbf{x}_0}$ introduced in Definition 2.2.30.

As already noticed by virtue of formula (2.25), the ellipticity of $\underline{L}_{\mathbf{x}_0}$ is equivalent to that of $L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})$. Since the strong coercivity estimate (4) also holds for $\mathbf{x}_0$ in the boundary of Ω by continuity, we deduce the ellipticity of $L^{\mathsf{pr}}(\mathbf{x}_0;\mathrm{D}_{\mathbf{x}})$ by the above arguments in part **a)** of the proof.

In order to check the covering condition, we first notice that the essential boundary condition $\Pi^D\gamma_0\mathbf{u} = 0$ writes $\pi_{N_0}M(\mathbf{x})\mathbf{u} = 0$ in a neighborhood of $\mathbf{x}_0$, with the invertible $N\times N$ matrix M appearing in Lemma 2.2.25. Let us smoothly extend the matrix valued function $\mathbf{x} \mapsto M(\mathbf{x})$ to a neighborhood of $\mathbf{x}_0$ inside Ω, keeping the property of invertibility of M. The change of unknown

$$\breve{\mathbf{u}}(\breve{\mathbf{x}}) = M\mathbf{u}\circ\phi_{\mathbf{x}_0}^{-1}(\breve{\mathbf{x}}), \quad \breve{\mathbf{v}}(\breve{\mathbf{x}}) = M\mathbf{v}\circ\phi_{\mathbf{x}_0}^{-1}(\breve{\mathbf{x}}),$$

allows to transform the essential boundary condition $\Pi^D\gamma_0\mathbf{u} = 0$ into $\pi_{N_0}\gamma_0\breve{\mathbf{u}} = 0$. Let $\breve{\mathsf{a}}_{\mathbf{x}_0}$ be the sesquilinear form defined by the identity

$$\breve{\mathsf{a}}_{\mathbf{x}_0}(\breve{\mathbf{u}},\breve{\mathbf{v}}) = \mathsf{a}(\mathbf{u},\mathbf{v}).$$

The coercivity of a on $\mathbf{V}$ implies the coercivity of $\breve{\mathsf{a}}_{\mathbf{x}_0}$ on $\mathbf{V}(V_R)$ (see Definition 2.2.17 for

V_R and B'_R) with

$$\mathbf{V}(V_R) = \{\mathbf{u} \in \mathbf{H}^1(V_R) :\ \pi_{N_0}\mathbf{u} = 0 \ \text{on}\ B'_R \quad \text{and} \quad \mathbf{u} = 0 \ \text{on}\ \partial V_R \setminus B'_R\}.$$

Let $\breve{\mathsf{a}}^{\mathsf{pr}}_{\mathbf{x}_0}$ be the principal part of $\breve{\mathsf{a}}_{\mathbf{x}_0}$ frozen at $\mathbf{0}$. Like in the step **a)** of the proof, we can deduce from the coercivity of $\breve{\mathsf{a}}_{\mathbf{x}_0}$ on $\mathbf{V}(V_R)$ that there exists a positive constant c_1 such that

$$\text{(9)} \qquad \operatorname{Re}\, \breve{\mathsf{a}}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u},\mathbf{u}) \geq c_1 |\mathbf{u}|^2_{1;\,V_R} \quad \forall \mathbf{u} \in \mathbf{V}(V_R),$$

compare with (3). By the same argument of homogeneity we obtain that (9) is still valid for any value of $R > 0$. We deduce that $\breve{\mathsf{a}}_{\mathbf{x}_0}$ still satisfies a coercivity inequality like (9) on the half-cylinder $\Sigma_+ := B'_1 \times \mathbb{R}_+$:

$$\text{(10)} \qquad \operatorname{Re}\, \breve{\mathsf{a}}^{\mathsf{pr}}_{\mathbf{x}_0}(\mathbf{u},\mathbf{u}) \geq c_1 |\mathbf{u}|^2_{1;\,\Sigma_+} \quad \forall \mathbf{u} \in \mathbf{V}(\Sigma_+),$$

with the space $\mathbf{V}(\Sigma_+)$ of functions $\mathbf{u} \in \mathbf{H}^1(\Sigma_+)$ satisfying the boundary conditions

$$\pi_{N_0}\mathbf{u} = 0 \ \text{on}\ B'_1 \quad \text{and} \quad \mathbf{u} = 0 \ \text{on}\ \partial B'_1 \times \mathbb{R}_+.$$

We can check that the interior and boundary operators associated with the principal part $\breve{\mathsf{a}}^{\mathsf{pr}}_{\mathbf{x}_0}$ frozen at $\mathbf{0}$, are simply $\underline{L}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$, $\underline{T}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$ and $\underline{D}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$.

As a result, we have reduced our problem to the homogeneous case with constant coefficients on the half-cylinder Σ_+, and we have to check the covering condition at $\mathbf{0}$. Thus we can rename $\breve{\mathsf{a}}^{\mathsf{pr}}_{\mathbf{x}_0}$ with a, $\underline{L}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$, $\underline{T}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$ and $\underline{D}_{\mathbf{x}_0} \circ M(\mathbf{x}_0)^{-1}$, with L, T and D, respectively.

Let $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$ be different from zero. According to Definition 2.2.6, we have to prove that the boundary operator $(T,D)(\boldsymbol{\xi}', \mathrm{D}_t)$ induces an isomorphism from $\mathfrak{M}_+[L;\boldsymbol{\xi}']$ onto $\mathbb{C}^N$. By homogeneity, it suffices to prove this for $|\boldsymbol{\xi}'| = 1$. So, let us pick up $\boldsymbol{\xi}'$ in the unit sphere $\mathbb{S}^{n-2}$, and let us prove that $(T,D)(\boldsymbol{\xi}', \mathrm{D}_t)$ is *injective* over $\mathfrak{M}_+[L;\boldsymbol{\xi}']$.

Let $s \mapsto \mathbf{U}(s)$ belong to the kernel of $(T,D)(\boldsymbol{\xi}', \mathrm{D}_s)$ in $\mathfrak{M}_+[L;\boldsymbol{\xi}']$. Then for any $\rho \geq 1$, the function $\mathbf{U}_\rho : t \mapsto \mathbf{U}(\rho t)$ belongs to the kernel of $(T,D)(\rho\boldsymbol{\xi}', \mathrm{D}_t)$ in $\mathfrak{M}_+[L;\rho\boldsymbol{\xi}']$. We set, with a cut-off function $\psi \in \mathscr{C}^\infty_0(B'_1)$ such that $\psi \equiv 1$ in $B'_{1/2}$, cf. (5):

$$\mathbf{u}(\breve{\mathbf{x}}) = \psi(\breve{\mathbf{x}}')\, \mathrm{e}^{i\breve{\mathbf{x}}'\cdot\rho\boldsymbol{\xi}'}\, \mathbf{U}(\rho t), \quad \breve{\mathbf{x}} = (\breve{\mathbf{x}}', t).$$

Since $D\mathbf{U} = 0$ on $t = 0$, the same holds for $\mathbf{u}$ on B'_1. Therefore estimate (10) holds for $\mathbf{u}$, which we write

$$\text{(11)} \qquad \operatorname{Re}\, \mathsf{a}(\mathbf{u},\mathbf{u}) \geq c_1 |\mathbf{u}|^2_{1;\,\Sigma_+}.$$

Like in (7), we find

$$\text{(12)} \qquad |\mathbf{u}|^2_{1;\,\Sigma_+} \geq \|\psi\|^2_{0;\,B'_1} \left(\rho^2 \|\mathbf{U}_\rho\|^2_{\mathbb{R}_+} + \|\mathbf{U}'_\rho\|^2_{\mathbb{R}_+}\right) - c_2\rho \|\mathbf{U}_\rho\|^2_{\mathbb{R}_+},$$

where $\mathbf{U}'_\rho = \mathrm{D}_t \mathbf{U}_\rho = \rho(\mathrm{D}_s \mathbf{U})_\rho$. Since $\mathbf{U}$ belongs to $\mathfrak{M}_+[L;\boldsymbol{\xi}']$, there holds $L(\mathrm{e}^{i\check{\mathbf{x}}'\cdot\rho\boldsymbol{\xi}'}\mathbf{U}_\rho) = 0$, and we find that

$$\|L\mathbf{u}\|_{\Sigma_+} \le C_2\big(\rho\|\mathbf{U}_\rho\|_{\mathbb{R}_+} + \|\mathbf{U}'_\rho\|_{\mathbb{R}_+}\big).$$

Since $T(\boldsymbol{\xi},\mathrm{D}_t)\mathbf{U} = 0$ on $t = 0$, there holds $T(\mathrm{e}^{i\check{\mathbf{x}}'\cdot\rho\boldsymbol{\xi}'}\mathbf{U}_\rho) = 0$ on B'_1. Integrating by parts we find finally that, cf. (6)

$$\text{(13)} \qquad \mathrm{Re}\;\mathsf{a}(\mathbf{u},\mathbf{u}) \le C_3\Big\{\rho\|\mathbf{U}_\rho\|^2_{\mathbb{R}_+} + \|\mathbf{U}'_\rho\|_{\mathbb{R}_+}\|\mathbf{U}_\rho\|_{\mathbb{R}_+} + |\mathbf{U}_\rho(0)|^2\Big\}.$$

For the term $|\mathbf{U}_\rho(0)|^2$ we use a Gagliardo-Nirenberg type estimate:
(14)

$$|\mathbf{U}_\rho(0)|^2 = -\int_0^\infty \mathrm{D}_t|\mathbf{U}_\rho|^2(t)\,dt = -2\,\mathrm{Re}\int_0^\infty \mathbf{U}_\rho(t)\bar{\mathbf{U}}'_\rho(t)\,dt \le 2\|\mathbf{U}_\rho\|_{\mathbb{R}_+}\|\mathbf{U}'_\rho\|_{\mathbb{R}_+}.$$

Putting (11)–(14) together, we obtain

$$\text{(15)} \qquad \|\psi\|^2_{0;B'_1}\big(\rho^2\|\mathbf{U}_\rho\|^2_{\mathbb{R}_+} + \|\mathbf{U}'_\rho\|^2_{\mathbb{R}_+}\big) \le C_4\Big\{\rho\|\mathbf{U}_\rho\|^2_{\mathbb{R}_+} + \|\mathbf{U}'_\rho\|_{\mathbb{R}_+}\|\mathbf{U}_\rho\|_{\mathbb{R}_+}\Big\}$$

Coming back to $\mathbf{U}$, we immediately deduce from (15)

$$\begin{aligned}\rho\|\psi\|^2_{0;B'_1}\big(\|\mathbf{U}\|^2_{\mathbb{R}_+} + \|\mathbf{U}'\|^2_{\mathbb{R}_+}\big) &\le C_4\big(\|\mathbf{U}\|^2_{\mathbb{R}_+} + \|\mathbf{U}'\|_{\mathbb{R}_+}\|\mathbf{U}\|_{\mathbb{R}_+}\big)\\ &\le C'_4\big(\|\mathbf{U}\|^2_{\mathbb{R}_+} + \|\mathbf{U}'\|^2_{\mathbb{R}_+}\big).\end{aligned}$$

Since this is true for all $\rho \ge 1$, we conclude that $\mathbf{U} \equiv 0$.

It remains to prove that the space $\mathfrak{M}_+[L;\boldsymbol{\xi}']$ has the dimension N. From the injectivity we deduce that $\dim\mathfrak{M}_+[L;\boldsymbol{\xi}'] \le N$. We prove that $\dim\mathfrak{M}_-[L;\boldsymbol{\xi}'] \le N$ by noticing that the space of conjugates of $\mathfrak{M}_-[L;\boldsymbol{\xi}']$ is equal to the space $\mathfrak{M}_+[\overline{L};\boldsymbol{\xi}']$, which is associated with the sesquilinear form $\bar{\mathsf{a}}$, itself coercive. Since

$$\dim\mathfrak{M}_+[L;\boldsymbol{\xi}'] + \dim\mathfrak{M}_-[L;\boldsymbol{\xi}'] = 2N$$

we conclude that $\dim\mathfrak{M}_+[L;\boldsymbol{\xi}'] = \dim\mathfrak{M}_-[L;\boldsymbol{\xi}'] = N$. □

Remark 3.2.7 As a consequence of the above statements, we note that the Dirichlet conditions cover any strongly elliptic operator L. In contrast, Neumann conditions depend on a and not only on L, and so does the coercivity on $\mathbf{H}^1(\Omega)$, see Section 4.8 for an example. △

3.3 Variational problems and solutions

As already mentioned, a variational problem over Ω is determined by the datum of a variational form a *and* of a variational space $\mathbf{V}$: Let a be a variational form according to (3.1) and $\mathbf{V}$ be a variational space according to (3.15). We recall the associated variational problem (3.10), formulated with a right hand side $\mathbf{q}$ given in the dual space $\mathbf{V}'$:

$$\textit{Find } \mathbf{u} \in \mathbf{V} \textit{ such that } \quad \forall \mathbf{v} \in \mathbf{V}, \quad \mathsf{a}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}}.$$

By the Riesz theorem, we can introduce A as the continuous operator from $\mathbf{V}$ into $\mathbf{V}'$ defined for any $\mathbf{u} \in \mathbf{V}$ by

$$\forall \mathbf{v} \in \mathbf{V}, \quad \langle A\mathbf{u}, \mathbf{v} \rangle_{\overline{\Omega}} = \mathsf{a}(\mathbf{u}, \mathbf{v}). \tag{3.26}$$

The function $\mathbf{u}$ solves the variational problem (3.10) if and only if $\mathbf{q} = A\mathbf{u}$. We note that the adjoint A^* of A is also defined from $\mathbf{V}$ into $\mathbf{V}'$, and satisfies

$$\forall \mathbf{u} \in \mathbf{V}, \quad \langle \mathbf{u}, A^*\mathbf{v} \rangle_{\overline{\Omega}} = \mathsf{a}(\mathbf{u}, \mathbf{v}).$$

Theorem 3.3.1 *Let Ω be a smooth bounded domain. Let a be a coercive sesquilinear form on the space $\mathbf{V}$. Then the operator A is Fredholm of index 0 from $\mathbf{V}$ into $\mathbf{V}'$, i.e., its kernel and cokernel are finite dimensional and have the same dimension. Moreover the condition of solvability of problem (3.10) for a given $\mathbf{q} \in \mathbf{V}'$ is*

$$\forall \mathbf{v} \in \ker A^*, \quad \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} = 0. \tag{3.27}$$

There holds the estimate

$$\|\mathbf{u}\|_{1;\Omega} \le C\big(\|\mathbf{q}\|_{\mathbf{V}'} + \|\mathbf{u}\|_{0;\Omega}\big), \tag{3.28}$$

with a constant C independent of $\mathbf{u}$ and $\mathbf{q}$.

Proof: Let λ be given, $\lambda > C$, with the constant C in the coercivity estimate (3.21). We define the sesquilinear form a_λ by

$$\mathsf{a}_\lambda(\mathbf{u}, \mathbf{v}) = \mathsf{a}(\mathbf{u}, \mathbf{v}) + \lambda \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}} \, \mathrm{d}\mathbf{x}.$$

The operator associated with a_λ is $A + \lambda\mathbb{I}$. Since there holds the strong coercivity estimate:

$$\forall \mathbf{u} \in \mathbf{V}, \quad \operatorname{Re} \mathsf{a}_\lambda(\mathbf{u}, \mathbf{u}) \ge c\|\mathbf{u}\|^2_{1;\Omega},$$

as a consequence of the Lax-Milgram lemma, $A+\lambda\mathbb{I}$ is an isomorphism from $\mathbf{V}$ onto $\mathbf{V}'$. Since the embedding of $\mathbf{V}$ into $\mathbf{L}^2(\Omega)$ is compact, A is a Fredholm operator and the alternative of Fredholm is true. □

Example 3.3.2 The Dirichlet problems for the Laplace and Lamé operators define invertible operators from $\mathbf{H}^1_0(\Omega)$ onto $\mathbf{H}^{-1}(\Omega)$. The Neumann problem for the Laplace operator has constant functions c as kernel and cokernel, whereas the Neumann problem for the Lamé system has the space of rigid motions $\mathbf{c} + \mathbf{d} \times \mathbf{x}$ ($\mathbf{c}$, $\mathbf{d}$ in $\mathbb{R}^n$) as kernel and cokernel, acting from $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^1(\Omega)'$. △

We summarize now the results on the interpretation of the variational problem (3.10) as an elliptic boundary value problem. As a consequence of standard distributional arguments, cf. proof of Lemma 3.1.5, and of the extension to $\mathbf{H}^1(\Omega; L)$ of the conormal trace B, cf. (3.8) we find:

Lemma 3.3.3 *Let* a *be a bounded sesquilinear form* (3.1)*. Let* Π^D *be a smooth field of orthogonal projection operators* $\mathbb{C}^N \to \mathbb{C}^N$*, and* $\Pi^T = \mathbb{I} - \Pi^D$*. Let* $\mathbf{V}$ *be the variational space defined by* (3.15)*.*
Let $\mathbf{f}$ *be given in* $\mathbf{L}^2(\Omega)$ *and* $\mathbf{g} = (g_1, \ldots, g_N)$ *be given in* $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$*. The expression*

$$\langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} = \int_\Omega \mathbf{f} \cdot \overline{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \langle \mathbf{g}, \Pi^T \gamma_0 \mathbf{v} \rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

defines an element $\mathbf{q}$ *of* $\mathbf{V}'$*. If* $\mathbf{u}$ *is a solution of the variational problem* (3.10)*, then* $\mathbf{u}$ *solves the boundary value problem*

$$\left\{ \begin{array}{rcll} L\mathbf{u} &=& \mathbf{f} & \textit{in } \Omega \\ \Pi^T B\mathbf{u} &=& \Pi^T \mathbf{g} & \textit{on } \partial\Omega \\ \Pi^D \gamma_0 \mathbf{u} &=& 0 & \textit{on } \partial\Omega, \end{array} \right. \tag{3.29}$$

with the trace operator γ_0 *on* $\partial\Omega$ *and the conormal system* B *associated with* a*.*

Remark 3.3.4 If, in addition, we give $\mathbf{h} = (h_1, \ldots, h_N)$ in $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, and, for a lifting $\mathbf{u}_0$ of $\mathbf{h}$ in $\mathbf{H}^1(\Omega)$, we set the variational problem

$$\textit{Find} \quad \mathbf{u}_1 \in \mathbf{V} \quad \textit{such that} \quad \forall \mathbf{v} \in \mathbf{V}, \;\; \mathsf{a}(\mathbf{u}_1, \mathbf{v}) = \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} - \mathsf{a}(\mathbf{u}_0, \mathbf{v}),$$

then its solutions $\mathbf{u}_1$ define solutions $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$ of problem (3.29) with the non-homogeneous essential boundary condition

$$\Pi^D \gamma_0 \mathbf{u} = \Pi^D \mathbf{h} \quad \text{on} \quad \partial\Omega.$$

The right hand side here is of the form (3.30) below if $\mathbf{h} \in \mathbf{H}^{\frac{3}{2}}(\partial\Omega)$ and $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$. △

3.4 Regularity of variational solutions

Our task in this section is to prove that the elliptic regularity results of Chapter 2 extend to variational solutions. We have to take into account that such solutions belong by definition to $\mathbf{H}^1$, and not a priori to $\mathbf{H}^2$. We have also to consider the possibility of less smooth right hand sides.

3.4.a H^2 and analytic regularity

If we consider $\mathbf{u}$ solution of problem (3.10) where the right hand side $\mathbf{q}$ is defined by smooth data $\mathbf{f}$ and $\mathbf{g}$, we can see immediately that the gap is the H^2 regularity of $\mathbf{u}$, which is the ground regularity in Theorems 2.3.2. Therefore we have to prove that if $\mathbf{f}$ belongs to L^2 and $\mathbf{g}$ to $\mathsf{H}^{\frac12}$, then the variational solution is H^2, which is the object of the next theorem.

Theorem 3.4.1 *Let Ω be a smooth bounded domain and* a *be a coercive sesquilinear form* (3.1) *on a subspace* $\mathbf{V}$ *of* $\mathsf{H}^1(\Omega)$ *according to Definition* 3.1.4. *The coefficients of the form* a *are assumed to belong to* $\mathscr{C}^1(\overline{\Omega})$. *Let* $\mathbf{u}\in\mathbf{V}$ *be a variational solution of problem* (3.10) *for a right hand side* $\mathbf{q}$ *defined by*

$$\big\langle \mathbf{q},\mathbf{v}\big\rangle_{\overline{\Omega}} = \int_\Omega \mathbf{f}\cdot\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \langle \mathbf{g}, \Pi^T\gamma_0\mathbf{v}\rangle_{\partial\Omega}\,, \tag{3.30}$$

with $\mathbf{f}$ *in* $\mathsf{L}^2(\Omega)$ *and* $\mathbf{g}$ *in* $\mathsf{H}^{\frac12}(\partial\Omega)$. *Then* $\mathbf{u}$ *belongs to* $\mathsf{H}^2(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{2;\,\Omega} \le C\Big(\|\mathbf{f}\|_{0;\,\Omega} + \|\mathbf{g}\|_{\frac12;\,\partial\Omega} + \|\mathbf{u}\|_{1;\,\Omega}\Big). \tag{3.31}$$

Proof: *(i) Localization.* Let $\mathbf{x}_0\in\overline{\Omega}$. Let $\mathcal{B}$ be a ball centered at $\mathbf{x}_0$ such that $\overline{\mathcal{B}}\subset\Omega$ if $\mathbf{x}_0$ is inside Ω, or contained in the domain of a local map of the boundary if $\mathbf{x}_0$ belongs to $\partial\Omega$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two smaller balls centered in $\mathbf{x}_0$, with $\overline{\mathcal{B}}_1\subset\mathcal{B}_2$ and $\overline{\mathcal{B}}_2\subset\mathcal{B}$, and χ be a smooth cut-off equal to 1 in $\mathcal{B}_1$ and 0 outside $\mathcal{B}_2$. The difference

$$\mathsf{a}(\chi\mathbf{u},\mathbf{v}) - \big\langle \chi\mathbf{q},\mathbf{v}\big\rangle_{\overline{\Omega}}$$

is a sum of terms of the form

$$\int_\Omega b(\mathbf{x})\,u_j(\mathbf{x})\,\partial^\gamma\overline{v}_i(\mathbf{x})\;\mathrm{d}\mathbf{x},\quad |\gamma|\le 1.$$

If $|\gamma|=1$, integrating by parts, we find that the above expression is equal to

$$-\int_\Omega \partial^\gamma(bu_j)\,\overline{v}_i\;\mathrm{d}\mathbf{x} + \int_{\partial\Omega} b\,u_j\,\overline{v}_i\,\mathbf{n}^\gamma\;\mathrm{d}\sigma.$$

Since $\mathbf{u}$ belongs to $\mathsf{H}^1(\Omega)$, the first part has the form $\int_\Omega \mathbf{f}'\cdot\overline{\mathbf{v}}$ with $\mathbf{f}'\in\mathsf{L}^2(\Omega)$, while the second term can be written in the form $\int_{\partial\Omega}\mathbf{g}'\cdot\overline{\mathbf{v}}$ with $\mathbf{g}'$ in $\mathsf{H}^{\frac12}(\partial\Omega)$. Moreover, since elements $\mathbf{v}$ of $\mathbf{V}$ satisfy $\Pi^T\mathbf{v}=\mathbf{v}$, this boundary integral is equal to $\langle\mathbf{g}',\Pi^T\gamma_0\mathbf{v}\rangle_{\partial\Omega}$. As a consequence,

$$\mathsf{a}(\chi\mathbf{u},\mathbf{v}) = \big\langle \chi\mathbf{q},\mathbf{v}\big\rangle_{\overline{\Omega}} + \big\langle \mathbf{q}',\mathbf{v}\big\rangle_{\overline{\Omega}}\,,$$

with $\mathbf{q}'$ defined like $\mathbf{q}$, cf. (3.30).

(ii) Local maps. Using a local map supported in $\mathcal{B}$ in the case when $\mathbf{x}_0$ belongs to $\partial\Omega$, we can assume that Ω coincides with the periodic half space $\mathbb{T}^n_+$ in $\mathcal{B}$. The projection operators

Π^D and Π^T are now defined on $\partial\mathbb{T}^n_+ \cap \mathcal{B}$. We perform a change of trial and test functions so that for all $\mathbf{x} \in \partial\mathbb{T}^n_+ \cap \mathcal{B}$, the projection operators are trivialized

$$\sum_{j=1}^{N_0} u_j(\mathbf{x})\,\mathbf{e}_j \in \mathrm{im}(\Pi^D(\mathbf{x})) \quad \text{and} \quad \sum_{j=N_0}^{N} u_j(\mathbf{x})\,\mathbf{e}_j \in \mathrm{im}(\Pi^T(\mathbf{x})).$$

Then, with a new coercive form, still denoted by a, defined on the new variational space

(1) $\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\mathcal{B}\cap\mathbb{T}^n_+) :\ u_1,\ldots,u_{N_0} = 0 \text{ on } \partial\mathbb{T}^n_+\cap\mathcal{B} \text{ and } \mathbf{u} = 0 \text{ on } \partial\mathcal{B}\setminus\partial\mathbb{T}^n_+\},$

and a new localized solution, still denoted by $\chi\mathbf{u}$, we are in the situation where the Dirichlet conditions are $u_j = 0$ for $j = 1,\ldots,N_0$. Also, we still denote by $\mathbf{q}$ the corresponding right hand side, which has still the form (3.30), so that

(2) $$\mathsf{a}(\chi\mathbf{u},\mathbf{v}) = \langle \mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}} \quad \forall\mathbf{v}\in\mathbf{V}.$$

(iii) Difference quotients. In the case when $\mathbf{x}_0 \in \partial\Omega$, thanks to point *(ii)* we can consider that Ω coincides with a subdomain $\mathcal{B}\cap\mathbb{T}^n_+$ of the periodic half-space and that $\chi\mathbf{u}$ is zero outside a smaller subdomain $\mathcal{B}_2\cap\mathbb{T}^n_+$. So we can use the difference quotients Δ^h_ℓ, with $|h| < h_0$ small enough, for $\ell = 1,\ldots,n-1$, cf. proof of Lemma 1.3.6 (p.42). We notice that $\Delta^h_\ell(\chi\mathbf{u})$ still belongs to the variational space $\mathbf{V}$ given in (1), and that using (2) we obtain

$$\mathsf{a}(\Delta^h_\ell(\chi\mathbf{u}),\mathbf{v}) = \langle \Delta^h_\ell\mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}} + \langle \mathbf{q}^h_\ell,\mathbf{v}\rangle_{\overline{\Omega}} \quad \forall\mathbf{v}\in\mathbf{V}.$$

Thanks to the $\mathscr{C}^1$ regularity of the coefficients of a, we can see that $\mathbf{q}^h_\ell$ is uniformly bounded in $\mathbf{V}'$ as h tends to 0:

$$|\langle \mathbf{q}^h_\ell,\mathbf{v}\rangle_{\overline{\Omega}}| \le C\,\|\chi\mathbf{u}\|_{1;\Omega}\,\|\mathbf{v}\|_{1;\Omega}, \quad |h| < h_0.$$

The representation (3.30) of the right hand side yields that

$$\langle \Delta^h_\ell\mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}} = \int_\Omega \mathbf{f}\cdot\Delta^{-h}_\ell\,\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \sum_{j=N_0+1}^{N}\int_{\partial\Omega} g_j\,\Delta^{-h}_\ell\,\overline{v}_j\,\mathrm{d}\sigma.$$

Since $\|\Delta^{-h}_\ell\,\mathbf{v}\|_{0;\Omega}$ and $\|\Delta^{-h}_\ell\,v_j\|_{-1/2;\partial\Omega}$ are uniformly bounded by $\|\mathbf{v}\|_{1;\Omega}$, the right hand sides $\mathbf{v}\mapsto\langle\Delta^h_\ell\mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}}$ are also uniformly bounded in $\mathbf{V}'$ as h tends to 0, with the estimates:

$$\|\Delta^h_\ell\mathbf{q}\|_{\mathbf{V}'} \le C\Big(\|\mathbf{f}\|_{0;\Omega} + \|\mathbf{g}\|_{\frac12;\partial\Omega}\Big).$$

Estimate (3.28) then gives

$$\|\Delta^h_\ell(\chi\mathbf{u})\|_{1;\Omega} \le C\Big(\|\Delta^h_\ell\mathbf{q}\|_{\mathbf{V}'} + \|\mathbf{q}^h_\ell\|_{\mathbf{V}'} + \|\Delta^h_\ell\mathbf{u}\|_{0;\Omega}\Big).$$

In the limit as $h \to 0$, we finally find for $\ell = 1, \ldots, n-1$:

$$\|\partial_\ell(\chi\mathbf{u})\|_{1;\Omega} \leq C\Big(\|\mathbf{f}\|_{0;\Omega} + \|\mathbf{g}\|_{\frac{1}{2};\partial\Omega} + \|\mathbf{u}\|_{1;\Omega}\Big).$$

In the situation where $\mathbf{x}_0$ is inside Ω, we can use difference quotients in all directions and the proof of local estimates ends here.

(iv) Normal derivatives. Using test functions in $\mathscr{C}_0^\infty(\Omega)^N$, we find that $L\mathbf{u} = \mathbf{f}$ as distributions in Ω. The coercivity of a implies the ellipticity of L. Therefore, in particular, the boundary $\{x_n = 0\}$ is not characteristic for L, i.e. (2.50) and (2.53) hold. Thus we find

$$\|\partial_{nn}(\chi\mathbf{u})\|_{0;\Omega} \leq C\Big(\|\mathbf{f}\|_{0;\Omega} + \sum_{\ell=1}^{n-1}\|\partial_\ell(\chi\mathbf{u})\|_{1;\Omega}\Big).$$

This ends the proof of the theorem. □

Owing to Theorem 3.2.6, we can apply elliptic regularity Theorems 2.3.2 and 2.7.1 to boundary value problems issued from a coercive formulation. Combining them with Theorem 3.4.1 we find optimal regularity results in Sobolev spaces and analytic classes. We state the analytic case first:

Theorem 3.4.2 *Let Ω be a bounded domain and* a *be a continuous coercive sesquilinear form on the subspace $\mathbf{V}$ of $\mathbf{H}^1(\Omega)$ characterized by the essential boundary conditions*

$$\Pi^D\gamma_0\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega.$$

Let $\mathbf{u} \in \mathbf{V}$ be a variational solution of problem (3.10) for a right hand side $\mathbf{q}$ defined by (3.30) with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$.

(i) Let two subdomains $\Omega_1 = \mathcal{U}_1 \cap \Omega$ and $\Omega_2 = \mathcal{U}_2 \cap \Omega$ be given with $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$. We set $\Gamma_2 := \overline{\Omega}_2 \cap \partial\Omega$ and assume that Γ_2 is analytic, that Π^D is analytic on Γ_2, and that the coefficients of the form a *are analytic up to the boundary of Ω_2. If $\mathbf{f}$ belongs to $\mathbf{A}(\Omega_2)$ and $\mathbf{g}$ to $\mathbf{A}(\Gamma_2)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega_1)$ with the estimates for all $m \in \mathbb{N}$:*

$$\frac{1}{m!}\,|\mathbf{u}|_{m;\Omega_1} \leq A^{m+1}\Big(\sum_{\ell=0}^{m-2}\frac{1}{\ell!}\,\big(|\mathbf{f}|_{\ell;\Omega_2} + \|\mathbf{g}\|_{\ell+\frac{1}{2};\Gamma_2}\big) + \sum_{\ell=0}^{1}|\mathbf{u}|_{\ell;\Omega_2}\Big). \tag{3.32}$$

(ii) We assume that Ω is analytic, that the coefficients of the form a *are analytic on $\overline{\Omega}$ and those of the projection operator Π^D are analytic on $\partial\Omega$. If $\mathbf{f} \in \mathbf{A}(\Omega)$ and $\mathbf{g} \in \mathbf{A}(\partial\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega)$.*

3.4.b Lower Sobolev regularity

Concerning regularity in the scale of Sobolev spaces, we see that the combination of Theorems 3.2.6 and 3.4.1 with Theorem 2.3.2 and Corollary 2.3.3 gives optimal regularity for solutions $\mathbf{u}$ of coercive variational problems in Sobolev spaces $\mathbf{H}^s(\Omega)$ for all real exponents $s \geq 2$. Nevertheless it can be also of some importance to consider less smooth data $\mathbf{q}$, which would correspond to solutions in $\mathbf{H}^s(\Omega)$ for $1 < s < 2$.

Thus we have to make precise the definition of what we understand for such data:

Notation 3.4.3 Let $\mathbf{V}$ be a subspace of $\mathbf{H}^1(\Omega)$ characterized by the essential boundary conditions $\Pi^D \gamma_0 \mathbf{u} = 0$ on $\partial\Omega$. We define source and target spaces as follows.

(i) For any real number $s > \frac{1}{2}$, let $\mathbf{H}^s_{\mathsf{V}}(\Omega)$ denote the space

$$\mathbf{H}^s_{\mathsf{V}}(\Omega) = \{\mathbf{u} \in \mathbf{H}^s(\Omega),\ \Pi^D \gamma_0 \mathbf{u} = 0 \text{ on } \partial\Omega\}. \tag{3.33}$$

(ii) For a real number $s \geq 1$, $s \neq \frac{3}{2}$, we define $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ differently depending on whether $s < \frac{3}{2}$ or not:

- ⋆ If $s \in [1, \frac{3}{2})$, $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ is the dual space of $\mathbf{H}^{2-s}_{\mathsf{V}}(\Omega)$.
- ⋆ If $s > \frac{3}{2}$, $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ is the space of $\mathbf{q} \in \mathbf{V}'$ for which there exists a couple $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{s-2}(\Omega) \times \prod_{\mathbf{s}\in\mathscr{S}} \Pi^T_{\mathbf{s}} \mathsf{H}^{s-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega)^N$ such that

$$\big\langle \mathbf{q}, \mathbf{v} \big\rangle_{\overline{\Omega}} = \big\langle \mathbf{f}, \mathbf{v} \big\rangle_{\Omega} + \big\langle \mathbf{g}, \Pi^T \gamma_0 \mathbf{v} \big\rangle_{\partial\Omega} \quad \forall \mathbf{v} \in \mathbf{V}. \tag{3.34}$$

(iii) If $\mathcal{V}$ is a subdomain of Ω, we define $\mathbf{H}^s_{\mathsf{V}}(\mathcal{V})$ and $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\mathcal{V})$ by the same formulas, starting from the convention that on the "side" $\partial_{\mathbf{s}_0}\mathcal{V} := \partial\mathcal{V} \setminus \partial\Omega$ of $\mathcal{V}$, we impose full Dirichlet boundary conditions $\Pi^D_{\mathbf{s}_0} = \mathbb{I}_N$. △

Remark 3.4.4 Let $\mathbf{u} \in \mathbf{V}$ be solution of problem (3.10).

(i) If $\mathbf{q} \in \mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ for $s < \frac{3}{2}$, we define a distribution $\mathbf{f} \in \mathbf{H}^{s-2}(\Omega)$ by restricting $\mathbf{q}$ to $\mathbf{H}^{2-s}_0(\Omega)$ which is continuously embedded in $\mathbf{H}^{2-s}_{\mathsf{V}}(\Omega)$, and we find that $L\mathbf{u} = \mathbf{f}$. However, the conormal trace $\Pi^T B\mathbf{u}$ does not make sense in general in this situation.

(ii) In contrast, if we are given a more regular $\mathbf{f}$, namely if $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the expression (3.34) makes sense for any element $\mathbf{g} \in \prod_{\mathbf{s}\in\mathscr{S}} \Pi^T_{\mathbf{s}} \mathsf{H}^{s-\frac{3}{2}}(\partial_{\mathbf{s}}\Omega)^N$, for any $s \in [1, 2]$, and defines an element $\mathbf{q}$ of $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$. In this case, $\mathbf{u}$ solves problem (3.29). △

We note that the operator $\mathbf{u} \mapsto \big(\mathbf{v} \mapsto \mathsf{a}(\mathbf{u}, \mathbf{v})\big)$ is continuous from $\mathbf{H}^s_{\mathsf{V}}(\Omega)$ into $\mathsf{R}\mathbf{H}^s_{\mathsf{V}}(\Omega)$ for all $s \geq 1$, with $s \neq \frac{3}{2}$. Conversely, we have the following general shift result.

Theorem 3.4.5 *Let Ω be a bounded domain and a be a continuous coercive sesquilinear form on the subspace $\mathbf{V}$ of $\mathbf{H}^1(\Omega)$ characterized by the essential boundary conditions*

$$\Pi^D \gamma_0 \mathbf{u} = 0 \quad \textit{on} \quad \partial\Omega.$$

Let $\mathbf{u} \in \mathbf{V}$ be a variational solution of problem (3.10) for a right hand side $\mathbf{q} \in \mathbf{V}'$.

(i) Let two subdomains $\Omega_1 = \mathcal{U}_1 \cap \Omega$ and $\Omega_2 = \mathcal{U}_2 \cap \Omega$ be given with $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$. We set $\Gamma_2 := \overline{\Omega}_2 \cap \partial\Omega$. Let $k \geq 2$ be an integer. We assume that Γ_2 is of class $\mathscr{C}^k$, that the coefficients of the form a *are $\mathscr{C}^{k-1}(\overline{\Omega}_2)$ and those of Π^D are in $\mathscr{C}^k(\Gamma_2)$. Let $s \in (1, k]$, $s \neq \frac{3}{2}$, be a real number. If* **q** *belongs to $\mathsf{RH}^s_\mathsf{V}(\Omega_2)$, then* **u** *belongs to $\mathbf{H}^s(\Omega_1)$ with the estimate*

$$\|\mathbf{u}\|_{s;\,\Omega_1} \leq c\Big(\|\mathbf{q}\|_{\mathsf{RH}^s_\mathsf{V}(\Omega_2)} + \|\mathbf{u}\|_{1;\,\Omega_2}\Big). \tag{3.35}$$

(ii) If the assumptions of (i) are satisfied on the whole domain Ω and its boundary $\partial\Omega$, then there holds for all real number $s \in (1, k]$, $s \neq \frac{3}{2}$,

$$\mathbf{q} \in \mathsf{RH}^s_\mathsf{V}(\Omega) \quad\Longrightarrow\quad \mathbf{u} \in \mathbf{H}^s(\Omega). \tag{3.36}$$

Proof: With Theorems 3.2.6, 3.4.1, 2.3.2, and Corollary 2.3.3 at hands, it remains to prove the above statement for $s \in (1, 2)$. We follow the same four steps as in the proof of Theorem 3.4.1. The first two steps *(i)* and *(ii)* are the same. Thus we are left with the situation of a compact support solution **u** of a coercive variational problem on the periodic half-space $\mathbb{T}^n_+$ (or on the periodic space $\mathbb{T}^n$). We use difference quotients, based on the new notations, for $\sigma = s - 1 \in (0, 1)$

(1) $$\Delta^{\mathbf{h}',\sigma} f(\mathbf{x}) = \frac{f(\mathbf{x} + \mathbf{h}') - f(\mathbf{x})}{|\mathbf{h}'|^{\sigma + \frac{n}{2} - \frac{1}{2}}}, \quad \mathbf{h}' \in \mathbb{T}^{n-1},$$

for the situation of the periodic half-space, and

(2) $$\Delta^{\mathbf{h},\sigma} f(\mathbf{x}) = \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{|\mathbf{h}|^{\sigma + \frac{n}{2}}}, \quad \mathbf{h} \in \mathbb{T}^n,$$

for the interior case. Obviously from the definitions of the spaces with Sobolev-Slobodeckii norms, there hold the following characterizations: For any $f \in \mathsf{L}^2(\mathbb{T}^n_+)$,

(3) $$f \in \mathsf{H}^\sigma(\mathbb{T}^{n-1}, \mathsf{L}^2(\mathbb{R}_+)) \quad\Longleftrightarrow\quad \mathbf{h}' \mapsto \|\Delta^{\mathbf{h}',\sigma} f\|_{0;\mathbb{T}^n_+} \text{ belongs to } \mathsf{L}^2(\mathbb{T}^{n-1}),$$

and, for any $f \in \mathsf{L}^2(\mathbb{T}^n)$,

(4) $$f \in \mathsf{H}^\sigma(\mathbb{T}^n) \quad\Longleftrightarrow\quad \mathbf{h} \mapsto \|\Delta^{\mathbf{h},\sigma} f\|_{0;\mathbb{T}^n} \text{ belongs to } \mathsf{L}^2(\mathbb{T}^n).$$

After localization around a boundary point, our variational space takes the form

$$\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\mathcal{B} \cap \mathbb{T}^n_+) :\ u_1, \ldots, u_{N_0} = 0 \text{ on } \partial\mathbb{T}^n_+ \cap \mathcal{B} \text{ and } \mathbf{u} = 0 \text{ on } \partial\mathcal{B} \setminus \partial\mathbb{T}^n_+\}$$

and we can start with the localized solution $\chi\mathbf{u}$ of the variational problem

(5) $$\mathsf{a}(\chi\mathbf{u}, \mathbf{v}) = \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} \quad \forall \mathbf{v} \in \mathbf{V},$$

where **q** belongs to $\mathsf{RH}^{1+\sigma}_\mathsf{V}(\mathcal{B} \cap \mathbb{T}^n_+)$. Now we use the difference quotients $\Delta^{\mathbf{h}',\sigma}$, with $|\mathbf{h}'| < h_0$ small enough. The term $\Delta^{\mathbf{h}',\sigma}(\chi\mathbf{u})$ still belongs to the variational space **V** and solves a variational problem of the form

$$\mathsf{a}(\Delta^{\mathbf{h}',\sigma}(\chi\mathbf{u}),\mathbf{v}) = \langle \Delta^{\mathbf{h}',\sigma}\mathbf{q},\mathbf{v}\rangle_{\overline{\Omega}} + \langle \mathbf{q}^{\mathbf{h}',\sigma},\mathbf{v}\rangle_{\overline{\Omega}} \quad \forall \mathbf{v}\in\mathbf{V}.$$

The coercivity of a yields the following estimates, with a constant C independent of $\mathbf{h}' \in \mathbb{T}^{n-1}$, $|\mathbf{h}'| < h_0$:

$$(6) \qquad \|\Delta^{\mathbf{h}',\sigma}(\chi\mathbf{u})\|_{1;\,\mathcal{B}\cap\mathbb{T}^n_+} \le C\Big(\|\Delta^{\mathbf{h}',\sigma}\mathbf{q}\|_{\mathbf{V}'} + \|\mathbf{q}^{\mathbf{h}',\sigma}\|_{\mathbf{V}'} + \|\Delta^{\mathbf{h}',\sigma}\mathbf{u}\|_{0;\,\mathcal{B}\cap\mathbb{T}^n_+}\Big).$$

We note that

1. The L^2 norm of the function $\mathbf{h}' \mapsto \|\Delta^{\mathbf{h}',\sigma}\mathbf{u}\|_{0;\,\mathcal{B}\cap\mathbb{T}^n_+}$ is estimated by $\|\mathbf{u}\|_{1;\,\mathcal{B}\cap\mathbb{T}^n_+}$.
2. The $\mathscr{C}^1$ regularity of the coefficients of a yields that the L^2 norm of the function $\mathbf{h}' \mapsto \|\mathbf{q}^{\mathbf{h}',\sigma}\|_{\mathbf{V}'}$ is estimated by $\|\mathbf{q}\|_{\mathbf{V}'}$.
3. The L^2 norm of the function $\mathbf{h}' \mapsto \|\Delta^{\mathbf{h}',\sigma}\mathbf{q}\|_{\mathbf{V}'}$ is estimated by $\|\mathbf{q}\|_{\mathsf{R}\mathbf{H}^{1+\sigma}_{\mathsf{V}}(\mathcal{B}\cap\mathbb{T}^n_+)}$.

With estimates (6) and characterization (3), we conclude that all first order derivatives of $\chi\mathbf{u}$ belong to $\mathsf{H}^\sigma(\mathbb{T}^{n-1}, \mathsf{L}^2(\mathbb{R}_+))$. This implies the $\mathbf{H}^\sigma$ regularity for tangential first order derivatives of $\chi\mathbf{u}$:

$$(7) \qquad \forall\alpha \text{ with } |\alpha| = 1 \text{ and } \alpha_n = 0, \quad \partial^\alpha\chi\mathbf{u} \in \mathbf{H}^\sigma(\mathcal{B}\cap\mathbb{T}^n_+).$$

It remains to prove that the normal derivative of $\chi\mathbf{u}$ also belongs to $\mathbf{H}^\sigma(\mathcal{B}\cap\mathbb{T}^n_+)$. For this, we are going to prove that

$$\forall\alpha,\ |\alpha| = 1, \quad \partial^\alpha\partial_n\chi\mathbf{u} \in \mathbf{H}^{\sigma-1}(\mathcal{B}\cap\mathbb{T}^n_+).$$

From (7), we see that $\partial^\alpha\chi\mathbf{u} \in \mathbf{H}^{\sigma-1}(\mathcal{B}\cap\mathbb{T}^n_+)$ for all $|\alpha| = 2$, except if $\alpha_n = 2$. Now, we end the proof like for Theorem 3.4.1, using the fact that $\partial_{nn}\chi\mathbf{u}$ can be obtained as a combination of $L\chi\mathbf{u}$, which belongs to $\mathbf{H}^{\sigma-1}(\mathcal{B}\cap\mathbb{T}^n_+)$ as a consequence of the regularity of $\mathbf{q}$, and all other second order derivatives, modulo lower order terms. □

As a corollary, we obtain optimal regularity for variational solutions of problem (3.29) if $\mathbf{f}\in\mathbf{L}^2(\Omega)$ and $\mathbf{g}\in\mathbf{H}^\sigma$ for all $\sigma\in[-\frac12,\frac12]$, including $\sigma = 0$:

Corollary 3.4.6 *Let Ω be a bounded domain of class $\mathscr{C}^2$ and a be a continuous coercive sesquilinear form on the subspace $\mathbf{V}$ of $\mathbf{H}^1(\Omega)$ defined by the essential boundary conditions $\Pi^D\mathbf{u} = 0$ on $\partial\Omega$. We assume that the coefficients of a are $\mathscr{C}^1(\overline{\Omega})$ and those of Π^D are in $\mathscr{C}^2(\partial\Omega)$. Let $\sigma\in[-\frac12,\frac12]$ be a real number. Let $\mathbf{u}\in\mathbf{V}$ be a variational solution of problem (3.10) for a right hand side $\mathbf{q}\in\mathbf{V}'$ of the form (3.34) with*

$$\mathbf{f}\in\mathbf{L}^2(\Omega) \quad \textit{and} \quad \mathbf{g}\in\prod_{\mathbf{s}\in\mathscr{S}}\Pi^T_{\mathbf{s}}\mathsf{H}^\sigma(\partial_{\mathbf{s}}\Omega)^N. \tag{3.37}$$

Then $\mathbf{u}$ belongs to $\mathbf{H}^{\sigma+\frac32}(\Omega)$ with corresponding a priori estimates.

Proof: In view of Remark 3.4.4, this result is a simple consequence of Theorem 3.4.5 when $\sigma \neq 0$ (take $s = \sigma + \frac{3}{2}$). We end the proof for the case $\sigma = 0$ by an interpolation argument: We can make the form a strongly coercive by the addition of a term of order 0, which does not alter regularity properties. Thus we can assume that the solution of problem (3.10) induces an isomorphism $\mathbb{B} : \mathbf{u} \mapsto \mathbf{q}$ from $\mathbf{V}$ onto $\mathbf{V}'$.

For $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$, $\sigma \neq 0$ we define the continuous operator

$$\begin{array}{rcl} \mathsf{R}_{(\sigma)} \; : \; \mathbf{L}^2(\Omega) \times \prod_{\mathbf{s}\in\mathscr{S}} \Pi^T_{\mathbf{s}} \mathsf{H}^\sigma(\partial_{\mathbf{s}}\Omega)^N & \longrightarrow & \mathbf{H}^{\sigma+\frac{3}{2}}(\Omega) \\ (\mathbf{f}, \mathbf{g}) & \longmapsto & \mathbf{u} = \mathbb{B}^{-1}\mathbf{q} \end{array}$$

where $\mathbf{q}$ is given by formula (3.34). Interpolating between $-\sigma$ and σ for a fixed positive σ gives the continuity of $\mathsf{R}_{(0)}$. □

3.5 Robin type boundary conditions

There is another type of boundary conditions, not covered by our formalism introduced in Section 3.1, which can also be written in variational form: Robin type boundary conditions, also called boundary conditions of the third kind. They specify a linear combination of a field value and its normal derivative. A simple example is the boundary value problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u + \alpha u &= g \quad \text{on } \partial\Omega. \end{cases}$$

This boundary condition can, for instance, describe heat conduction by convection on the surface. A variational formulation of this problem requires the addition of a boundary integral to the standard variational form a, see §4.1.c for specifics of this example.

For a more general setting, we use the framework of Section 3.1 but introduce an additional bilinear form

$$\mathsf{b}(\mathbf{u}, \mathbf{v}) = \int_{\partial\Omega} (Z\gamma_0\mathbf{u}) \cdot \gamma_0\overline{\mathbf{v}} \, \mathrm{d}\sigma, \tag{3.38}$$

where Z is a smooth function from $\partial\Omega$ into $\mathbb{C}^{N\times N}$. With the sesquilinear form a defined by (3.1), we now consider the new sesquilinear form $\tilde{\mathsf{a}} = \mathsf{a} + \mathsf{b}$:

$$\tilde{\mathsf{a}}(\mathbf{u}, \mathbf{v}) = \mathsf{a}(\mathbf{u}, \mathbf{v}) + \int_{\partial\Omega} (Z\gamma_0\mathbf{u}) \cdot \gamma_0\overline{\mathbf{v}} \, \mathrm{d}\sigma. \tag{3.39}$$

We use the variational space $\mathbf{V} \subset \mathbf{H}^1(\Omega)$ defined by essential boundary conditions as in Definition 3.1.4 and can then consider the problem (compare with (3.10))

$$\textit{Find} \quad \mathbf{u} \in \mathbf{V} \quad \textit{such that} \quad \forall \mathbf{v} \in \mathbf{V}, \quad \tilde{\mathsf{a}}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}}\,, \tag{3.40}$$

for $\mathbf{q}$ given by

$$\langle \mathbf{q}, \mathbf{v} \rangle_{\overline{\Omega}} = \int_{\Omega} \mathbf{f} \cdot \overline{\mathbf{v}} \, \mathrm{d}\mathbf{x} + \langle \mathbf{g}, \gamma_0\mathbf{v} \rangle_{\partial\Omega}\,, \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.41}$$

with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} = (g_1, \ldots, g_N) \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$.

The additional sesquilinear form b is of lower order and therefore the coercivity (in the sense of Definition 3.2.1) of the sesquilinear form $\tilde{\mathsf{a}}$ is equivalent to the coercivity of the principal part a, as the following lemma shows.

Lemma 3.5.1 *The sesquilinear form $\tilde{\mathsf{a}}$ is coercive on $\mathbf{V}$, i.e. there exist $C_1, C_2 > 0$ such that*

$$\forall \mathbf{u} \in \mathbf{V}, \quad \operatorname{Re} \tilde{\mathsf{a}}(\mathbf{u}, \mathbf{u}) \geq C_1 \|\mathbf{u}\|^2_{1;\Omega} - C_2 \|\mathbf{u}\|^2_{0;\Omega},$$

if and only if the sesquilinear form a is coercive on $\mathbf{V}$.

Proof: We use the well-known trace inequality, valid for all $\mathbf{u} \in \mathbf{H}^1(\Omega)$,

$$\|\gamma_0 \mathbf{u}\|^2_{0;\partial\Omega} \leq c_1 \|\mathbf{u}\|_{0;\Omega} \|\mathbf{u}\|_{1;\Omega}. \tag{3.42}$$

For lack of a good reference, we sketch a proof of (3.42): By using the density of $C^1(\overline{\Omega})$ in $\mathsf{H}^1(\Omega)$ and employing a partition of unity and local diffeomorphisms, we can reduce the problem to the case where Ω is the half-space $\mathbb{R}^n_+$ and $u \in C^1_0(\mathbb{R}^n)$. For this case, we get (3.42) from the Cauchy-Schwarz inequality with a constant $c_1 = 2$:

$$\begin{aligned}\|\gamma_0 \mathbf{u}\|^2_{0;\partial\Omega} &= \int_{\mathbb{R}^{n-1}} |\mathbf{u}(\mathbf{x}', 0)|^2 \mathrm{d}\mathbf{x}' = -\int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{d}{dt}\big(|\mathbf{u}(\mathbf{x}', t)|^2\big) \mathrm{d}t \, \mathrm{d}\mathbf{x}' \\ &= -2 \operatorname{Re} \int_{\mathbb{R}^n_+} u(\mathbf{x}) \partial_{x_n} \overline{\mathbf{u}}(\mathbf{x}) \mathrm{d}\mathbf{x} \leq 2 \|\mathbf{u}\|_{0;\Omega} \|\partial_{x_n} \mathbf{u}\|_{0;\Omega}.\end{aligned}$$

A consequence of (3.42) is the estimate

(1) $$\|\gamma_0 \mathbf{u}\|^2_{0;\partial\Omega} \leq c_1 \big(\varepsilon \|\mathbf{u}\|^2_{1;\Omega} + \varepsilon^{-1} \|\mathbf{u}\|^2_{0;\Omega}\big)$$

for any $\varepsilon > 0$. Using (1) and the smoothness of Z, we find c_2 and c_3 such that

(2) $$|\mathsf{b}(\mathbf{u}, \mathbf{u})| \leq c_2 \varepsilon \|\mathbf{u}\|^2_{1;\Omega} + c_3 \varepsilon^{-1} \|\mathbf{u}\|^2_{0;\Omega}$$

holds for all $\varepsilon > 0$.

Now if we suppose that a is coercive on $\mathbf{V}$, we get $c, C > 0$ such that

$$\operatorname{Re} \mathsf{a}(\mathbf{u}, \mathbf{u}) \geq c \|\mathbf{u}\|^2_{1;\Omega} - C \|\mathbf{u}\|^2_{0;\Omega}.$$

Together with estimate (2)this implies

$$c \|\mathbf{u}\|^2_{1,\Omega} \quad \leq \quad \operatorname{Re} \tilde{\mathsf{a}}(\mathbf{u}, \mathbf{u}) + c_2 \varepsilon \|\mathbf{u}\|^2_{1;\Omega} + c_3 \varepsilon^{-1} \|\mathbf{u}\|^2_{0;\Omega} + C \|\mathbf{u}\|^2_{0;\Omega}$$

for all $\varepsilon > 0$. Choosing ε small enough (take $c_2 \varepsilon = \frac{c}{2}$), we finally obtain the coercivity of $\tilde{\mathsf{a}}$ in the sense defined above.

The converse follows from the same arguments. □

As a consequence of this Lemma we find that if a is coercive on $\mathbf{V}$, then the operator $\tilde{A}$ associated with $\tilde{\mathsf{a}}$ is Fredholm of index 0 from $\mathbf{V}$ into $\mathbf{V}'$. This follows from the same arguments as in the proof of Theorem 3.3.1.

A direct consequence of Green's formula (3.9) is the following characterization of the boundary value problem solved by any solution $\mathbf{u}$ of (3.40), compare with Lemma 3.3.3.

Lemma 3.5.2 *Let $\tilde{\mathsf{a}}$ be the bounded sesquilinear form defined by $\tilde{\mathsf{a}} = \mathsf{a} + \mathsf{b}$, with a defined by* (3.1) *and b by* (3.38). *If $\mathbf{u} \in \mathbf{V}$ is a solution of the variational problem* (3.40), *then $\mathbf{u}$ solves the boundary value problem*

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \Pi^{\mathsf{T}} B\mathbf{u} + \Pi^{\mathsf{T}} Z\gamma_0\mathbf{u} = \Pi^{\mathsf{T}}\mathbf{g} & \text{on } \partial\Omega, \\ \Pi^{\mathsf{D}}\gamma_0\mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.43}$$

where we recall that B is the conormal system associated with a and defined by (3.5), *Π^{D} is the projection operator defining $\mathbf{V}$ according to* (3.15) *and $\Pi^{\mathsf{T}} = \mathbb{I}_N - \Pi^{\mathsf{D}}$.*

From this Lemma (or from the definition of b), we see that the regularity of a solution of (3.40) is a direct consequence of the regularity of the solution of the associated standard problem (3.29), since the boundary condition

$$\Pi^{\mathsf{T}} B\mathbf{u} + \Pi^{\mathsf{T}} Z\gamma_0\mathbf{u} = \Pi^{\mathsf{T}}\mathbf{g} \quad \text{on } \partial\Omega,$$

can be equivalently written

$$\Pi^{\mathsf{T}} B\mathbf{u} = \Pi^{\mathsf{T}}\mathbf{g} - \Pi^{\mathsf{T}} Z\gamma_0\mathbf{u} \quad \text{on } \partial\Omega.$$

Hence if $\mathbf{u}$ is a solution of the variational problem (3.40), then $\mathbf{u}$ solves problem (3.29) with interior datum $\mathbf{f}$ and boundary datum $\mathbf{g} - Z\gamma_0\mathbf{u} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Furthermore if $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$, then $\mathbf{g} - Z\gamma_0\mathbf{u}$ belongs to $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ as well, due to the continuity of the trace operator from $\mathbf{H}^1(\Omega)$ into $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ (cf. (2.1)). Applying Theorem 3.4.1, we obtain the following regularity result.

Theorem 3.5.3 *Let Ω be a smooth bounded domain and a be a coercive sesquilinear form* (3.1) *on $\mathbf{V}$. Assume that the coefficients of the form a belong to $\mathscr{C}^1(\overline{\Omega})$ and that the coefficients of the matrix Z are in $\mathscr{C}^1(\partial\Omega)$. Let $\mathbf{u} \in \mathbf{V}$ be a variational solution of problem* (3.40) *for a right hand side $\mathbf{q}$ defined by* (3.41) *with $\mathbf{f}$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{g}$ in $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then $\mathbf{u}$ belongs to $\mathbf{H}^2(\Omega)$ with the estimate* (3.31).

Higher order regularities are obtained without any efforts since the principal part of the operator $B + Z\gamma_0$ is the same as the one of B. Then by combining Theorem 3.5.3 with Theorems 2.3.2 and 2.7.1 we find the following global regularity result, in Sobolev spaces and analytic classes:

Theorem 3.5.4 *Let the assumptions of Theorem 3.5.3 be satisfied. Let* $\mathbf{u} \in \mathbf{V}$ *be a variational solution of problem* (3.40) *for a right hand side* $\mathbf{q}$ *defined by* (3.41) *with* $\mathbf{f}$ *in* $\mathbf{L}^2(\Omega)$ *and* $\mathbf{g}$ *in* $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$.

(i) Let k *be a non-negative integer. We assume that* Ω *is of class* $\mathscr{C}^{k+2}$*, that the coefficients of the form* a *are in* $\mathscr{C}^{k+1}(\overline{\Omega})$ *and those of the matrix* Z *are in* $\mathscr{C}^{k+1}(\partial\Omega)$. *If* $\mathbf{f} \in \mathbf{H}^k(\Omega)$ *and* $\mathbf{g} \in \mathbf{H}^{k+\frac{1}{2}}(\partial\Omega)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{H}^{k+2}(\Omega)$.

(ii) We assume that Ω *is analytic, that the coefficients of the form* a *are analytic up to the boundary of* Ω*, and that the projector* Π^{D} *and the matrix* Z *are analytic on* $\partial\Omega$. *If* $\mathbf{f} \in \mathbf{A}(\Omega)$ *and* $\mathbf{g} \in \mathbf{A}(\partial\Omega)$*, then* $\mathbf{u}$ *belongs to* $\mathbf{A}(\Omega)$.

Chapter 4

Examples

Introduction

In this chapter, we introduce various equations or systems occurring in particular in physical modelling, and which enter our general framework of second order elliptic boundary value problems. This gives the opportunity of defining the classical examples which we will address all along our work, and also to explain how the assumptions on ellipticity and covering can be verified. All mathematical results in this chapter are well known, with the exception of the results about the variational solution of the impedance problem (see Section 4.5.d), which are new, as far as we know.

Plan of Chapter 4

§1 The Laplace operator Δ, with Dirichlet, Neumann or Robin boundary conditions.

§2 More general second order scalar operator in divergence form.

§3 The Lamé system, for linearly elastic isotropic materials.

§4 More general elasticity systems for anisotropic inhomogeneous materials.

§5 Regularized Maxwell equations (in one-field formulations using the electric field only, or the magnetic field only for perfectly conducting boundary conditions). Two field formulations for impedance boundary conditions.

§6 The Reissner-Mindlin plate model, which is an asymptotic model for thin plates, formulated on their mid-surfaces.

§7 The piezoelectric system, where the equations of linear elasticity are coupled with an electric potential.

§8 Conclusion about the influence of variational formulations on natural boundary conditions: We investigate distinct variational formulations corresponding to the same second order system, but to distinct boundary conditions and, accordingly, distinct covering properties.

4.1 The Laplace operator

In this section we consider the Laplace operator $\Delta = \partial_1^2 + \ldots + \partial_n^2$. It is elliptic because its symbol is equal to $-(\xi_1^2 + \ldots + \xi_n^2) = -|\boldsymbol{\xi}|^2$ and is therefore different from zero if $\boldsymbol{\xi} \neq 0$. Clearly $-\Delta$ is even strongly elliptic.

As boundary conditions, we consider the classical ones, namely the Dirichlet and Neumann conditions.

4.1.a The Dirichlet problem

We start with the Dirichlet problem:

$$\left\{\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.1}$$

We prove that this system forms an elliptic boundary value problem: As described in Chapter 2, the tangent system $(\underline{L}_{\mathbf{x}_0}, \underline{C}_{\mathbf{x}_0})$ associated with (4.1) at a boundary point $\mathbf{x}_0$ represents the Dirichlet problem on the half space $\mathbb{R}^n_+$ for the operator

$$\underline{L}_{\mathbf{x}_0} = \mathrm{D}_{\check{\mathbf{x}}}^\top J_{\mathbf{x}_0} J_{\mathbf{x}_0}^\top \mathrm{D}_{\check{\mathbf{x}}}$$

where $J_{\mathbf{x}_0}$ is the Jacobian of the local diffeomorphism $\phi_{\mathbf{x}_0}$ in the point $\mathbf{x}_0$, see (2.34) in Definition 2.2.31. This will again be the Laplacian if $J_{\mathbf{x}_0}$ is an orthogonal matrix which can always be achieved by a suitable choice of basis in $\mathbb{R}^n$. Thus we are reduced to the problem (4.1) set in $\mathbb{R}^n_+$.

According to Definition 2.2.6, the covering condition is satisfied if the system

$$\left\{\begin{array}{rcll} |\boldsymbol{\xi}'|^2 U - \partial_t^2 U &=& 0 & \text{in } \mathbb{R}_+, \\ U\big|_{t=0} &=& H, & \end{array}\right. \tag{4.2}$$

has a unique exponentially decaying solution U for all non-zero $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$. As the two linearly independent solutions of the differential equation $|\boldsymbol{\xi}'|^2 U - \partial_t^2 U = 0$ are $\mathrm{e}^{-|\boldsymbol{\xi}'|t}$ and $\mathrm{e}^{|\boldsymbol{\xi}'|t}$, the unique exponentially decaying solution of (4.2) is

$$U(t) = H\,\mathrm{e}^{-|\boldsymbol{\xi}'|t}.$$

Note finally that (4.1) enters into the variational setting of Chapter 3 by taking

$$\mathsf{V} = \mathsf{H}^1_0(\Omega), \tag{4.3}$$

$$\mathsf{a}(u,v) = \int_\Omega \nabla u \cdot \nabla \bar{v}\,\mathrm{d}\mathbf{x}, \quad \forall u,v \in \mathsf{V}. \tag{4.4}$$

The Poincaré inequality (1.10) guarantees the strong coerciveness of a on $\mathsf{H}^1_0(\Omega)$, namely there exists $c > 0$ such that

$$\forall u \in \mathsf{H}^1_0(\Omega), \quad \mathsf{a}(u,u) = |u|^2_{1,\Omega} \geq c\|u\|^2_{1,\Omega}.$$

By the Lax-Milgram lemma, this further implies that the associated operator A defined by (3.26) is an isomorphism from $\mathsf{H}^1_0(\Omega)$ onto $\mathsf{H}^{-1}(\Omega)$.

As all the assumptions of Theorem 3.4.5 are satisfied for problem (4.1), the shift theorem in standard Sobolev or analytic spaces is valid. More precisely we have the

Theorem 4.1.1 *Let Ω be a smooth bounded domain and a be the sesquilinear form defined by (4.4) on $\mathsf{H}^1_0(\Omega)$. For $f \in \mathsf{H}^{-1}(\Omega)$ let $u \in \mathsf{H}^1_0(\Omega)$ be the unique variational solution of*

$$\mathsf{a}(u, v) = \langle f\,,\,\bar v\rangle, \quad \forall v \in \mathsf{H}^1_0(\Omega).$$

(i) Let k be a non-negative integer. We assume that Ω is of class $\mathscr{C}^{k+2}$. If $f \in \mathsf{H}^k(\Omega)$, then u belongs to $\mathsf{H}^{k+2}(\Omega)$.

(ii) We assume that Ω is analytic. If $f \in \mathsf{A}(\Omega)$, then u belongs to $\mathsf{A}(\Omega)$.

4.1.b The Neumann problem

Here we consider the following problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $\partial_n u = \mathbf{n} \cdot \nabla u$ is the normal derivative of u.

As before this system forms an elliptic boundary value problem: Again if the local Jacobian J_{x_0} is orthogonal, the tangent system is still problem (4.5) set in $\mathbb{R}^n_+$. Then indeed (cf. Definition 2.2.6) the covering condition is satisfied since the system

$$\begin{cases} |\boldsymbol{\xi}'|^2 U - \partial_t^2 U &= 0 \quad \text{in } \mathbb{R}_+, \\ \partial_t U\big|_{t=0} &= H, \end{cases} \tag{4.6}$$

has a unique exponentially decaying solution U for all non-zero $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$ given by

$$U(t) = -\frac{H}{|\boldsymbol{\xi}'|}\,\mathrm{e}^{-|\boldsymbol{\xi}'|t}.$$

The variational formulation of (4.5) consists in taking the sesquilinear form a defined by (4.4) but on $\mathsf{V} = \mathsf{H}^1(\Omega)$. This form is coercive in the sense of Definition 3.2.1 because

$$\mathsf{a}(u, u) = |u|^2_{1,\Omega} = \|u\|^2_{1,\Omega} - \|u\|^2_{0,\Omega}.$$

Denote by A, the operator associated with a defined by (3.26). It is easy to check that $\ker A = \ker A^\star = \mathbb{C}$, since

$$\mathsf{a}(u, u) = 0 \quad \Longleftrightarrow \quad u \text{ is a constant function.}$$

By the Lax-Milgram lemma, this implies that A is an isomorphism from $\mathsf{H}^1(\Omega)/\mathbb{C}$ onto $\{f \in \mathsf{H}^{-1}(\Omega) : \langle f\,,\,1\rangle = 0\}$.

Again all the assumptions of Theorem 3.4.5 are satisfied for problem (4.5) and therefore the shift theorem in standard Sobolev or analytic spaces holds:

Theorem 4.1.2 *Let Ω be a smooth bounded domain and a be the sesquilinear form defined by (4.4) on $\mathsf{H}^1(\Omega)$. Let $u \in \mathsf{H}^1(\Omega)$ be a (unique up to a constant) variational solution of problem*

$$\mathsf{a}(u,v) = \int_\Omega f\bar{v}\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} g\bar{v}\,\mathrm{d}\sigma, \quad \forall v \in \mathsf{H}^1(\Omega),$$

with $f \in \mathsf{L}^2(\Omega)$ and $g \in \mathsf{H}^{\frac{1}{2}}(\partial\Omega)$ such that $\int_\Omega f\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} g\,\mathrm{d}\sigma = 0$.

(i) Let k be a non-negative integer. If Ω is of class $\mathscr{C}^{k+2}$, $f \in \mathsf{H}^k(\Omega)$ and $g \in \mathsf{H}^{k+\frac{1}{2}}(\partial\Omega)$, then u belongs to $\mathsf{H}^{k+2}(\Omega)$.

(ii) If we assume that Ω is analytic, that $f \in \mathsf{A}(\Omega)$ and $g \in \mathsf{A}(\partial\Omega)$, then u belongs to $\mathsf{A}(\Omega)$.

4.1.c Robin boundary conditions

Now we consider the Laplace equation with Robin boundary conditions

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ \partial_n u + \alpha u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where α is a complex number such that $\operatorname{Re}\alpha > 0$.

According to the framework studied in Section 3.5, this system forms an elliptic boundary value problem whose principal part is the Neumann problem (4.5). Moreover its variational formulation consists in taking the sesquilinear form $\tilde{\mathsf{a}}$ given by

$$\tilde{\mathsf{a}}(u,v) = \int_\Omega \nabla u \cdot \nabla\bar{v}\,\mathrm{d}\mathbf{x} + \alpha \int_{\partial\Omega} \gamma_0 u \gamma_0 \bar{v}\,\mathrm{d}\sigma.$$

The assumption $\operatorname{Re}\alpha > 0$ guarantees that $\tilde{\mathsf{a}}$ is strongly coercive on $\mathsf{H}^1(\Omega)$, in the sense that there exists $c > 0$ such that

$$\forall u \in \mathsf{H}^1(\Omega), \quad \operatorname{Re}\tilde{\mathsf{a}}(u,u) \geq c\|u\|^2_{1,\Omega}.$$

Indeed by a contradiction argument and the compact embedding of $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$, one readily shows that

$$\forall u \in \mathsf{H}^1(\Omega), \quad \operatorname{Re}\tilde{\mathsf{a}}(u,u) = |u|^2_{1,\Omega} + \operatorname{Re}\alpha|\gamma_0 u|^2_{0,\partial\Omega} \geq c\|u\|^2_{1,\Omega}.$$

Hence by the Lax-Milgram lemma, there exists a unique solution $u \in \mathsf{H}^1(\Omega)$ of

$$\tilde{\mathsf{a}}(u,v) = \int_\Omega f\bar{v}\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} g\bar{v}\,\mathrm{d}\sigma, \quad \forall v \in \mathsf{H}^1(\Omega), \tag{4.8}$$

with $f \in \mathsf{L}^2(\Omega)$ and $g \in \mathsf{H}^{\frac{1}{2}}(\partial\Omega)$.

Since the assumptions of Theorem 3.5.4 are satisfied for problem (4.7), the shift theorem in standard Sobolev or analytic spaces holds:

Theorem 4.1.3 *Let Ω be a smooth bounded domain and $\tilde{\mathsf{a}}$ be the sesquilinear form defined above on $\mathsf{H}^1(\Omega)$. Let $u \in \mathsf{H}^1(\Omega)$ be the unique solution of problem (4.8) with $f \in \mathsf{L}^2(\Omega)$ and $g \in \mathsf{H}^{\frac{1}{2}}(\partial\Omega)$.*

(i) Let k be a non-negative integer. If Ω is of class $\mathscr{C}^{k+2}$, $f \in \mathsf{H}^k(\Omega)$ and $g \in \mathsf{H}^{k+\frac{1}{2}}(\partial\Omega)$, then u belongs to $\mathsf{H}^{k+2}(\Omega)$.

(ii) If we assume that Ω is analytic, that $f \in \mathsf{A}(\Omega)$ and $g \in \mathsf{A}(\partial\Omega)$, then u belongs to $\mathsf{A}(\Omega)$.

Note that the condition $\operatorname{Re}\alpha > 0$ can be weakened. Let $\operatorname{Re}\alpha \leq 0$ and $\operatorname{Im}\alpha \neq 0$. Then, as we will show, the statements of Theorem 4.1.3 remain valid. Only the case $\operatorname{Re}\alpha < 0$ and $\operatorname{Im}\alpha = 0$ gives rise to a non zero kernel.

The trick consists in representing α as

$$\alpha = |\alpha|\, \mathrm{e}^{2i\theta} \quad \text{with} \quad -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}$$

and defining the sesquilinear form

$$\tilde{\mathsf{a}}_\theta = \mathrm{e}^{-i\theta}\, \tilde{\mathsf{a}}\,.$$

With such a choice, the new form $\tilde{\mathsf{a}}_\theta$ is strongly coercive on $\mathsf{H}^1(\Omega)$ in the above sense and by the Lax-Milgram lemma, there exists a unique solution $u \in \mathsf{H}^1(\Omega)$ of

$$\tilde{\mathsf{a}}_\theta(u,v) = \mathrm{e}^{-i\theta} \int_\Omega f\bar{v}\,\mathrm{d}\mathbf{x} + \mathrm{e}^{-i\theta} \int_{\partial\Omega} g\bar{v}\,\mathrm{d}\sigma, \quad \forall v \in \mathsf{H}^1(\Omega),$$

with $f \in \mathsf{L}^2(\Omega)$ and $g \in \mathsf{H}^{\frac{1}{2}}(\partial\Omega)$. This is the same problem as (4.8).

4.1.d The Helmholtz operator

All the regularity results of the above subsections can be generalized to the Helmholtz operator $\Delta + k^2$, with $k \in \mathbb{C}$, since its principal part is the Laplace operator. The only difference concerns the kernel of the associated operator A_k defined by (3.26), when the sesquilinear form is clearly given by

$$\mathsf{a}(u,v) = \int_\Omega (\nabla u \cdot \nabla \bar{v} - k^2 u\bar{v})\,\mathrm{d}\mathbf{x}.$$

If k^2 is not an eigenvalue of the operator A_0, then $\ker A_k = \ker A_k^\star = \{0\}$. On the other hand, if k^2 is an eigenvalue of the operator A_0, then $\ker A_k = \ker A_k^\star$ is equal to the set of eigenvectors of A_0 associated with the eigenvalue k^2.

4.2 Second order scalar operator

We here concentrate on the case of a general second order scalar operator in divergence form:

$$L = -\operatorname{div}(A\nabla) = -\sum_{k,\ell=1}^{n} \partial_k (A^{k\ell}\partial_\ell),$$

where $A(\mathbf{x}) = (A^{k\ell}(\mathbf{x}))_{1\le k,\ell\le n}$ is a real positive definite $n\times n$ matrix depending smoothly on $\mathbf{x}$, i.e., the mapping $\mathbf{x} \mapsto A(\mathbf{x})$ is a smooth function on $\bar\Omega$ with values in the space of positive definite $n \times n$ matrices. Without loss of generality we may assume that A is symmetric. Its principal part frozen at a point $\mathbf{x}_0$ is given by

$$L^{\mathsf{pr}}_{\mathbf{x}_0}(\mathrm{D}_{\mathbf{x}}) = -\sum_{k,\ell=1}^{n} A^{k\ell}(\mathbf{x}_0)\partial_k\partial_\ell.$$

Its symbol is then

$$L^{\mathsf{pr}}_{\mathbf{x}_0}(\boldsymbol{\xi}) = \sum_{k,\ell=1}^{n} A^{k\ell}(\mathbf{x}_0)\xi_k\xi_\ell.$$

Due to the positive definiteness of $A(\mathbf{x}_0)$, we have

$$\sum_{k,\ell=1}^{n} A^{k\ell}(\mathbf{x}_0)\xi_k\xi_\ell \ge c|\boldsymbol{\xi}|^2, \tag{4.9}$$

for some $c > 0$. This shows that L is elliptic at any point of $\bar\Omega$ (it is even strongly elliptic). In dimension $n = 2$, L is properly elliptic because

$$L^{\mathsf{pr}}_{\mathbf{x}_0}(\xi',\tau) = a_{11}(\xi')^2 + 2a_{12}\xi'\tau + a_{22}\tau^2,$$

where for shortness we have set $a_{k\ell} = A^{k\ell}(\mathbf{x}_0)$. The positive definiteness of $A(\mathbf{x}_0)$ is here equivalent to the condition

$$\delta = a_{12}^2 - a_{11}a_{22} < 0 \ \text{ and } \ a_{11} > 0,\ a_{22} > 0.$$

Therefore for $\xi' \ne 0$, the roots $\tau_\pm$ of $L^{\mathsf{pr}}_{\mathbf{x}_0}(\xi',\tau) = 0$ are of the form

$$\tau_\pm = \frac{-a_{12} \pm i\sqrt{-\delta}}{a_{22}}\,\xi'.$$

In other words, the equation $L^{\mathsf{pr}}_{\mathbf{x}_0}(\xi',\tau) = 0$ has exactly one complex root with a positive imaginary part (in fact it has two conjugate complex roots).

In Ω, we can complement L with either Dirichlet or Neumann boundary conditions.

4.2.a Dirichlet boundary conditions

The problem is then

$$\begin{cases} -\operatorname{div}(A\nabla)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

Using the local diffeomorphism $\phi_{\mathbf{x}_0}$ at a point $\mathbf{x}_0$ of the boundary of Ω, we obtain a tangent system of the same form as (4.10) but set in $\mathbb{R}^n_+$. In detail, we have with the components $(\phi_\ell)_{\ell=1\ldots n}$ of $\phi_{\mathbf{x}_0}$

$$\begin{cases} \underline{L}_{\mathbf{x}_0}(\mathrm{D}_\mathbf{x})v = -\displaystyle\sum_{k,\ell=1}^{n} \breve{a}_{k\ell}\partial_k\partial_\ell v &= \breve{f} \quad \text{in } \mathbb{R}^n_+, \\ v &= 0 \quad \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

where the coefficients $\breve{a}_{k\ell}$ are given by

$$\breve{a}_{k\ell} = \sum_{k',\ell'=1}^{n} A^{k'\ell'}(\mathbf{x}_0)(\partial_{k'}\phi_k)(\mathbf{x}_0)(\partial_{\ell'}\phi_\ell)(\mathbf{x}_0).$$

Accordingly the ellipticity condition has to be checked for the symbol

$$\underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}) = \sum_{k,\ell=1}^{n}\sum_{k',\ell'=1}^{n} A^{k'\ell'}(\mathbf{x}_0)(\partial_{k'}\phi_k)(\partial_{\ell'}\phi_\ell)\xi_k\xi_\ell = \sum_{k',\ell'=1}^{n} A^{k'\ell'}(\mathbf{x}_0)\eta_{k'}\eta_{\ell'},$$

where $\boldsymbol{\eta} = J^\top_{\mathbf{x}_0}\boldsymbol{\xi}$, the matrix $J_{\mathbf{x}_0}$ being the Jacobi matrix of $\phi_{\mathbf{x}_0}$:

$$J_{\mathbf{x}_0} = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{pmatrix}.$$

Since $\underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}) = L^{\mathsf{pr}}_{\mathbf{x}_0}(J^\top_{\mathbf{x}_0}\boldsymbol{\xi})$, the ellipticity of $\underline{L}_{\mathbf{x}_0}$ is a consequence of (4.9).

According to Definition 2.2.6, the covering condition is satisfied if the system

$$\begin{cases} \underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}', \frac{1}{i}\partial_t) &= 0 \quad \text{in } \mathbb{R}_+, \\ U\big|_{t=0} &= H, \end{cases} \tag{4.11}$$

has a unique exponentially decaying solution U for all non-zero $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$. The two linearly independent solutions of the differential equation $\underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}', \frac{1}{i}\partial_t) = 0$ are $\mathrm{e}^{i\tau_+ t}$ and $\mathrm{e}^{i\tau_- t}$, where $\tau_\pm$ are the two roots of the equation

$$\underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}', \tau) = 0.$$

For a fixed non-zero $\boldsymbol{\xi}'$, the above equation is of the form

$$\alpha + 2\beta\tau + \breve{a}_{n,n}\tau^2 = 0,$$

where α and β depend on $\boldsymbol{\xi}'$. As in the 2D case, the positive definiteness of A implies that $\alpha > 0$, $\breve{a}_{n,n} > 0$ and $\delta = \beta^2 - \alpha\breve{a}_{n,n} < 0$. Therefore, the two roots are

$$\tau_\pm = \frac{-\beta \pm i\sqrt{-\delta}}{\alpha}$$

and the unique exponentially decaying solution of (4.11) is then $U(t) = H\, \mathrm{e}^{i\tau_+ t}$.

The variational setting of (4.10) consists in taking $\mathsf{V} = \mathsf{H}^1_0(\Omega)$ and

$$\mathsf{a}(u,v) = \int_\Omega A\nabla u \cdot \nabla \bar{v}\,\mathrm{d}\mathbf{x}, \quad \forall u,v \in \mathsf{V}. \tag{4.12}$$

The positive definiteness of A and its continuity imply that there exists $\alpha > 0$ such that

$$\mathsf{a}(u,u) = \int_\Omega A\nabla u \cdot \nabla \bar{u}\,\mathrm{d}\mathbf{x} \geq \alpha \int_\Omega |\nabla u|^2\,\mathrm{d}\mathbf{x}, \quad \forall u \in \mathsf{H}^1(\Omega). \tag{4.13}$$

Invoking the Poincaré inequality (1.10) we deduce the strong coerciveness of a on $\mathsf{H}^1_0(\Omega)$, i.e., the existence of $c > 0$ such that

$$\mathsf{a}(u,u) \geq c\|u\|^2_{1,\Omega}, \quad \forall u \in \mathsf{V}.$$

By the Lax-Milgram lemma, the associated operator A defined by (3.26) is an isomorphism from $\mathsf{H}^1_0(\Omega)$ onto $\mathsf{H}^{-1}(\Omega)$.

We have checked all the assumptions of Theorem 3.4.5 for problem (4.10), consequently the shift theorem in standard Sobolev or analytic spaces is valid. This means that all the results stated in Theorem 4.1.1 hold in this more general case.

4.2.b Neumann boundary conditions

The problem is here

$$\begin{cases} -\operatorname{div}(A\nabla)u = f & \text{in } \Omega, \\ (A\nabla u)\cdot\mathbf{n} = g & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Passing to the corresponding tangent system at a point $\mathbf{x}_0$ of the boundary of Ω, we obtain a system of the same form as (4.14) but set in $\mathbb{R}^n_+$, namely

$$\begin{cases} \underline{L}_{\mathbf{x}_0}(\mathrm{D}_\mathbf{x})v = -\displaystyle\sum_{k,\ell=1}^n \breve{a}_{k\ell}\partial_k\partial_\ell v = \breve{f} & \text{in } \mathbb{R}^n_+, \\ \displaystyle\sum_{k=1}^n \breve{a}_{kn}\partial_k v = \breve{g} & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

where the coefficients $\breve{a}_{k\ell}$ are given as in the previous subsection. The ellipticity being already checked, we can directly pass to the covering condition. It will be satisfied (cf. Definition 2.2.6) if the system

$$\left\{\begin{array}{rcll} \underline{L}_{\mathbf{x}_0}(\boldsymbol{\xi}', \frac{1}{i}\partial_t) &=& 0 & \text{in } \mathbb{R}_+, \\ \big(\sum_{k=1}^{n-1} \breve{a}_{kn}\xi'_k v + \breve{a}_{nn}\dfrac{1}{i}\partial_t v\big)\big|_{t=0} &=& H, & \end{array}\right. \tag{4.15}$$

has a unique exponentially decaying solution U for all non-zero $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$. As in the previous subsection this unique exponentially decaying solution is given by

$$U(t) = \kappa\, \mathrm{e}^{i\tau_+ t} \quad \text{with} \quad \kappa = \frac{H}{\sum_{k=1}^{n-1}\breve{a}_{kn}\xi'_k + \breve{a}_{nn}\tau_+},$$

the denominator being non-zero because the imaginary part of τ_+ is positive.

For the variational setting of (4.14) we take the sesquilinear form (4.12) defined on $\mathsf{V} = \mathsf{H}^1(\Omega)$. Using (4.13), we directly get

$$\mathsf{a}(u,u) \geq \alpha|u|^2_{1,\Omega} = \alpha\|u\|^2_{1,\Omega} - \alpha\|u\|^2_{0,\Omega},$$

which means that a is coercive in the sense of Definition 3.2.1. Let A be the operator associated with a via (3.26). Using the above estimate we directly deduce that $\ker A = \ker A^\star = \mathbb{C}$. Again by the Lax-Milgram lemma, this implies that A is an isomorphism from $\mathsf{H}^1(\Omega)/\mathbb{C}$ onto $\{f \in \mathsf{H}^{-1}(\Omega) : \langle f\,,\,1\rangle = 0\}$.

Since all the assumptions of Theorem 3.4.5 hold for problem (4.14), the shift theorem in standard Sobolev or analytic spaces is valid, i. e., all the results stated in Theorem 4.1.2 hold in this more general case.

4.3 The Lamé system

Here we consider the Lamé system that corresponds to linear elasticity with an isotropic material law, which is a classical system of mathematical physics [71, 21, 33, 35, 77]. It is a $n \times n$ system that can be written as

$$L(\mathrm{D}_{\mathbf{x}}) = -\mu\Delta\mathbb{I}_n - (\lambda+\mu)\nabla\,\mathrm{div}\,. \tag{4.16}$$

This means that the components L_{ij} of L are given by

$$L_{ij}(\mathrm{D}_{\mathbf{x}}) = -\mu\delta_{ij}\Delta - (\lambda+\mu)\partial_i\partial_j.$$

The coefficients λ and μ are called the Lamé coefficients. They are linked to the Poisson coefficient ν and the Young modulus E by the formulas

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \quad \text{and} \quad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \tag{4.17}$$

Conversely λ and μ are determined by ν and E by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}. \tag{4.18}$$

"Real" materials satisfy $E > 0$ and $0 < \nu < 1/2$,[1] that is, equivalently:

$$\mu > 0 \quad \text{and} \quad \lambda > 0.$$

We will see below that this condition is sufficient, but not necessary to guarantee the ellipticity assumptions, and, in fact, leads to the strong ellipticity of L.

Coming back to the ellipticity of L, we need to check that its symbol is invertible. Here this symbol is equal to the matrix $L(\boldsymbol{\xi}) = (L_{ij}(\boldsymbol{\xi}))_{1\le i,j\le n}$, where

$$L_{ij}(\boldsymbol{\xi}) = \mu\delta_{ij}|\boldsymbol{\xi}|^2 + (\lambda+\mu)\xi_i\xi_j.$$

First we remark that μ cannot be equal to zero. Indeed if $\mu = 0$, then $L_{ij}(\boldsymbol{\xi}) = \lambda\xi_i\xi_j$ and therefore each line is a multiple of the vector $\boldsymbol{\xi}$, consequently $L(\boldsymbol{\xi})$ is not invertible. In the case $\mu \neq 0$ setting $\gamma = \frac{\lambda_L+\mu_L}{\mu_L} = \frac{1}{1-2\nu}$, the matrix $L(\boldsymbol{\xi})$ can be equivalently written

$$L(\boldsymbol{\xi}) = \mu \begin{pmatrix} |\boldsymbol{\xi}|^2+\gamma\xi_1^2 & \gamma\xi_1\xi_2 & \dots & \gamma\xi_1\xi_n \\ \gamma\xi_1\xi_2 & |\boldsymbol{\xi}|^2+\gamma\xi_2^2 & \dots & \gamma\xi_2\xi_n \\ & & \ddots & \\ \gamma\xi_1\xi_{n-1} & \dots & |\boldsymbol{\xi}|^2+\gamma\xi_{n-1}^2 & \gamma\xi_{n-1}\xi_n \\ \gamma\xi_1\xi_n & \dots & \gamma\xi_{n-1}\xi_n & |\boldsymbol{\xi}|^2+\gamma\xi_n^2 \end{pmatrix}. \tag{4.19}$$

We can show by induction that the determinant of this matrix is equal to

$$\det L(\boldsymbol{\xi}) = \mu^n(1+\gamma)|\boldsymbol{\xi}|^{2n},$$

and therefore $L(\boldsymbol{\xi})$ is invertible if and only if $\gamma \neq -1$ and $\mu \neq 0$ or equivalently if and only $\mu \neq 0$ and $\lambda \neq -2\mu$.

To show the strong ellipticity of L, we need to show that the matrix $L(\boldsymbol{\xi})$ is positive definite: For any $\boldsymbol{\xi} \neq 0$ and $\boldsymbol{\eta} \in \mathbb{R}^n$, we consider the expression

$$\begin{aligned} \boldsymbol{\eta}^\top L(\boldsymbol{\xi})\boldsymbol{\eta} &= \sum_{i,j=1}^n L_{ij}(\boldsymbol{\xi})\eta_i\eta_j = \mu|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 + (\lambda+\mu)\sum_{i,j=1}^n \xi_i\xi_j\eta_i\eta_j \\ &= \mu|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2 + (\lambda+\mu)(\boldsymbol{\xi}\cdot\boldsymbol{\eta})^2 \\ &= |\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2\big(\mu + (\lambda+\mu)\cos^2\theta\big) \\ &= |\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2\big(\mu\sin^2\theta + (\lambda+2\mu)\cos^2\theta\big), \end{aligned}$$

where θ is the angle between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. From this last identity, we see that if $\mu > 0$ and $\lambda + 2\mu > 0$, then $\boldsymbol{\eta}^\top L(\boldsymbol{\xi})\boldsymbol{\eta} \geq 0$ and $\boldsymbol{\eta}^\top L(\boldsymbol{\xi})\boldsymbol{\eta} = 0$ if and only if $\boldsymbol{\eta} = 0$. It is also easy

[1] The limit $\nu = \frac{1}{2}$ corresponds to incompressibility.

to check that these two conditions are necessary for the positive definiteness of $L(\boldsymbol{\xi})$. In other words $L(\boldsymbol{\xi})$ is a positive definite matrix if and only if $\mu > 0$ and $\lambda + 2\mu > 0$.

For the Lamé system, the classical boundary conditions are the hard clamped and stress free ones [71, 20, 21, 33, 35, 77]. We further consider less standard conditions involving normal or tangential components of the displacement field.

4.3.a Hard Clamped boundary conditions

The problem is

$$\left\{ \begin{array}{rcll} L\mathbf{u} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{0} & \text{on } \partial\Omega. \end{array} \right. \tag{4.20}$$

If we assume that $\mu > 0$ and $\lambda > 0$, then this system is an elliptic boundary value problem as a consequence of the coerciveness (proved below) of the sesquilinear form associated with this problem, by virtue of Theorem 3.2.6. On the other hand if $\mu > 0$ and $\lambda + 2\mu > 0$, then the Lamé system is strongly elliptic and we can alternatively invoke [4]. Let us now consider the general case in dimension 3 in order to obtain the largest set of values of λ and μ guaranteeing the covering condition. First we may remark that the elasticity system is invariant by rotation by using the Piola transformation. Namely if we perform the rotation $\mathbf{y} = R\mathbf{x}$, and set

$$\mathbf{v}(\mathbf{y}) = R\mathbf{u}(\mathbf{x}), \quad \mathbf{g}(\mathbf{y}) = R\mathbf{f}(\mathbf{x}),$$

then we see that $\mathbf{u}$ satisfies

$$\mu\Delta_{\mathbf{x}}\mathbf{u} + (\lambda + \mu)\nabla_{\mathbf{x}} \operatorname{div}_{\mathbf{x}} \mathbf{u} = \mathbf{f},$$

if and only if $\mathbf{v}$ satisfies

$$\mu\Delta_{\mathbf{y}}\mathbf{v} + (\lambda + \mu)\nabla_{\mathbf{y}} \operatorname{div}_{\mathbf{y}} \mathbf{u} = \mathbf{g},$$

Consequently we are reduced to checking the covering condition in the half-space $\mathbb{R}^3_+$. In this case for any $\boldsymbol{\xi}' = (\xi_1, \xi_2) \in \mathbb{R}^2$, the associated operator $L(\boldsymbol{\xi}', \mathrm{D}_t)$ takes the form

$$L(\boldsymbol{\xi}', \mathrm{D}_t) = \mu \begin{pmatrix} |\boldsymbol{\xi}'|^2 + \gamma\xi_1^2 - \partial_t^2 & \gamma\xi_1\xi_2 & -i\gamma\xi_1\partial_t \\ \gamma\xi_1\xi_2 & |\boldsymbol{\xi}'|^2 + \gamma\xi_2^2 - \partial_t^2 & -i\gamma\xi_2\partial_t \\ -i\gamma\xi_1\partial_t & -i\gamma\xi_2\partial_t & |\boldsymbol{\xi}'|^2 - (1+\gamma)\partial_t^2 \end{pmatrix}.$$

Hence if $\mu \neq 0$ and $\gamma \neq 0$,[2] the set $\mathfrak{M}_+[L^{\mathsf{pr}}; \boldsymbol{\xi}']$ of stable solutions is obtained using the classical theory of ordinary differential systems. After some calculations one finds that it is spanned by the functions $\mathbf{U}^{(j)} = \mathbf{V}^{(j)} \mathrm{e}^{-|\boldsymbol{\xi}'|t}$, $j = 1, 2, 3$, with (see [53, §4.2.5])

$$\mathbf{V}^{(1)} = \begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \end{pmatrix}, \quad \mathbf{V}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ i|\boldsymbol{\xi}'| \end{pmatrix}, \quad \mathbf{V}^{(3)} = \begin{pmatrix} \xi_1 t \\ \xi_2 t \\ i(2\gamma^{-1} + 1 + |\boldsymbol{\xi}'|t) \end{pmatrix}.$$

[2] If $\mu \neq 0$ and $\gamma = 0$, then the system reduces to the vector Laplace operator and therefore the Dirichlet boundary conditions cover it.

Hence for $\boldsymbol{\xi}' \neq \mathbf{0}$ the mapping

$$\mathfrak{M}_+[L^{\mathsf{pr}}; \boldsymbol{\xi}'] \longrightarrow \mathbb{C}^3 \ : \ \mathsf{U} \longrightarrow \mathsf{U}\big|_{t=0},$$

is an isomorphism if and only if the matrix

$$\begin{pmatrix} \xi_2 & \xi_1 & 0 \\ -\xi_1 & \xi_2 & 0 \\ 0 & i|\boldsymbol{\xi}'| & i(2\gamma^{-1}+1) \end{pmatrix}$$

is invertible. Since its determinant is equal to $i(2\gamma^{-1}+1)|\boldsymbol{\xi}'|^2$, we deduce that the covering condition holds if and only if $\mu \neq 0$ and $\gamma \neq -2$ or equivalently if and only $\mu \neq 0$ and $\lambda \neq -3\mu$.

In conclusion, the system (4.20) is elliptic on $\overline{\Omega}$ if and only if

$$\mu \neq 0, \quad \lambda \neq -2\mu, \quad \text{and} \quad \lambda \neq -3\mu \tag{4.21}$$

The fact that the Lamé system in dimension 3 is not covered by its 3 Dirichlet boundary conditions when $\lambda + 3\mu = 0$ was first pointed out in [63].

Now in the remainder of this subsection we assume that $\mu > 0$ and $\lambda \geq 0$.

The variational formulation of problem (4.20) is quite standard. We take

$$\mathsf{V} \ = \ \mathsf{H}^1_0(\Omega), \tag{4.22}$$

$$\mathsf{a}(\mathsf{u},\mathsf{v}) \ = \ \int_\Omega \Big(2\mu \sum_{i,j=1}^n \epsilon_{ij}(\mathsf{u})\epsilon_{ij}(\bar{\mathsf{v}}) + \lambda \operatorname{div}\mathsf{u} \operatorname{div}\bar{\mathsf{v}}\Big)\, \mathrm{d}\mathbf{x}, \quad \forall \mathsf{u},\mathsf{v} \in \mathsf{V}, \tag{4.23}$$

where the strain tensor $\epsilon(\mathsf{u}) = (\epsilon_{ij}(\mathsf{u}))_{1\leq i,j\leq n}$ is defined by $\epsilon_{ij}(\mathsf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.

Here and below we shall use the Korn inequality (see [33, 35, 20] for smooth domains and [40] for a bounded domain with a Lipschitz boundary) which asserts that there exists $c > 0$ such that

$$|\mathsf{u}|^2_{1,\Omega} \leq c\Big(\int_\Omega \sum_{i,j=1}^n |\epsilon_{ij}(\mathsf{u})|^2 \, \mathrm{d}\mathbf{x} + |\mathsf{u}|^2_{0,\Omega}\Big), \quad \forall \mathsf{u} \in \mathsf{H}^1(\Omega). \tag{4.24}$$

This estimate and the compact embedding of $\mathsf{H}^1(\Omega)$ into $\mathsf{L}^2(\Omega)$ implies that there exists $c' > 0$ such that

$$|\mathsf{u}|^2_{1,\Omega} \leq c' \int_\Omega \sum_{i,j=1}^n |\epsilon_{ij}(\mathsf{u})|^2 \, \mathrm{d}\mathbf{x}, \quad \forall \mathsf{u} \in \mathsf{H}^1_0(\Omega). \tag{4.25}$$

This implies that the form a defined by (4.23) is strongly coercive on $\mathsf{H}^1_0(\Omega)$. By the Lax-Milgram lemma, the associated operator A defined by (3.26) is an isomorphism from $\mathsf{H}^1_0(\Omega)$ onto $\mathsf{H}^{-1}(\Omega)$.

We have checked that all the assumptions of Theorem 3.4.5 are satisfied for problem (4.20), and therefore the shift theorem in standard Sobolev or analytic spaces is valid for this problem.

4.3.b Stress free boundary conditions

The problem is

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \sigma(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

Here the boundary conditions are the natural ones associated with the sesquilinear form a from (4.23) defined on $\mathbf{V} = \mathbf{H}^1(\Omega)$: There holds for all $\mathbf{u} \in \mathbf{H}^2(\Omega)$, $\mathbf{v} \in \mathbf{H}^1(\Omega)$

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega L\mathbf{u} \cdot \bar{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \{\sigma(\mathbf{u})\mathbf{n}\} \cdot \bar{\mathbf{v}} \,\mathrm{d}\sigma \tag{4.27}$$

where the stress tensor $\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))_{1\le i,j\le n}$ is defined by

$$\sigma_{ij}(\mathbf{u}) = 2\mu\, \epsilon_{ij}(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u}\, \delta_{ij},$$

and the traction $\sigma(\mathbf{u})\mathbf{n}$ is the vector with components $\sum_j \sigma_{ij} n_j$, $i = 1, \ldots, n$. Thus (4.27) shows that $\{\sigma(\mathbf{u})\mathbf{n}\}\big|_{\partial\Omega}$ defines the associated conormal system B applied to $\mathbf{u}$.

By Korn inequality (4.24), this form a is coercive on $\mathbf{H}^1(\Omega)$ if $\mu > 0$ and $\lambda \ge 0$, see also [76, Ch.3] (Prop. 7.2 and Th. 7.9). Denote by A the operator defined by (3.26). We have $\ker A = \ker A^\star = \mathbf{R}$, where $\mathbf{R}$ is the set of rigid motions:

$$\mathbf{R} = \{\mathbf{u} : \mathbf{x} \to \mathbf{c} + \mathbf{d} \times \mathbf{x} : \mathbf{c}, \mathbf{d} \in \mathbb{R}^n\}.$$

Indeed it is well known that, see for example Lemma II.1 of [46],

$$\mathsf{a}(\mathbf{u}, \mathbf{u}) = 0 \quad \Longleftrightarrow \quad \mathbf{u} \in \mathbf{R}.$$

By the Lax-Milgram lemma, A is an isomorphism from $\mathbf{H}^1(\Omega)/\mathbf{R}$ onto its dual space $\{\mathbf{f} \in \widetilde{\mathbf{H}}^{-1}(\Omega) : \langle \mathbf{f}\,,\, \mathbf{r}\rangle = 0,\ \forall \mathbf{r} \in \mathbf{R}\}$.

Again all the assumptions of Theorem 3.4.5 are satisfied for problem (4.26) – with non-zero boundary data $\mathbf{g}$ – and therefore the shift theorem in standard Sobolev or analytic spaces holds:

Theorem 4.3.1 *Let Ω be a smooth bounded domain and a be the sesquilinear form defined by (4.23) on $\mathbf{H}^1(\Omega)$. Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ be a (unique up to a element in $\mathbf{R}$) variational solution of problem*

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}} \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \bar{\mathbf{v}} \,\mathrm{d}\sigma, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{\frac12}(\partial\Omega)$ such that $\int_\Omega \mathbf{f} \cdot \mathbf{r} \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{r} \,\mathrm{d}\sigma = 0$, for all $\mathbf{r} \in \mathbf{R}$.

(i) Let k be a non-negative integer. If Ω is of class $\mathscr{C}^{k+2}$, $\mathbf{f} \in \mathbf{H}^k(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{k+\frac12}(\partial\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{H}^{k+2}(\Omega)$.

(ii) If we assume that Ω is analytic, that $\mathbf{f} \in \mathbf{A}(\Omega)$ and $\mathbf{g} \in \mathbf{A}(\partial\Omega)$, then $\mathbf{u}$ belongs to $\mathbf{A}(\Omega)$.

4.3.c Simply supported boundary conditions

Here we consider the problem

$$\begin{cases} L\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u_t} &= \mathbf{0} \quad \text{on } \partial\Omega, \\ (\sigma(\mathbf{u})\mathbf{n})\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.28}$$

where we recall that $\mathbf{u_t}$ is the tangential component of $\mathbf{u}$, namely $\mathbf{u_t} = \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$. Introducing the projection $\Pi^D\mathbf{u} = \mathbf{u_t}$, its variational formulation consists in taking the sesquilinear form a from (4.23) defined on the following closed subspace of $\mathbf{H}^1(\Omega)$:

$$\mathbf{V} = \mathbf{H}^1_N(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \Pi^D\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega\}.$$

By Korn inequality (4.24), this form is strongly coercive on $\mathbf{H}^1_N(\Omega)$ if $\mu > 0$ and $\lambda \geq 0$ because Lemma 4.3.2 below shows that any $\mathbf{r} \in \mathbf{R}$ satisfying the tangential boundary condition

$$\mathbf{r_t} = \mathbf{0} \quad \text{on } \partial\Omega$$

is equal to zero. This means that the operator A defined by (3.26) is an isomorphism from $\mathbf{H}^1_N(\Omega)$ onto $\mathbf{H}^1_N(\Omega)'$.

Again all the assumptions of Theorem 3.4.5 are satisfied for problem (4.28) and therefore the shift theorem in standard Sobolev or analytic spaces holds.

Lemma 4.3.2 *If $\mathbf{r} \in \mathbf{R}$ satisfies the homogeneous tangential boundary condition*

$$\mathbf{r_t} = \mathbf{0} \quad \text{on } \partial\Omega,$$

then it is equal to zero.

Proof: We use an argument which is related to the construction of finite elements of Whitney or lowest order Nédélec type, where the finite element space is exactly $\mathbf{R}$, with degrees of freedom related to the tangential components of the vector fields on the boundary [74, 67]. Using Green's formula, there holds

$$\int_\Omega \mathbf{curl}\, \mathsf{r}(\mathbf{x})\, \mathrm{d}\mathbf{x} = -\int_{\partial\Omega} \mathsf{r} \times \mathbf{n}\, \mathrm{d}\sigma = 0,$$

by assumption. Since $\mathsf{r}(\mathbf{x}) = \mathbf{c} + \mathbf{d} \times \mathbf{x}$ for some $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$, we get $\mathbf{curl}\, \mathsf{r}(\mathbf{x}) = (n-1)\mathbf{d}$ and therefore $\mathbf{d} = \mathbf{0}$. This means that $\mathsf{r}(\mathbf{x})$ is the constant vector $\mathbf{c} \in \mathbb{R}^n$. Finding two points on the boundary such that the two tangent hyperplanes generate all of $\mathbb{R}^n$, we deduce that $\mathsf{r} = \mathbf{0}$. □

4.3.d Soft Clamped (sliding) boundary conditions

Here we consider the problem

$$\left\{\begin{array}{rcll} L\mathbf{u} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} & = & 0 & \text{on } \partial\Omega, \\ (\sigma(\mathbf{u})\mathbf{n})_t & = & \mathbf{0} & \text{on } \partial\Omega. \end{array}\right. \tag{4.29}$$

For the variational setting of this problem, it suffices to consider the sesquilinear form a from (4.23) defined on (closed subspace of $\mathbf{H}^1(\Omega)$):

$$\mathbf{V} = \mathbf{H}^1_T(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \Pi^D\mathbf{u} = (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega\}.$$

As above by the Korn inequality (4.24), this form is coercive on $\mathbf{H}^1_T(\Omega)$ if $\mu > 0$ and $\lambda \geq 0$. Its strong coerciveness is, in general, not valid, but it holds under some geometric assumptions on the domain Ω. As a counterexample, if Ω is axisymmetric with respect to the x_n axis, then the rigid motion $\mathbf{r}(\mathbf{x}) = (0, 0, \ldots, 0, 1) \times \mathbf{x}$ satisfies the normal boundary condition

$$\mathbf{r}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega$$

and defines therefore a solution of the homogeneous problem associated with (4.29). For the general case, we simply notice that the operator A defined by (3.26) is an isomorphism from $\mathbf{H}^1_T(\Omega)/\ker A$ onto $\{\mathbf{v} \in \mathbf{H}^1_T(\Omega)' : \langle \mathbf{v}, \mathbf{r}\rangle = 0, \forall \mathbf{r} \in \ker A\}$, where $\ker A$ is obviously a subspace of $\mathbf{R}$.

Again all the assumptions of Theorem 3.4.5 are satisfied for problem (4.29) and therefore the shift theorem in standard Sobolev or analytic spaces holds.

4.4 The anisotropic elasticity system

The anisotropic elasticity system corresponds to the linear elasticity with anisotropic material law. More precisely it is a $n \times n$ system that can be written as

$$L(\mathbf{x}, \mathrm{D}_{\mathbf{x}})\mathbf{u} = -\Big(\sum_{j=1}^{n} \partial_j \sigma_{ij}(\mathbf{u})\Big)_{1\leq i\leq n}, \tag{4.30}$$

where the stress tensor is here given by Hooke's law

$$\sigma_{ij}(\mathbf{u}) = \sum_{m,n=1}^{n} C_{ijmn}(\mathbf{x})\epsilon_{mn}(\mathbf{u}),$$

the elasticity moduli $C_{ijmn}(\mathbf{x})$ are real valued smooth functions that satisfy the symmetry relations

$$C_{ijmn}(\mathbf{x}) = C_{mnij}(\mathbf{x}) = C_{jimn}(\mathbf{x}) = C_{ijnm}(\mathbf{x}), \tag{4.31}$$

and the strong ellipticity condition: there exists $M > 0$ such that

$$\forall \big(\epsilon_{ij}\big)_{1\le i,j\le n} \in \mathbb{R}^{n\times n}_{\rm sym},\ \ \forall \mathbf{x}\in\bar\Omega, \qquad \sum_{i,j,m,n=1}^{n} C_{ijmn}(\mathbf{x})\epsilon_{ij}\epsilon_{mn} \ge M \sum_{i,j=1}^{n} |\epsilon_{ij}|^2, \tag{4.32}$$

where $\mathbb{R}^{n\times n}_{\rm sym}$ is the space of symmetric $n\times n$ matrices.

Obviously we recover the Lamé system from the previous section if

$$C_{ijmn}(\mathbf{x}) = \mu(\delta_{mi}\delta_{nj} + \delta_{mj}\delta_{ni}) + \lambda\delta_{ij}\delta_{mn},$$

which satisfy the above assumptions if $\mu > 0$ and $\lambda \ge 0$.

Note that due to the symmetry constraint in (4.32), the sesquilinear form corresponding to the operator L defined in (4.30) is never *formally positive* in the sense of Definition 3.2.4. Indeed, thanks to the symmetries (4.31), rigid motions again belong to the kernel of the operator.

Like for the Lamé system, we can complement the anisotropic elasticity with hard clamped boundary conditions (system (4.20)), stress free boundary conditions (system (4.26)), simply supported boundary conditions (system (4.28)) and finally soft clamped boundary condition (system (4.29)). The variational settings of these systems are exactly the same as the corresponding ones for the Lamé system, where now the sesquilinear form a can generally be written as

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \int_\Omega \sum_{i,j=1}^{n} \sigma_{ij}(\mathbf{u})\,\epsilon_{ij}(\bar{\mathbf{v}})\,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}.$$

Due to the ellipticity assumption (4.32), the relevant properties of this sesquilinear form on the variational spaces $\mathbf{V}$ are in close analogy with the corresponding ones for the Lamé system and therefore the same regularity results hold.

4.5 The regularized Maxwell system

A rich source for examples and counter-examples of elliptic boundary value problems is provided by the equations of electrodynamics. In their simplest form, the time-harmonic Maxwell equations for electromagnetic waves in a bounded domain Ω of $\mathbb{R}^3$ filled by an isotropic homogeneous material can be written as a symmetric first order 6×6 system

$$\mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H} = 0 \quad\text{and}\quad \mathbf{curl}\,\mathbf{H} + i\omega\mathbf{E} = \mathbf{J} \quad \text{in } \Omega.$$

Here $\mathbf{E}$ is the electric part and $\mathbf{H}$ is the magnetic part of the electromagnetic field, and the constant ω corresponds to the wave number or frequency. The right hand side $\mathbf{J}$ is the current density which – in the absence of free electric charges – is divergence free, namely

$$\operatorname{div}\mathbf{J} = \mathbf{0} \quad \text{in } \Omega.$$

If the frequency ω is different from zero, we can eliminate $\mathbf{H}$ by the first relation $\mathbf{H} = \frac{1}{i\omega}\,\mathbf{curl}\,\mathbf{E}$ and therefore the second one yields the second order system

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\mathbf{E} = i\omega\mathbf{J} \quad \text{in } \Omega.$$

Since $\mathbf{J}$ is supposed to be divergence free, we deduce that $\mathbf{E}$ is also divergence free. Consequently, the solutions of this system are also solutions of the regularized Maxwell system

$$L_{\omega,s}(\mathrm{D}_{\mathbf{x}})\mathbf{E} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} - \omega^2\mathbf{E} = i\omega\mathbf{J} \quad \text{in } \Omega, \tag{4.33}$$

where s is an arbitrary parameter. Conversely, any solution $\mathbf{E}$ of the second order system (4.33) defines via $\mathbf{H} = \mathbf{curl}\,\mathbf{E}/(i\omega)$ a solution of the original first order system, provided $\mathbf{E}$ is divergence free.

The principal part of the second order system (4.33) is

$$L_{0,s}(\mathrm{D}_{\mathbf{x}}) = \mathbf{curl}\,\mathbf{curl} - s\nabla\operatorname{div} = -\mathbb{I}_3\Delta + (1-s)\nabla\operatorname{div}.$$

It is clearly strongly elliptic if $s > 0$, which we will assume from now on.

Note that this system can be obtained as a particular case of the Lamé system with $\mu = 1$ and $\lambda = s - 2$. Nevertheless, the standard sesquilinear form a associated with the regularized Maxwell system (4.33) is, cf. [23, 45, 31]

$$\mathsf{a}_{\omega,s}(\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\bar{\mathbf{v}} + s\operatorname{div}\mathbf{u}\operatorname{div}\bar{\mathbf{v}} - \omega^2\mathbf{u}\cdot\mathbf{v})\,\mathrm{d}\mathbf{x}, \tag{4.34}$$

and its principal part $\mathsf{a}_{0,s}$ is distinct from the Lamé sesquilinear form (4.23), even with the right choice of parameters $\mu = 1$ and $\lambda = s - 2$. This fact leads to distinct natural boundary conditions, in general, see §4.8 for more comments.

Another distinct (and sometimes disturbing) feature of the regularized Maxwell system, when written in variational form, is that variational spaces are not naturally subspaces of $\mathbf{H}^1(\Omega)$, but of the larger functional space

$$\mathbf{X}(\Omega) = \mathbf{H}(\mathbf{curl},\Omega) \cap \mathbf{H}(\operatorname{div},\Omega) \tag{4.35}$$

where $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\operatorname{div},\Omega)$ are the spaces of fields $\mathbf{u}$ in $\mathbf{L}^2(\Omega)$ with square integrable $\mathbf{curl}\,\mathbf{u}$ and $\operatorname{div}\mathbf{u}$, respectively. Its natural norm is

$$\|\mathbf{u}\|^2_{\mathbf{X}(\Omega)} = \|\mathbf{u}\|^2_{0;\Omega} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{0;\Omega} + \|\operatorname{div}\mathbf{u}\|^2_{0;\Omega}. \tag{4.36}$$

The space $\mathbf{X}(\Omega)$ is contained in $\mathbf{H}^1_{\mathsf{loc}}(\Omega)$, but never in $\mathbf{H}^1(\Omega)$, see [39].

However, in contrast with elastic displacements, electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are always supposed to satisfy some essential boundary conditions on the boundary $\partial\Omega$. There are several of them. We can quote: the perfectly conducting conditions, the perfectly insulating conditions, and, more generally, impedance or transparent boundary conditions.

From the mathematical point of view of elliptic boundary value problems, one can, of course, associate the second order system (4.33) also with *Dirichlet* boundary conditions, i.e. require that all components of $\mathbf{E}$ vanish on $\partial\Omega$. This obviously defines an elliptic boundary value problem if $s \neq -1$, see (4.21), so that one can apply all the elliptic regularity theory of the previous chapters in this situation. Unfortunately, in this case one will not be able to prove that $\mathbf{E}$ is divergence free, so that one does not get a solution of the original non-regularized Maxwell system. What one rather obtains is just a solution of an elasticity problem with hard clamped boundary conditions.

In the next subsections, we consider in more details the perfectly conducting conditions, and then the impedance boundary conditions. We show in both cases how they can be made to enter our general framework of elliptic boundary value problems in coercive variational form, leading to optimal Sobolev and analytic regularity results.

4.5.a Perfectly conducting electric boundary condition

This condition specifies that the tangential component $\mathbf{E} \times \mathbf{n}$ of the electric field is zero on $\partial\Omega$. The variational space is then

$$\mathbf{X}_N(\Omega) = \{\mathbf{u} \in \mathbf{X}(\Omega) : \ \mathbf{u} \times \mathbf{n} = 0 \ \text{ on } \ \partial\Omega\}.$$

In this definition, the boundary condition has to be understood in the weak sense. For $\mathbf{u} \in \mathbf{X}(\Omega)$, the tangential trace $\mathbf{u} \times \mathbf{n}$ can be defined as an element of $\mathbf{H}^{-1/2}(\Omega)$ by means of the Green's formula

$$\forall \boldsymbol{\varphi} \in \mathbf{H}^1(\Omega), \qquad \int_\Omega (\mathbf{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} - \mathbf{curl}\, \mathbf{u} \cdot \boldsymbol{\varphi})\, \mathrm{d}\mathbf{x} = \langle \mathbf{u} \times \mathbf{n}, \boldsymbol{\varphi} \rangle_{\partial\Omega} \tag{4.37}$$

similarly to the way we defined the weak conormal derivatives in the previous chapter, equation (3.8), see again [39]. The vanishing of this trace is then equivalent to the validity of the Green formula

$$\forall \mathbf{v} \in \mathbf{H}^1(\Omega), \qquad \int_\Omega (\mathbf{curl}\, \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v})\, \mathrm{d}\mathbf{x} = 0\,.$$

In this part we consider domains with smooth boundaries: Namely, if $\partial\Omega$ is of class $\mathscr{C}^2$ (in fact, as soon as we have H^2 regularity of the Dirichlet problem for the Laplacian, see [14]), the space $\mathbf{X}_N(\Omega)$ is included in $\mathbf{H}^1(\Omega)$, thus coincides with the space $\mathbf{H}^1_N(\Omega)$ introduced in §4.3.c. Thus, in the case of smooth domains we are back to our general framework of Chapter 3 with the variational space

$$\mathbf{V} = \mathbf{H}^1_N(\Omega).$$

It is known (see [23]) that the sesquilinear form $\mathsf{a}_{0,s}$ is strongly coercive on $\mathbf{H}^1_N(\Omega)$, in other words, there exists $c > 0$ such that

$$\mathsf{a}_{0,s}(\mathbf{u}, \mathbf{u}) \geq c \|\mathbf{u}\|^2_{1,\Omega}, \quad \forall \mathbf{u} \in \mathbf{H}^1_N(\Omega).$$

By the compact embedding of $\mathbf{H}^1_N(\Omega)$ in $\mathbf{L}^2(\Omega)$, there holds the Fredholm alternative for the variational problem

$$\forall \mathbf{v} \in \mathbf{H}^1_N(\Omega), \quad \mathsf{a}_{\omega,s}(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}} \, \mathrm{d}\mathbf{x}. \tag{4.38}$$

By virtue of Theorem 3.4.5, for $\mathbf{f} \in \mathbf{L}^2(\Omega)$ any solution $\mathbf{u} \in \mathbf{H}^1_N(\Omega)$ is more regular, namely $\mathbf{u} \in \mathbf{H}^2(\Omega)$. In order to determine the natural boundary condition, we use integration by parts like in the general formulas (3.6) and (3.14), and obtain, with the tangential component $\mathbf{v_t} = \mathbf{n} \times (\mathbf{v} \times \mathbf{n})$ of $\mathbf{v}$,

$$\begin{aligned} \mathsf{a}_{\omega,s}(\mathbf{u},\mathbf{v}) &= \int_\Omega L_{\omega,s}\mathbf{u} \cdot \bar{\mathbf{v}} \, \mathrm{d}\mathbf{x} \\ &+ \int_{\partial\Omega} ((\mathbf{curl}\,\mathbf{u}) \times \mathbf{n}) \cdot \bar{\mathbf{v}}_\mathbf{t} + s \operatorname{div} \mathbf{u} \, (\bar{\mathbf{v}} \cdot \mathbf{n}) \, \mathrm{d}\sigma, \end{aligned} \tag{4.39}$$

this identity being valid for all $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $\mathbf{v} \in \mathbf{H}^1(\Omega)$. Note that formula (4.39) shows that the associated conormal system B is given by

$$B\mathbf{u} = \big\{(\mathbf{curl}\,\mathbf{u}) \times \mathbf{n} + s(\operatorname{div}\mathbf{u})\,\mathbf{n}\big\}\big|_{\partial\Omega}. \tag{4.40}$$

Thus we see that the strong form (3.16) of the variational problem (4.38) is (compare with (3.16) in the general case)

$$\begin{cases} L_{\omega,s}\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{on } \partial\Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{cases} \tag{4.41}$$

Since all the assumptions of Theorem 3.4.5 are satisfied for problem (4.41), the shift theorem in standard Sobolev or analytic spaces is valid for this problem.

Let us finally show a specific property for the regularized formulation that allows us to go back to the original first order Maxwell system if $\omega \neq 0$, namely the divergence free property of $\mathbf{u}$ under the condition that the right-hand side has the same property:

Lemma 4.5.1 *If the right-hand side $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is divergence free:*

$$\operatorname{div}\mathbf{f} = 0 \quad \textit{in } \Omega,$$

and $-\omega^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions, then any solution $\mathbf{u} \in \mathbf{H}^1_N(\Omega)$ of problem (4.38) is divergence free as well, i.e.,

$$\operatorname{div}\mathbf{u} = 0 \quad \textit{in } \Omega.$$

Proof: Taking test functions $\mathbf{v}$ in (4.38) in the form $\mathbf{v} = \nabla\bar{\varphi}$, with an arbitrary $\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega)$ (this directly implies that $\mathbf{v} \in \mathbf{H}^1_N(\Omega)$), we get

$$\forall\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega), \quad s\int_\Omega \operatorname{div}\mathbf{u}\ \operatorname{div}\nabla\varphi\,\mathrm{d}\mathbf{x} - \omega^2\int_\Omega \mathbf{u}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}.$$

This right-hand side is equal to zero as a consequence of Green's formula and the divergence free property of $\mathbf{f}$. Therefore $\mathbf{u}$ satisfies

$$\forall\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega), \quad \int_\Omega \operatorname{div}\mathbf{u}\ (s\Delta + \omega^2)\varphi\,\mathrm{d}\mathbf{x} = 0.$$

Now assume that $\operatorname{div}\mathbf{u}$ is not zero. This means that the operator defined by the elliptic boundary system with the strongly elliptic operator $s\Delta+\omega^2$ and Dirichlet boundary conditions is not surjective from $\mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega)$ to $\mathsf{L}^2(\Omega)$. According to Theorems 2.2.39, 3.3.1 and 4.1.1, this operator is Fredholm of index zero. Since it is not surjective, it is not injective either, and this means that there exists an eigenfunction $\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega)$ of the equation $(s\Delta + \omega^2)\varphi = 0$, and hence $-\omega^2/s$ is an eigenvalue of the Laplace operator with Dirichlet boundary conditions, contrary to the assumption. □

4.5.b Perfectly conducting magnetic boundary conditions

This condition specifies the normal component $\mathbf{H}\cdot\mathbf{n}$ of the magnetic field $\mathbf{H}$ to be zero on the boundary of the domain. The variational space is then

$$\mathbf{X}_T(\Omega) = \{\mathbf{u} \in \mathbf{X}(\Omega) :\ \mathbf{u}\cdot\mathbf{n} = 0 \ \text{ on } \ \partial\Omega\}.$$

The weak definition of the boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ means in this case:

$$\forall\varphi \in \mathsf{H}^1(\Omega), \quad \int_\Omega (\operatorname{div}\mathbf{u}\,\varphi + \mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}\mathbf{x} = 0\,. \tag{4.42}$$

Like in the electric case, if $\partial\Omega$ is of class $\mathscr{C}^2$, the space $\mathbf{X}_T(\Omega)$ is contained in $\mathbf{H}^1(\Omega)$; it coincides with the space $\mathbf{H}^1_T(\Omega)$ introduced in §4.3.d. Thus, in the case of smooth domains we are back to our general framework with the variational space

$$\mathbf{V} = \mathbf{H}^1_T(\Omega).$$

Since the magnetic field satisfies the same second order system as the electric field (with a different right hand side and different boundary conditions), for the problem in magnetic field formulation we can take the same sesquilinear form (4.34), now defined on $\mathbf{H}^1_T(\Omega)$. Like before this form is coercive on $\mathbf{H}^1_T(\Omega)$, cf. [23].

Thus, for $\mathbf{f} \in \mathbf{L}^2(\Omega)$, we consider solutions $\mathbf{u} \in \mathbf{H}^1_T(\Omega)$ of the problem

$$\mathsf{a}_{\omega,s}(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x}, \quad \forall\mathbf{v} \in \mathbf{H}^1_T(\Omega). \tag{4.43}$$

We know that there is at most a finite number of linearly independent solutions $\mathbf{u}$, and Theorem 3.4.5 then yields that $\mathbf{u} \in \mathbf{H}^2(\Omega)$. Using the Green formula (4.39), we find that $\mathbf{u}$ solves the problem in strong form

$$\begin{cases} L_{\omega,s}\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ (\mathbf{curl}\,\mathbf{u}) \times \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \\ \mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.44}$$

Like in the electric formulation, since all the assumptions of Theorem 3.4.5 are satisfied for problem (4.44), the shift theorem in standard Sobolev or analytic spaces is valid for this problem.

As previously, let us show how H^2 regularity results for the Helmholtz equation can be used to prove the divergence free property:

Lemma 4.5.2 *If the right-hand side $\mathbf{f} \in \mathbf{L}^2(\Omega)$ is divergence free and satisfies $\mathbf{f}\cdot\mathbf{n} = 0$ in the weak sense on $\partial\Omega$, that is*

$$\int_\Omega \mathbf{f}\cdot\nabla\varphi \,\mathrm{d}\mathbf{x} = 0, \quad \forall\varphi \in \mathsf{H}^1(\Omega),$$

and if $-\omega^2/s$ is not a non-zero eigenvalue of the Laplace operator Δ with Neumann boundary conditions, then any solution $\mathbf{u} \in \mathbf{H}^1_T(\Omega)$ of problem (4.43) is divergence free.

Proof: Like in the proof of Lemma 4.5.1, the main idea is to show that $\operatorname{div}\mathbf{u}$ is orthogonal in $\mathsf{L}^2(\Omega)$ to all right hand sides in the Neumann problem for the operator $s\Delta+\omega^2$ on $\mathsf{H}^2(\Omega)$. For this, consider

(1) $$\varphi \in \mathsf{H}^2(\Omega) \quad \text{satisfying} \quad \partial_n\varphi = 0 \quad \text{on } \partial\Omega\,.$$

We see that then $\mathbf{v} = \nabla\varphi$ is in $\mathbf{H}^1_T(\Omega)$ and can be used as a test function in the variational formulation (4.43). This gives with (4.42)

(2) $$\int_\Omega \operatorname{div}\mathbf{u}\,(s\Delta+\omega^2)\varphi\,\mathrm{d}\mathbf{x} = s\int_\Omega \operatorname{div}\mathbf{u}\,\operatorname{div}\mathbf{v}\,\mathrm{d}\mathbf{x} - \omega^2\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} = 0\,.$$

Now if $-\omega^2/s$ is not a Neumann eigenvalue, then thanks to the Fredholm alternative, the operator $s\Delta+\omega^2$ is an isomorphism from the space of φ satisfying (1) onto $\mathsf{L}^2(\Omega)$. We can therefore find a $\varphi \in \mathsf{H}^2(\Omega)$ that satisfies

$$\begin{cases} (s\Delta+\omega^2)\varphi &= \operatorname{div}\bar{\mathbf{u}} \quad \text{in } \Omega, \\ \partial_n\varphi &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

If $\omega = 0$ (which is a Neumann eigenvalue), we still can find such a φ: In this case, the right hand sides do not fill all of $\mathsf{L}^2(\Omega)$, but only the subspace $\mathsf{L}^2_0(\Omega) = \{g \in \mathsf{L}^2(\Omega) : \int_\Omega g\,\mathrm{d}\mathbf{x} = 0\}$, see Theorem 4.1.2. But $\operatorname{div}\mathbf{u}$ belongs to this subspace, as we can see from (4.42) by taking a constant test function. Therefore, unless $-\omega^2/s$ is a non-zero Neumann eigenvalue, we find from (2)

$$\int_\Omega |\operatorname{div} \mathbf{u}|^2 \,\mathrm{d}\mathbf{x} = 0\,,$$

and hence $\mathbf{u}$ is divergence free. □

4.5.c Regularity of the electromagnetic field with perfectly conducting boundary conditions

Relying on the ellipticity of the electric and magnetic problems (4.41) and (4.44) we are now able to deduce regularity results for the electromagnetic field $(\mathbf{E}, \mathbf{H})$ solution of the harmonic Maxwell system with the perfectly conducting boundary conditions

$$\begin{cases} \mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H} = \mathbf{0} \quad \text{and} \quad \mathbf{curl}\,\mathbf{H} + i\omega\mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{E}\times\mathbf{n} = 0 \quad \text{and} \quad \mathbf{H}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.45}$$

We can drop the condition $\operatorname{div}\mathbf{J} = 0$, replacing it by regularity assumptions on $\operatorname{div}\mathbf{J}$.

Theorem 4.5.3 *Let Ω be a smooth bounded domain. Let $(\mathbf{E}, \mathbf{H}) \in \mathbf{H}(\mathbf{curl};\Omega)^2$ be solution of the harmonic Maxwell system (4.45) with a non-zero frequency ω and $\mathbf{J} \in \mathbf{H}(\operatorname{div};\Omega)$.*

(i) Let k be a non-negative integer and let Ω be of class $\mathscr{C}^{k+2}$.
- ⋆ *If $\mathbf{J}\in\mathbf{H}^k(\Omega)$ and $\operatorname{div}\mathbf{J}\in \mathrm{H}^{k+1}(\Omega)$, then $\mathbf{E}\in\mathbf{H}^{k+2}(\Omega)$ and $\mathbf{H}\in\mathbf{H}^{k+1}(\Omega)$.*
- ⋆ *If moreover $\mathbf{curl}\,\mathbf{J}\in\mathbf{H}^k(\Omega)$ and $\mathbf{J}\times\mathbf{n}\in\mathbf{H}^{k+\frac12}(\partial\Omega)$, then $\mathbf{H}\in\mathbf{H}^{k+2}(\Omega)$.*

(ii) If we assume that Ω is analytic and $\mathbf{J}\in\mathbf{A}(\Omega)$, then $\mathbf{E}$ and $\mathbf{H}$ belong to $\mathbf{A}(\Omega)$.

Proof: Since $\operatorname{div}\mathbf{E} = \frac{1}{i\omega}\operatorname{div}\mathbf{J}$, we deduce that $\operatorname{div}\mathbf{E}\in \mathrm{L}^2(\Omega)$. Since, moreover, $\mathbf{E}\in\mathbf{H}(\mathbf{curl};\Omega)$ and $\mathbf{E}\times\mathbf{n} = 0$ on $\partial\Omega$, we have $\mathbf{E}\in\mathbf{X}_N(\Omega)$, and, as Ω is smooth enough, $\mathbf{E}\in\mathbf{H}^1_N(\Omega)$. With the form $\mathsf{a}_{\omega,s}$ defined in (4.34), the electric field $\mathbf{E}$ is solution of the variational problem

$$\begin{aligned} \forall\mathbf{E}'\in\mathbf{H}^1_N(\Omega),\quad \mathsf{a}_{\omega,s}(\mathbf{E},\mathbf{E}') &= \int_\Omega i\omega\mathbf{J}\cdot\overline{\mathbf{E}}' + \tfrac{s}{i\omega}\operatorname{div}\mathbf{J}\,\operatorname{div}\overline{\mathbf{E}}'\,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \big(i\omega\mathbf{J} - \tfrac{s}{i\omega}\nabla\operatorname{div}\mathbf{J}\big)\cdot\overline{\mathbf{E}}'\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\tfrac{s}{i\omega}\operatorname{div}\mathbf{J}\,\overline{\mathbf{E}}'\cdot\mathbf{n}\,\mathrm{d}\sigma, \end{aligned}$$

which writes in distributional form, cf. (4.41)

$$\begin{cases} L_{\omega,s}\mathbf{E} &= i\omega\mathbf{J} - \tfrac{s}{i\omega}\nabla\operatorname{div}\mathbf{J} \quad \text{in } \Omega, \\ \operatorname{div}\mathbf{E} &= \tfrac{1}{i\omega}\operatorname{div}\mathbf{J} \quad \text{on } \partial\Omega, \\ \mathbf{E}\times\mathbf{n} &= \mathbf{0} \quad \text{on } \partial\Omega. \end{cases}$$

As already mentioned, this problem satisfies the assumptions of Theorem 3.4.5, therefore $\mathbf{E} \in \mathbf{H}^{k+2}(\Omega)$ if $\mathbf{J} \in \mathbf{H}^k(\Omega)$ and $\operatorname{div} \mathbf{J} \in \mathrm{H}^{k+1}(\Omega)$. The regularity of $\mathbf{H}$ comes from the equality $\mathbf{H} = \frac{1}{i\omega} \operatorname{\mathbf{curl}} \mathbf{E}$.

If $\operatorname{\mathbf{curl}} \mathbf{J} \in \mathbf{H}^k(\Omega)$, we use the magnetic formulation: $\mathbf{H} \in \mathbf{H}^1_T(\Omega)$ solves

$$\forall \mathbf{H}' \in \mathbf{H}^1_T(\Omega), \quad \mathsf{a}_{\omega,s}(\mathbf{H}, \mathbf{H}') = \int_\Omega \mathbf{J} \cdot \operatorname{\mathbf{curl}} \overline{\mathbf{H}}' \, \mathrm{d}\mathbf{x} = \int_\Omega \operatorname{\mathbf{curl}} \mathbf{J} \cdot \overline{\mathbf{H}}' \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} (\mathbf{J} \times \mathbf{n}) \cdot \overline{\mathbf{H}}' \, \mathrm{d}\sigma$$

which gives

$$\begin{cases} L_{\omega,s}\mathbf{H} = \operatorname{\mathbf{curl}} \mathbf{J} & \text{in } \Omega, \\ (\operatorname{\mathbf{curl}} \mathbf{H}) \times \mathbf{n} = \mathbf{J} \times \mathbf{n} & \text{on } \partial\Omega, \\ \mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

and satisfies the assumptions of Theorem 3.4.5 as well, and deduce that $\mathbf{H} \in \mathbf{H}^{k+2}(\Omega)$.

The analytic regularity is a direct consequence of Theorem 3.4.5. □

4.5.d Imperfectly conducting or impedance boundary conditions

This condition, also called Leontovich boundary condition, specifies that the tangential component $\mathbf{E_t} = \mathbf{n} \times (\mathbf{E} \times \mathbf{n})$ of the electric field is proportional to $\mathbf{H} \times \mathbf{n}$ (tangential component of the magnetic field rotated by $90°$ clockwise) on $\partial\Omega$, more precisely it imposes that $\mathbf{H} \times \mathbf{n} - \lambda \, \mathbf{E_t} = \mathbf{0}$ on $\partial\Omega$, where the impedance λ is a smooth function defined on $\partial\Omega$ satisfying

$$\lambda : \partial\Omega \to \mathbb{C}, \quad \text{such that} \quad \forall \mathbf{x} \in \partial\Omega, \;\; \lambda(\mathbf{x}) \neq 0, \tag{4.46}$$

see for instance [75, 67]. The case $\lambda \equiv 1$ is also called the Silver-Müller boundary condition [12].

So we are now interested in properties of solutions of the Maxwell system with impedance boundary conditions

$$\begin{cases} \operatorname{\mathbf{curl}} \mathbf{E} - i\omega \mathbf{H} = \mathbf{0} \quad \text{and} \quad \operatorname{\mathbf{curl}} \mathbf{H} + i\omega \mathbf{E} = \mathbf{J} \quad \text{in } \Omega, \\ \mathbf{H} \times \mathbf{n} - \lambda \, \mathbf{E_t} = \mathbf{0} \quad \text{on } \partial\Omega, \end{cases} \tag{4.47}$$

Eliminating $\mathbf{H}$ by the relation $\mathbf{H} = \frac{1}{i\omega} \operatorname{\mathbf{curl}} \mathbf{E}$, we can write the impedance condition in the form:

$$(\operatorname{\mathbf{curl}} \mathbf{E}) \times \mathbf{n} - i\omega\lambda \, \mathbf{E_t} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{4.48}$$

We can see that, at first glance, this boundary condition enters the framework of general Robin boundary conditions (3.43) $B\mathbf{E} + Z\gamma_0\mathbf{E} = 0$ with $Z\mathbf{E} = -i\omega\lambda\mathbf{E_t}$, since in our case $B\mathbf{E} = \operatorname{\mathbf{curl}} \mathbf{E} \times \mathbf{n} + (\operatorname{div} \mathbf{E})\mathbf{n}$, see (4.40).

The new difficulty is that the space $\mathbf{X}(\Omega)$ (4.35) without any boundary condition is not included in $\mathbf{H}^1(\Omega)$, and does not give sense to a Robin type boundary term. This is the reason for the introduction of the new variational space associated with this boundary condition

$$\mathbf{X}_{\mathsf{imp}}(\Omega) = \{\mathbf{u} \in \mathbf{X}(\Omega) : \ \gamma_0 \mathbf{u}_{\mathbf{t}} \in \mathbf{L}^2(\partial\Omega)\},$$

equipped with the natural norm

$$\|\mathbf{u}\|^2_{\mathbf{X}_{\mathsf{imp}}(\Omega)} = \|\mathbf{u}\|^2_{0;\,\Omega} + \|\,\mathbf{curl}\,\mathbf{u}\|^2_{0;\,\Omega} + \|\,\mathrm{div}\,\mathbf{u}\|^2_{0;\,\Omega} + \|\gamma_0\mathbf{u}_{\mathbf{t}}\|^2_{0;\,\partial\Omega}\,.$$

Assuming that $\mathbf{J} \in \mathbf{H}(\mathrm{div};\Omega)$, and introducing a positive regularizing parameter s, we can prove that the electric part $\mathbf{E}$ of any solution $(\mathbf{E},\mathbf{H}) \in \mathbf{H}(\mathbf{curl};\Omega)^2$ to system (4.47) is solution of the variational problem

Find $\mathbf{E} \in \mathbf{X}_{\mathsf{imp}}(\Omega)$ *such that* $\forall \mathbf{E}' \in \mathbf{X}_{\mathsf{imp}}(\Omega)$,

$$\mathsf{a}_{\omega,s}(\mathbf{E},\mathbf{E}') - i\omega \int_{\partial\Omega} \lambda\, \mathbf{E}_{\mathbf{t}} \cdot \overline{\mathbf{E}}'_{\mathbf{t}}\, \mathrm{d}\sigma = \int_\Omega i\omega \mathbf{J}\cdot\overline{\mathbf{E}}' + \tfrac{s}{i\omega}\,\mathrm{div}\,\mathbf{J}\,\mathrm{div}\,\overline{\mathbf{E}}'\,\mathrm{d}\mathbf{x}. \tag{4.49}$$

Here $\mathsf{a}_{\omega,s}$ is the sesquilinear form (4.34). The variational problem (4.49) is similar to problem (3.40) with the general sesquilinear form (3.39).

But, even if we take a domain with smooth boundary, the space $\mathbf{X}_{\mathsf{imp}}(\Omega)$ is not embedded in $\mathbf{H}^1(\Omega)$. The best which can be proved within the scale of standard Sobolev spaces is the embedding of $\mathbf{X}_{\mathsf{imp}}(\Omega)$ in $\mathbf{H}^{1/2}(\Omega)$ [22, Theorem 2]. It is known however that $\mathbf{H}^1(\Omega)$ is a dense subset of $\mathbf{X}_{\mathsf{imp}}(\Omega)$ [19, 29].

Moreover, there is no hope of applying the theory of elliptic boundary value problems to (4.49): With the second order operator $L_{\omega,s}$ introduced in (4.33), problem (4.49) writes in distributional form

$$\begin{cases} L_{\omega,s}\mathbf{E} &=& i\omega\mathbf{J} - \frac{s}{i\omega}\nabla\,\mathrm{div}\,\mathbf{J} & \text{in } \Omega,\\ (\mathbf{curl}\,\mathbf{E})\times\mathbf{n} - i\omega\lambda\,\mathbf{E}_{\mathbf{t}} &=& \mathbf{0} & \text{on } \partial\Omega,\\ \mathrm{div}\,\mathbf{E} &=& \frac{1}{i\omega}\,\mathrm{div}\,\mathbf{J} & \text{on } \partial\Omega, \end{cases} \tag{4.50}$$

its principal part is problem (4.66), which is not elliptic, as explained in §4.8.

Note that the variational formulation for the magnetic field $\mathbf{H} \in \mathbf{X}_{\mathsf{imp}}(\Omega)$ uses the same space and reads

$$\forall \mathbf{H}' \in \mathbf{X}_{\mathsf{imp}}(\Omega), \quad \mathsf{a}_{\omega,s}(\mathbf{H},\mathbf{H}') - i\omega \int_{\partial\Omega} \frac{1}{\lambda}\,\mathbf{H}_{\mathbf{t}}\cdot\overline{\mathbf{H}}'_{\mathbf{t}}\,\mathrm{d}\sigma = \int_\Omega \mathbf{J}\cdot\mathbf{curl}\,\overline{\mathbf{H}}'\,\mathrm{d}\mathbf{x}, \tag{4.51}$$

which does not define an elliptic problem either.

A much better idea, inspired by [75, §5.4.3], is to consider a coupled regularized formulation for the full electromagnetic field $(\mathbf{E},\mathbf{H})$, using the variational space

$$\mathbf{V} = \big\{(\mathbf{E},\mathbf{H}) \in \big(\mathbf{H}(\mathbf{curl},\Omega)\cap\mathbf{H}(\mathrm{div},\Omega)\big)^2 : \ \mathbf{H}\times\mathbf{n} = \lambda\mathbf{E}_{\mathbf{t}} \ \text{ on } \partial\Omega\big\}, \tag{4.52}$$

considering now the impedance condition in (4.47) as an essential boundary condition. The new sesquilinear form $\tilde{\mathsf{a}}(\mathbf{E},\mathbf{H};\mathbf{E}',\mathbf{H}')$ is now the sum of the sesquilinear forms appearing in (4.49) and (4.51). Thus $\tilde{\mathsf{a}}$ has the Robin form $\mathsf{a}+\mathsf{b}$ with

$$\mathsf{a}(\mathbf{E},\mathbf{H};\mathbf{E}',\mathbf{H}') = \mathsf{a}_{\omega,s}(\mathbf{E},\mathbf{E}') + \mathsf{a}_{\omega,s}(\mathbf{H},\mathbf{H}') \tag{4.53a}$$

where $\mathsf{a}_{\omega,s}$ is the sesquilinear form (4.34), and

$$\mathsf{b}(\mathbf{E},\mathbf{H};\mathbf{E}',\mathbf{H}') = -i\omega\int_{\partial\Omega}\lambda\,\mathbf{E}_{\mathbf{t}}\cdot\overline{\mathbf{E}}'_{\mathbf{t}}\,\mathrm{d}\sigma - i\omega\int_{\partial\Omega}\frac{1}{\lambda}\,\mathbf{H}_{\mathbf{t}}\cdot\overline{\mathbf{H}}'_{\mathbf{t}}\,\mathrm{d}\sigma. \tag{4.53b}$$

The derivation of the following lemma is standard:

Lemma 4.5.4 *Any solution $(\mathbf{E},\mathbf{H})\in\mathbf{H}(\mathbf{curl};\Omega)^2$ of the full impedance problem (4.47) belongs to the space $\mathbf{V}$ defined in (4.52), and solves the variational problem*

Find $(\mathbf{E},\mathbf{H})\in\mathbf{V}$ *such that* $\forall(\mathbf{E}',\mathbf{H}')\in\mathbf{V}$,

$$\tilde{\mathsf{a}}(\mathbf{E},\mathbf{H};\mathbf{E}',\mathbf{H}') = \int_\Omega\Big(i\omega\mathbf{J}\cdot\overline{\mathbf{E}}' + \tfrac{s}{i\omega}\operatorname{div}\mathbf{J}\,\operatorname{div}\overline{\mathbf{E}}'\Big)\,\mathrm{d}\mathbf{x} + \int_\Omega\mathbf{J}\cdot\mathbf{curl}\,\overline{\mathbf{H}}'\,\mathrm{d}\mathbf{x}, \tag{4.54}$$

with the choice (4.53) for $\tilde{\mathsf{a}}=\mathsf{a}+\mathsf{b}$.

The new fact is that $\mathbf{V}$ is embedded in $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ and that the sesquilinear form a is coercive on $\mathbf{V}$. We prove these two assertions in the following statements.

Lemma 4.5.5 *If the boundary of Ω is of class $\mathscr{C}^2$ and the impedance function λ satisfies (4.46), the space $\mathbf{V}$ defined by (4.52) is contained in $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$.*

Proof: Let $(\mathbf{E},\mathbf{H})\in\mathbf{V}$. Let us prove that $\mathbf{E}\in\mathbf{H}^1(\Omega)$. The proof for $\mathbf{H}$ is similar. Since for any $\psi\in\mathscr{C}^\infty(\overline{\Omega})$, $(\psi\mathbf{E},\psi\mathbf{H})$ also belongs to $\mathbf{V}$, we can assume that Ω is topologically trivial.

There exists a vector potential $\mathbf{w}\in\mathbf{X}_N(\Omega)$ such that, cf. [7]

$$\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{E} \quad \text{in} \quad \Omega.$$

Thus, there exists a potential $\varphi\in\mathsf{H}^1(\omega)$ such that

(1) $$\nabla\varphi = \mathbf{E}-\mathbf{w}.$$

Since Ω is smooth enough, $\mathbf{X}_N(\Omega)\subset\mathbf{H}^1(\Omega)$. Therefore, as a consequence of $\operatorname{div}\mathbf{E}\in\mathsf{L}^2(\Omega)$ we find that

(2) $$\operatorname{div}\nabla\varphi\in\mathsf{L}^2(\Omega).$$

The trace $\mathbf{E}_{\mathbf{t}}$ coincides with $\nabla_{\mathbf{t}}\varphi$, where $\nabla_{\mathbf{t}}$ is the tangential gradient. But by the impedance condition, $\mathbf{H}\times\mathbf{n} = \lambda\mathbf{E}_{\mathbf{t}}$. As $\mathbf{H}$ belongs to $\mathbf{H}(\mathbf{curl};\Omega)$, its trace $\mathbf{H}\times\mathbf{n}$ belongs to

$\mathbf{H}^{-1/2}(\mathrm{div};\partial\Omega)$, [75, Theorem 5.4.2], and so does $\lambda\mathbf{E}_\mathbf{t}$. Finally

$$\mathrm{div}_\mathbf{t}\,\lambda\nabla_\mathbf{t}\varphi \in \mathsf{H}^{-\frac{1}{2}}(\partial\Omega).$$

Since λ is smooth and never 0 on $\partial\Omega$, the operator $\mathrm{div}_\mathbf{t}\,\lambda\nabla_\mathbf{t}$ is a scalar elliptic operator on $\partial\Omega$ which is a smooth manifold without boundary. By elliptic regularity we obtain that

$$\varphi\big|_{\partial\Omega} \in \mathsf{H}^{\frac{3}{2}}(\partial\Omega). \tag{3}$$

Now, using the elliptic regularity for φ solution of the Dirichlet problem (2)-(3) on Ω, we find $\varphi \in \mathsf{H}^2(\Omega)$. Coming back to (1), we have obtained that $\mathbf{E} \in \mathbf{H}^1(\Omega)$. □

Theorem 4.5.6 *If the boundary of Ω is of class $\mathscr{C}^2$ and the impedance function λ satisfies (4.46), the sesquilinear form* a *(4.53a) is coercive on* $\mathbf{V}$ *in the sense of Definition 3.2.1, i.e., there exists $C>0$ and $c>0$ such that for all $(\mathbf{E},\mathbf{H})\in\mathbf{V}$*

$$\mathrm{Re}\;\mathsf{a}(\mathbf{E},\mathbf{H};\mathbf{E},\mathbf{H}) \ge c\big(\|\mathbf{E}\|^2_{1;\Omega} + \|\mathbf{H}\|^2_{1;\Omega}\big) - C\big(\|\mathbf{E}\|^2_{0;\Omega} + \|\mathbf{H}\|^2_{0;\Omega}\big).$$

Proof: It relies on the following formula which holds for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{H}^1(\Omega)$, cf. [30, Lemma 2.2], [75, Lemma 5.4.2] and the earlier reference [23]:

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} - \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} - \mathrm{div}\,\mathbf{u}\;\mathrm{div}\,\mathbf{v}\,\mathrm{d}\mathbf{x} =$$
$$- \int_{\partial\Omega} \mathrm{div}_\mathbf{t}\,\mathbf{u}_\mathbf{t}\,(\mathbf{v}\cdot\mathbf{n}) + (\mathbf{u}\cdot\mathbf{n})\;\mathrm{div}_\mathbf{t}\,\mathbf{v}_\mathbf{t} + \mathrm{tr}\,\mathscr{B}\,(\mathbf{u}\cdot\mathbf{n})\,(\mathbf{v}\cdot\mathbf{n}) + \mathscr{B}\mathbf{u}_\mathbf{t}\cdot\mathbf{v}_\mathbf{t}\,\mathrm{d}\sigma \tag{4.55}$$

In (4.55) $\mathscr{B}$ is the second fundamental form of the surface $\partial\Omega$ and $\mathrm{div}_\mathbf{t}\,\mathbf{u}_\mathbf{t}$ is the surface divergence of $\mathbf{u}_\mathbf{t}$: There holds $\mathrm{div}_\mathbf{t}\,\mathbf{u}_\mathbf{t} = \mathrm{div}\,\mathbf{u} - \partial_\mathbf{n}\mathsf{u}_\mathbf{n} - \mathsf{u}_\mathbf{n}\,\mathrm{div}\,\mathbf{n}$, with $\mathsf{u}_\mathbf{n} = \mathbf{u}\cdot\mathbf{n}$. This surface divergence is the operator which defines the space $\mathbf{H}^{-\frac{1}{2}}(\mathrm{div};\partial\Omega)$:

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{div};\partial\Omega) = \{\mathbf{u}\in\mathbf{H}^{-\frac{1}{2}}(\partial\Omega) :\ \mathbf{u}=\mathbf{u}_\mathbf{t} \quad\text{and}\quad \mathrm{div}_\mathbf{t}\,\mathbf{u}_\mathbf{t}\in\mathsf{H}^{-\frac{1}{2}}(\partial\Omega)\}. \tag{1}$$

Let $(\mathbf{E},\mathbf{H})\in\mathbf{V}$. Since by Lemma 4.5.5 $\mathbf{E}$ and $\mathbf{H}$ belong to $\mathbf{H}^1(\Omega)$, we can use (4.55) for $\mathbf{u}=\mathbf{v}=\mathbf{E}$ and $\mathbf{u}=\mathbf{v}=\mathbf{H}$ and sum up the obtained equalities: We find that

$$\|\mathbf{E}\|^2_{1;\Omega} + \|\mathbf{H}\|^2_{1;\Omega} = \|\mathbf{E}\|^2_{\mathbf{X}(\Omega)} + \|\mathbf{H}\|^2_{\mathbf{X}(\Omega)} + D \tag{2}$$

where the difference term D can be estimated as

$$|D| \lesssim \|\,\mathrm{div}_\mathbf{t}\,\mathbf{E}_\mathbf{t}\|_{-\frac{1}{2};\partial\Omega}\,\|\mathsf{E}_\mathbf{n}\|_{\frac{1}{2};\partial\Omega} + \|\,\mathrm{div}_\mathbf{t}\,\mathbf{H}_\mathbf{t}\|_{-\frac{1}{2};\partial\Omega}\,\|\mathsf{H}_\mathbf{n}\|_{\frac{1}{2};\partial\Omega}$$
$$+ \|\mathsf{E}_n\|^2_{0;\partial\Omega} + \|\mathsf{H}_n\|^2_{0;\partial\Omega} + \|\mathbf{E}_\mathbf{t}\|^2_{0;\partial\Omega} + \|\mathbf{H}_\mathbf{t}\|^2_{0;\partial\Omega}.$$

Here $\mathsf{E}_\mathbf{n}$ and $\mathsf{H}_\mathbf{n}$ are the normal component $\mathbf{E}\cdot\mathbf{n}$ and $\mathbf{H}\cdot\mathbf{n}$ of $\mathbf{E}$ and $\mathbf{H}$, respectively. With the help of inequality (3.42), we find that the last four terms can be estimated by

$$C\big(\|\mathbf{E}\|_{0;\Omega}\,\|\mathbf{E}\|_{1;\Omega} + \|\mathbf{H}\|_{0;\Omega}\,\|\mathbf{H}\|_{1;\Omega}\big)$$

while, with definition (1) the first two terms are estimated by

$$C\big(\|\mathbf{E_t}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)}\,\|\mathbf{E}\|_{1;\,\Omega} + \|\mathbf{H_t}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)}\,\|\mathbf{H}\|_{1;\,\Omega}\big).$$

We need the impedance relations $\mathbf{H}\times\mathbf{n} = \lambda\mathbf{E_t}$ (and $\mathbf{E}\times\mathbf{n} = -\lambda^{-1}\mathbf{H_t}$, which is equivalent) in order to bound $\mathbf{E_t}$ and $\mathbf{H_t}$ in the norm $\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)$: We find

$$\|\mathbf{E_t}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)} \lesssim \|\mathbf{H}\times\mathbf{n}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)}$$

and

$$\|\mathbf{H_t}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)} \lesssim \|\mathbf{E}\times\mathbf{n}\|_{\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)}.$$

We use the continuity of the trace $\mathbf{v}\mapsto(\mathbf{v}\times\mathbf{n})\big|_{\partial\Omega}$ from $\mathbf{H}(\mathbf{curl};\Omega)$ into $\mathbf{H}^{-\frac12}(\mathrm{div};\partial\Omega)$ [75, Theorem 5.4.2] and find finally that

$$|D| \lesssim \|\mathbf{E}\|_{0;\,\Omega}\,\|\mathbf{E}\|_{1;\,\Omega} + \|\mathbf{H}\|_{0;\,\Omega}\,\|\mathbf{H}\|_{1;\,\Omega} + \|\mathbf{H}\|_{\mathbf{H}(\mathbf{curl};\Omega)}\,\|\mathbf{E}\|_{1;\,\Omega} + \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl};\Omega)}\,\|\mathbf{H}\|_{1;\,\Omega}.$$

We deduce that for all $\varepsilon > 0$

$$|D| \lesssim \varepsilon(\|\mathbf{E}\|^2_{1;\,\Omega} + \|\mathbf{H}\|^2_{1;\,\Omega}) + \varepsilon^{-1}(\|\mathbf{E}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} + \|\mathbf{H}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)})$$

Coming back to (2) and choosing ε small enough we finally find that there exists $C > 0$ such that for all $(\mathbf{E},\mathbf{H}) \in \mathbf{V}$,

(3) $$\|\mathbf{E}\|^2_{1;\,\Omega} + \|\mathbf{H}\|^2_{1;\,\Omega} \le C\big(\|\mathbf{E}\|^2_{\mathbf{X}(\Omega)} + \|\mathbf{H}\|^2_{\mathbf{X}(\Omega)}\big).$$

Using the definition of the sesquilinear form (4.53a), we see that (3) yields the desired coercivity estimate. □

It is clear now that, thanks to Theorem 4.5.6, the regularity Theorem 3.5.4 for elliptic Robin-type boundary value problems can be applied directly to this case, providing Fredholm properties and optimal elliptic regularity for solutions of problem (4.54):

Theorem 4.5.7 *Let Ω be a smooth bounded domain. Let $\omega \neq 0$ be the frequency and λ be an impedance factor satisfying (4.46). Let $(\mathbf{E},\mathbf{H}) \in \mathbf{H}(\mathbf{curl};\Omega)^2$ be an electromagnetic field solution of the harmonic Maxwell system with impedance boundary conditions (4.47) for data $\mathbf{J} \in \mathbf{H}(\mathrm{div};\Omega)$.*

(i) Let k be a non-negative integer. If Ω is of class $\mathscr{C}^{k+2}$, and $\mathbf{J} \in \mathbf{H}^k(\Omega)$ is such that $\mathrm{div}\,\mathbf{J} \in \mathrm{H}^{k+1}(\Omega)$, $\mathbf{curl}\,\mathbf{J} \in \mathbf{H}^k(\Omega)$ and $\mathbf{J}\times\mathbf{n} \in \mathbf{H}^{k+\frac12}(\partial\Omega)$, then $(\mathbf{E},\mathbf{H})$ belongs to $\mathbf{H}^{k+2}(\Omega)^2$.

(ii) If we assume that Ω is analytic and $\mathbf{J} \in \mathbf{A}(\Omega)$, then $\mathbf{E}$ and $\mathbf{H}$ belong to $\mathbf{A}(\Omega)$.

Now we use the general tools from Section 3.5 to obtain the strong formulation of the variational problem (4.54). This is then the standard second order elliptic boundary value problem satisfied by the solution of the Maxwell impedance boundary value problem.

According to Lemma 3.5.2 we have to identify the projectors Π^D, Π^T and the conormal system B together with the matrix Z defining the Robin-type boundary condition. The zero order boundary condition is then just the essential boundary condition, in our case the one defining the space $\mathbf{V}$.

The projector Π^D is the orthogonal projector in $\mathbb{C}^6$ the kernel of which is the subspace defined by the impedance condition:

$$\forall(\mathbf{u},\mathbf{v})\in\mathbb{C}^6\ :\quad \Pi^D(\mathbf{u},\mathbf{v}) = (1+|\lambda|^2)^{-1}\big(|\lambda|^2\mathbf{u_t} - \overline{\lambda}\,\mathbf{v}\times\mathbf{n},\ \mathbf{v_t}+\lambda\,\mathbf{u}\times\mathbf{n}\big)$$

Therefore the complementary projector $\Pi^T = \mathbb{I} - \Pi^D$ is given by

$$\forall(\mathbf{u},\mathbf{v})\in\mathbb{C}^6\ :\quad \Pi^T(\mathbf{u},\mathbf{v}) = \big(\mathsf{u_n}\mathbf{n},\mathsf{v_n}\mathbf{n}\big) + (1+|\lambda|^2)^{-1}\big(\mathbf{u_t}+\overline{\lambda}\,\mathbf{v}\times\mathbf{n},\ |\lambda|^2\mathbf{v_t}-\lambda\,\mathbf{u}\times\mathbf{n}\big)\,.$$

The conormal system is given as in (4.40) by

$$B(\mathbf{E},\mathbf{H}) = \gamma_0\big((\mathbf{curl}\,\mathbf{E})\times\mathbf{n} + s(\operatorname{div}\mathbf{E})\,\mathbf{n},\ (\mathbf{curl}\,\mathbf{H})\times\mathbf{n}+s(\operatorname{div}\mathbf{H})\,\mathbf{n}\big)\,.$$

Thus the principal part of the first order boundary operator is given by

$$\Pi^T B(\mathbf{E},\mathbf{H}) = \gamma_0\Big\{\big(s(\operatorname{div}\mathbf{E})\mathbf{n},\ s(\operatorname{div}\mathbf{H})\mathbf{n}\big) + \\ (1+|\lambda|^2)^{-1}\big((\mathbf{curl}\,\mathbf{E})\times\mathbf{n} - \overline{\lambda}\,(\mathbf{curl}\,\mathbf{H})_\mathbf{t},\ |\lambda|^2(\mathbf{curl}\,\mathbf{H})\times\mathbf{n}+\lambda\,(\mathbf{curl}\,\mathbf{E})_\mathbf{t}\big)\Big\}.$$

From (4.53b), we deduce that the operator Z defining the sesquilinear form b according to (3.38) is given by

$$Z(\mathbf{E},\mathbf{H}) = \big(-i\omega\lambda\,\mathbf{E_t},\ -i\omega\frac{1}{\lambda}\,\mathbf{H_t}\big)$$

and, therefore,

$$\Pi^T Z\gamma_0(\mathbf{E},\mathbf{H}) = \gamma_0\Big\{\frac{i\omega}{1+|\lambda|^2}\big(-\lambda\,\mathbf{E_t} - \frac{\overline{\lambda}}{\lambda}\,\mathbf{H}\times\mathbf{n},\ -\overline{\lambda}\,\mathbf{H_t}+\lambda^2\mathbf{E}\times\mathbf{n}\big)\Big\}\,.$$

We recall that the natural boundary condition of a general Robin-type problem is given by $\Pi^T BBl\mathbf{u} + \Pi^T Z\gamma_0 Bl\mathbf{u} = \Pi^T\mathbf{g}$. In the case of problem (4.54) we have for $\mathbf{g}$

$$\mathbf{g} = \tfrac{s}{i\omega}\big((\operatorname{div}\mathbf{J})\,\mathbf{n},\ 0\big) + \big(\mathbf{0},\mathbf{J}\times\mathbf{n}\big),$$

which gives for $\Pi^T\mathbf{g}$

$$\Pi^T\mathbf{g} = \tfrac{s}{i\omega}\big((\operatorname{div}\mathbf{J})\,\mathbf{n},\ 0\big) + (1+|\lambda|^2)^{-1}\big(-\overline{\lambda}\,\mathbf{J_t},\ |\lambda|^2\mathbf{J}\times\mathbf{n}\big)\,.$$

Putting all this together, we obtain the following elliptic second order boundary value problem for the two vector functions $\mathbf{E},\mathbf{H}$:

$$\left\{\begin{array}{rcll}
\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - s\nabla\operatorname{div}\mathbf{E} - \omega^2\mathbf{E} &=& i\omega\mathbf{J} - \frac{s}{i\omega}\nabla\operatorname{div}\mathbf{J} & \text{in }\ \Omega,\\
\mathbf{curl}\,\mathbf{curl}\,\mathbf{H} - s\nabla\operatorname{div}\mathbf{H} - \omega^2\mathbf{H} &=& \mathbf{curl}\,\mathbf{J} & \text{in }\ \Omega,\\
\mathbf{H}\times\mathbf{n} - \lambda\,\mathbf{E_t} &=& \mathbf{0} & \text{on }\ \partial\Omega,\\
\operatorname{div}\mathbf{E} &=& \frac{1}{i\omega}\operatorname{div}\mathbf{J} & \text{on }\ \partial\Omega,\\
\operatorname{div}\mathbf{H} &=& 0 & \text{on }\ \partial\Omega,\\
|\lambda|^2(\mathbf{curl}\,\mathbf{H})\times\mathbf{n} + \lambda(\mathbf{curl}\,\mathbf{E})_\mathbf{t} && &\\
-\,i\omega\overline{\lambda}\mathbf{H_t} + i\omega\lambda^2\mathbf{E}\times\mathbf{n} &=& |\lambda|^2\mathbf{J}\times\mathbf{n} & \text{on }\ \partial\Omega.
\end{array}\right. \tag{4.56}$$

Note that, using the impedance condition in the form $\mathbf{H_t} + \lambda\,\mathbf{E} \times \mathbf{n} = 0$, the last boundary condition becomes

$$|\lambda|^2(\mathbf{curl}\,\mathbf{H}) \times \mathbf{n} + \lambda(\mathbf{curl}\,\mathbf{E})_{\mathbf{t}} + i\omega|\lambda|^2\mathbf{E} \times \mathbf{n} - i\omega\lambda\mathbf{H_t} = |\lambda|^2\mathbf{J} \times \mathbf{n}$$

or, equivalently

$$\overline{\lambda}(\mathbf{curl}\,\mathbf{H} + i\omega\mathbf{E}) \times \mathbf{n} + (\mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H})_{\mathbf{t}} = \overline{\lambda}\,\mathbf{J} \times \mathbf{n} \tag{4.57}$$

which clearly displays a linear combination of the harmonic Maxwell equations in (4.47).

For the sake of completeness, we address now the question of knowing whether our coupled regularized formulation (4.54) allows us to go back to the original first order Maxwell system (4.47).

Let us start with the divergence properties of $\mathbf{E}$ and $\mathbf{H}$: We prove that the identities valid on the boundary for $\operatorname{div}\mathbf{E}$ and $\operatorname{div}\mathbf{H}$ in (4.56) are, in fact, valid inside Ω.

Lemma 4.5.8 *If the right-hand side $\mathbf{J}$ belongs to $\mathbf{H}(\operatorname{div};\Omega)$ and $-\omega^2/s$ is not an eigenvalue of the Laplace operator Δ with Dirichlet boundary conditions in $\mathsf{H}^2(\Omega)$, then any solution $(\mathbf{E},\mathbf{H}) \in \mathbf{V}$ of the variational problem (4.54) satisfies*

$$\operatorname{div}\mathbf{E} = \tfrac{1}{i\omega}\operatorname{div}\mathbf{J} \quad \textit{and} \quad \operatorname{div}\mathbf{H} = 0 \quad \textit{in}\ \ \Omega. \tag{4.58}$$

Proof: In (4.54) we first take test functions in the form $(\nabla\bar{\varphi},\mathbf{0})$ with an arbitrary $\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega)$. This directly implies that $(\nabla\bar{\varphi},\mathbf{0})$ belongs to $\mathbf{V}$, and therefore we get

$$\forall\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega),$$
$$s\int_\Omega \operatorname{div}\mathbf{E}\ \operatorname{div}\nabla\varphi\,\mathrm{d}\mathbf{x} - \omega^2\int_\Omega \mathbf{E}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} = \int_\Omega \Big(i\omega\mathbf{J}\cdot\nabla\varphi + \tfrac{s}{i\omega}\operatorname{div}\mathbf{J}\ \operatorname{div}\nabla\varphi\Big)\,\mathrm{d}\mathbf{x}.$$

We then conclude as in Lemma 4.5.1 that $\mathbf{E} - \frac{1}{i\omega}\mathbf{J}$ is divergence free.
Similarly choosing test functions in the form $(\mathbf{0},\nabla\bar{\varphi})$ with an arbitrary $\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega)$ (which belongs to $\mathbf{V}$), we have

$$\forall\varphi \in \mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega), \quad s\int_\Omega \operatorname{div}\mathbf{H}\ \operatorname{div}\nabla\varphi\,\mathrm{d}\mathbf{x} - \omega^2\int_\Omega \mathbf{H}\cdot\nabla\varphi\,\mathrm{d}\mathbf{x} = 0.$$

This leads to the divergence free property of $\mathbf{H}$. □

It remains to see whether any $(\mathbf{E},\mathbf{H}) \in \mathbf{V}$ solution of (4.54) satisfies the harmonic Maxwell system (4.47).

Lemma 4.5.9 *Let $\omega \neq 0$. Assume that the conjugate homogeneous first order Maxwell system with impedance condition*

$$\begin{cases} \mathbf{curl}\,\mathbf{E}_0 + i\omega\mathbf{H}_0 = \mathbf{0} \quad \textit{and} \quad \mathbf{curl}\,\mathbf{H}_0 - i\omega\mathbf{E}_0 = \mathbf{0} \quad \textit{in}\ \ \Omega, \\ \mathbf{H}_0 \times \mathbf{n} - \lambda\,\mathbf{E}_{0,\mathbf{t}} = \mathbf{0} \quad \textit{on}\ \ \partial\Omega, \end{cases} \tag{4.59}$$

admits $(\mathbf{E}_0, \mathbf{H}_0) = (\mathbf{0}, \mathbf{0})$ *as only solution in* $\mathbf{H}(\mathbf{curl};\Omega)^2$*, and that* $-\omega^2/s$ *is not an eigenvalue of the Laplace operator* Δ *with Dirichlet boundary conditions. If the right-hand side* $\mathbf{J}$ *belongs to* $\mathbf{H}(\mathrm{div};\Omega)$*, then any solution* $(\mathbf{E}, \mathbf{H}) \in \mathbf{V}$ *of* (4.54) *satisfies* (4.47)*.*

Proof: Taking (4.57) and (4.58) into account we see that (4.56) reduces here to

$$(1)\qquad \left\{\begin{array}{rcll} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\mathbf{E} &=& i\omega\mathbf{J} & \text{in } \Omega,\\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{H} - \omega^2\mathbf{H} &=& \mathbf{curl}\,\mathbf{J} & \text{in } \Omega,\\ \mathrm{div}\,\mathbf{E} &=& \frac{1}{i\omega}\mathrm{div}\,\mathbf{J} & \text{in } \Omega,\\ \mathrm{div}\,\mathbf{H} &=& 0 & \text{in } \Omega,\\ \mathbf{H}\times\mathbf{n} - \lambda\,\mathbf{E_t} &=& \mathbf{0} & \text{on } \partial\Omega,\\ \overline{\lambda}(\mathbf{curl}\,\mathbf{H} + i\omega\mathbf{E})\times\mathbf{n} + (\mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H})_{\mathbf{t}} &=& \overline{\lambda}\,\mathbf{J}\times\mathbf{n} & \text{on } \partial\Omega. \end{array}\right.$$

Let us set

$$(2)\qquad \mathbf{U} = \mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H} \quad\text{and}\quad \mathbf{T} = \mathbf{curl}\,\mathbf{H} - \mathbf{J} + i\omega\mathbf{E}.$$

From the first two equations from (1), we write

$$\begin{array}{rcll} \omega^2\mathbf{E} + i\omega\mathbf{J} &=& \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} & \text{in } \Omega,\\ \omega^2\mathbf{H} &=& \mathbf{curl}(\mathbf{curl}\,\mathbf{H} - \mathbf{J}) & \text{in } \Omega. \end{array}$$

Therefore, taking the curl of $\mathbf{U}$ and $\mathbf{T}$ we get

$$(3)\qquad \begin{array}{rcl} \mathbf{curl}\,\mathbf{U} &=& \omega^2\mathbf{E} + i\omega\mathbf{J} - i\omega\,\mathbf{curl}\,\mathbf{H}\\ \mathbf{curl}\,\mathbf{T} &=& \omega^2\mathbf{H} + i\omega\,\mathbf{curl}\,\mathbf{E}. \end{array}$$

Identities (2) and (3) give immediately

$$(4)\qquad \mathbf{curl}\,\mathbf{U} + i\omega\mathbf{T} = 0 \quad\text{and}\quad \mathbf{curl}\,\mathbf{T} - i\omega\mathbf{U} = 0 \quad\text{in } \Omega.$$

Now we notice that the boundary condition

$$\overline{\lambda}(\mathbf{curl}\,\mathbf{H} + i\omega\mathbf{E})\times\mathbf{n} + (\mathbf{curl}\,\mathbf{E} - i\omega\mathbf{H})_{\mathbf{t}} = \overline{\lambda}\,\mathbf{J}\times\mathbf{n} \quad\text{on } \partial\Omega,$$

from (1) is fully equivalent to

$$\overline{\lambda}\,\mathbf{T}\times\mathbf{n} + \mathbf{U_t} = \mathbf{0} \quad\text{on } \partial\Omega.$$

Taking the conjugate of this identity, we get $\lambda\overline{\mathbf{T}}\times\mathbf{n} + \overline{\mathbf{U}}_{\mathbf{t}} = \mathbf{0}$, which is equivalent to

$$(5)\qquad \overline{\mathbf{U}}\times\mathbf{n} - \lambda\,\overline{\mathbf{T}}_{\mathbf{t}} = \mathbf{0} \quad\text{on } \partial\Omega.$$

The equalities (2) and (5) show that the couple $(\overline{\mathbf{T}}, \overline{\mathbf{U}})$ is solution of the impedance problem (4.47) with $\mathbf{J} = 0$ and ω replaced with $-\omega$, i.e. problem (4.59). The assumption of uniqueness yields that $(\overline{\mathbf{T}}, \overline{\mathbf{U}}) = 0$, which proves that the couple $(\mathbf{E}, \mathbf{H})$ satisfies the original impedance problem (4.47). □

4.6 The Reissner-Mindlin plate model

The system of Reissner-Mindlin is a standard engineering model describing the bending of elastic structures of small to moderate thickness. More precisely, this model describes the bending of a plate $P = \Omega \times (-\varepsilon, \varepsilon)$ in equilibrium subject to transverse loading, in terms of the deflection w of its midplane $\Omega \subset \mathbb{R}^2$ and of the rotation $\boldsymbol{\psi} = (\psi_1, \psi_2)^\top$ of fibers transverse to this midplane [65, 82, 84]. Both unknowns are assumed to be independent of the transverse variable x_3. They are then determined as solution of the system

$$\begin{cases} \frac{\varepsilon^2}{3}\tilde{L}\boldsymbol{\psi} + \mu k(\nabla w + \boldsymbol{\psi}) &= \varepsilon^2 \mathbf{p} \quad \text{in } \Omega, \\ -\mu k(\Delta w + \operatorname{div} \boldsymbol{\psi}) &= \varepsilon^2 p_3 \quad \text{in } \Omega, \end{cases} \tag{4.60}$$

where 2ε is the thickness of the plate, $\tilde{L}$ is the two-dimensional Lamé system (in the plane stress model) given here by (compare with (4.16))

$$\tilde{L}(\mathrm{D}_{\mathbf{x}}) = -\mu\Delta\mathbb{I}_n - (\tilde{\lambda} + \mu)\nabla \operatorname{div},$$

with $\tilde{\lambda} = \frac{2\mu\lambda}{\lambda+2\mu}$ the homogenized Lamé coefficient, $k > 0$ is the shear correction factor and finally $\mathbf{p}$ and p_3 are loadings related to the volume and traction forces applied to the three-dimensional plate. The associated system of partial differential operators is a 3×3 system with constant coefficients given by

$$L(\mathrm{D}_{\mathbf{x}}) = \begin{pmatrix} -\frac{\varepsilon^2}{3}\big(\mu\Delta + (\tilde{\lambda}+\mu)\partial_1^2\big) + \mu k & -\frac{\varepsilon^2}{3}(\tilde{\lambda}+\mu)\partial_1\partial_2 & \mu k\partial_1 \\ -\frac{\varepsilon^2}{3}(\tilde{\lambda}+\mu)\partial_1\partial_2 & -\frac{\varepsilon^2}{3}\big(\mu\Delta + (\tilde{\lambda}+\mu)\partial_2^2\big) + \mu k & \mu k\partial_2 \\ -\mu k\partial_1 & -\mu k\partial_2 & -\mu k\Delta \end{pmatrix}.$$

Its principal part is

$$L^{\mathsf{pr}}(\mathrm{D}_{\mathbf{x}}) = \begin{pmatrix} -\frac{\varepsilon^2}{3}\big(\mu\Delta + (\tilde{\lambda}+\mu)\partial_1^2\big) & -\frac{\varepsilon^2}{3}(\tilde{\lambda}+\mu)\partial_1\partial_2 & 0 \\ -\frac{\varepsilon^2}{3}(\tilde{\lambda}+\mu)\partial_1\partial_2 & -\frac{\varepsilon^2}{3}\big(\mu\Delta + (\tilde{\lambda}+\mu)\partial_1^2\big) & 0 \\ 0 & 0 & -\mu k\Delta \end{pmatrix}$$

and splits into the Lamé system for the first two variables and to the Laplace operator for the third one up to some negative factor. Therefore this system is strongly elliptic provided $\mu > 0$ and $\tilde{\lambda} + \mu > 0$, or equivalently, provided $\mu > 0$ and $\frac{3\lambda+2\mu}{\lambda+2\mu} > 0$.

Eight canonical boundary conditions can be considered on the boundary of Ω [65, 82, 84], corresponding to the 4 types of boundary conditions of the two-dimensional Lamé system described in section 4.3 and to either Dirichlet boundary condition for w or to a Neumann type boundary condition:

$$q_n := \mu k(\boldsymbol{\psi} + \nabla w) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

In other words, with the stress $\sigma(\boldsymbol{\psi})$ given by

$$\sigma_{\alpha\beta}(\boldsymbol{\psi}) = 2\mu\, \epsilon_{\alpha\beta}(\boldsymbol{\psi}) + \tilde{\lambda} \operatorname{div} \boldsymbol{\psi}\, \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2,$$

we can have either hard clamped boundary condition

$$\boldsymbol{\psi} = 0\,; \quad w = 0 \quad \text{on } \partial\Omega,$$

or soft clamped boundary condition

$$\boldsymbol{\psi}\cdot\mathbf{n} = 0\,; \quad (\sigma(\boldsymbol{\psi})\mathbf{n})\cdot\mathbf{t} = 0\,; \quad w = 0 \quad \text{on } \partial\Omega,$$

or hard simply supported boundary condition

$$\boldsymbol{\psi}\cdot\mathbf{t} = 0\,; \quad (\sigma(\boldsymbol{\psi})\mathbf{n})\cdot\mathbf{n} = 0\,; \quad w = 0 \quad \text{on } \partial\Omega,$$

or soft simply supported boundary condition

$$\sigma(\boldsymbol{\psi})\mathbf{n} = 0\,; \quad w = 0 \quad \text{on } \partial\Omega,$$

or frictional I boundary condition

$$\boldsymbol{\psi} = 0\,; \quad q_n = 0 \quad \text{on } \partial\Omega,$$

or sliding edge boundary condition

$$\boldsymbol{\psi}\cdot\mathbf{n} = 0\,; \quad (\sigma(\boldsymbol{\psi})\mathbf{n})\cdot\mathbf{t} = 0\,; \quad q_n = 0 \quad \text{on } \partial\Omega,$$

or frictional II boundary condition

$$\boldsymbol{\psi}\cdot\mathbf{t} = 0\,; \quad (\sigma(\boldsymbol{\psi})\mathbf{n})\cdot\mathbf{n} = 0\,; \quad q_n = 0 \quad \text{on } \partial\Omega,$$

or stress free boundary condition

$$\sigma(\boldsymbol{\psi})\mathbf{n} = 0\,; \quad q_n = 0 \quad \text{on } \partial\Omega,$$

It was shown in Lemma 4.2 of [84] that each of these 8 boundary conditions covers the system (4.60) on $\partial\Omega$ provided $\mu > 0$ and $\frac{3\lambda+2\mu}{\lambda+2\mu} > 0$. Note that this covering condition also follows from the variational formulation below and Theorem 3.2.6.

The variational formulation of problem (4.60) with each of the 8 boundary conditions is easily deduced from the natural energy functional associated with the RM model:

$$\begin{aligned}
\mathbf{V} &= \{(\boldsymbol{\psi}, w) \in \mathbf{H}^1(\Omega) \ \text{satisfying the essential boundary conditions}\},\\
\mathsf{a}\big((\boldsymbol{\psi}, w), (\boldsymbol{\chi}, z)\big) &= \frac{\varepsilon^2}{3}\int_\Omega (2\mu \sum_{\alpha,\beta=1}^{2} \epsilon_{\alpha\beta}(\boldsymbol{\psi})\, \epsilon_{\alpha\beta}(\overline{\boldsymbol{\chi}}) + \tilde{\lambda}\,\mathrm{div}\,\boldsymbol{\psi}\,\mathrm{div}\,\overline{\boldsymbol{\chi}})\,\mathrm{d}\mathbf{x}\\
&+ \mu k \int_\Omega (\nabla w + \boldsymbol{\psi})\cdot(\nabla\overline{z} + \overline{\boldsymbol{\chi}})\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

This form is coercive on $\mathbf{V}$ in the sense of Definition 3.2.1 under the above assumptions on μ and λ: Indeed, invoking the ellipticity assumption (4.32) and the Korn inequality (4.24), there exists $\alpha > 0$ such that

$$\mathsf{a}((\boldsymbol{\psi}, w), (\boldsymbol{\psi}, w)) \geq \alpha(|\boldsymbol{\psi}|^2_{1,\Omega} - |\boldsymbol{\psi}|^2_{0,\Omega}) + \mu k|\nabla w + \boldsymbol{\psi}|^2_{0,\Omega}.$$

Now using the triangular inequality, we may write

$$|\nabla w|_{0,\Omega}^2 \leq (|\nabla w + \boldsymbol{\psi}|_{0,\Omega} + |\boldsymbol{\psi}|_{0,\Omega})^2 \leq 2(|\nabla w + \boldsymbol{\psi}|_{0,\Omega}^2 + |\boldsymbol{\psi}|_{0,\Omega}^2).$$

This estimate in the previous one directly yields

$$\mathsf{a}((\boldsymbol{\psi}, w), (\boldsymbol{\psi}, w)) \geq \alpha |\boldsymbol{\psi}|_{1,\Omega}^2 + \frac{\mu k}{2} |w|_{1,\Omega}^2 - (\alpha + \mu k) |\boldsymbol{\psi}|_{0,\Omega}^2,$$

which leads to the coerciveness property.

For the injectivity property of the operator A associated with a by (3.26) we look for a pair $(\boldsymbol{\psi}, w) \in \mathbf{V}$ such that

$$\mathsf{a}((\boldsymbol{\psi}, w), (\boldsymbol{\psi}, w)) = 0.$$

In view of the form of a this means that

$$\epsilon_{\alpha\beta}(\boldsymbol{\psi}) = 0\ ; \quad \operatorname{div} \boldsymbol{\psi} = 0\ ; \quad \nabla w + \boldsymbol{\psi} = 0 \quad \text{in } \Omega,$$

for all $\alpha, \beta = 1$ or 2. Hence $\boldsymbol{\psi}$ is a rigid motion, in other words, there exist $a, b, c \in \mathbb{C}$ such that

$$\boldsymbol{\psi}(x_1, x_2) = (a, b)^\top + c(-x_2, x_1)^\top.$$

As $\nabla w = -\boldsymbol{\psi}$, we then find that $\boldsymbol{\psi}$ is curl free, hence $c = 0$, and w must be of the form

$$w = -ax_1 - bx_2 + d,$$

for some $d \in \mathbb{C}$. We thus have found that

$$\ker A = \mathbf{V} \cap \{(a, b, -ax_1 - bx_2 + d)^\top \ :\ a, b, d \in \mathbb{C}\}.$$

This implies that $\ker A = \{0\}$ for the first four boundary conditions. On the other hand, for the stress free boundary condition one readily sees that

$$\ker A = \{(a, b, -ax_1 - bx_2 + d)^\top \ :\ a, b, d \in \mathbb{C}\},$$

while for the remaining three, we have

$$\ker A = (0, 0)^\top \times \mathbb{C}.$$

The Lax-Milgram lemma then furnishes an isomorphism between $\mathbf{V}/\ker A$ and $\{\mathbf{p} \in \mathbf{V} : \langle \mathbf{p}\ ,\ \mathbf{r} \rangle = 0,\ \forall \mathbf{r} \in \ker A\}$. Hence shift results in Sobolev spaces or spaces of analytic functions are valid in the same form as in Theorem 3.4.5.

4.7 The piezoelectric system

Smart structures made of piezoelectric and piezomagnetic materials are gaining interest in applications since they are able to transform the energy from one type to another (among magnetic, electric and mechanical energy), allowing to use them as sensors or actuators. Commonly used piezoelectric materials are ceramics and quartz. The mathematical model of this system starts to be well established [13, 34, 51, 81] and corresponds to a coupling between the elasticity system and Maxwell equations. A full mathematical analysis is not yet done, except in some particular cases [50, 60, 62]. Usually, the electric field $\mathbf{E}$ is assumed to be static, hence curl free, i.e., $\mathbf{E} = \nabla\Phi$, where Φ is an electric potential. After this reduction, the obtained system is a coupling between the elasticity system and a second order scalar equation.

If we restrict ourselves to the 3-dimensional case, it is a 4×4 system involving the vector valued function $\mathbf{u}$, corresponding to the displacement field and the scalar valued function Φ, corresponding to the electric potential. It can be written as

$$L(\mathrm{D}_{\mathbf{x}}) \begin{pmatrix} \mathbf{u} \\ \Phi \end{pmatrix} = \begin{pmatrix} \left(-\sum_{j=1}^{3} \partial_j \sigma_{ij}(\mathbf{u}, \Phi)\right)_{1\le i\le 3} \\ \sum_{j=1}^{3} \partial_j D_j(\mathbf{u}, \Phi) \end{pmatrix},$$

where the stress tensor is here given by generalized Hooke's law (actuatoric effect)

$$\sigma_{ij}(\mathbf{u}, \Phi) = \sum_{m,n=1}^{3} C_{ijmn}\epsilon_{mn}(\mathbf{u}) + \sum_{k=1}^{3} e_{kij}\partial_k\Phi,$$

and the electric displacement $\mathbf{D} = (D_1, D_2, D_3)$ is given by (sensoric effect)

$$D_i(\mathbf{u}, \Phi) = \sum_{m,n=1}^{3} e_{imn}\epsilon_{mn}(\mathbf{u}) - \sum_{k=1}^{3} \varepsilon_{ik}\partial_k\Phi,$$

where the elasticity moduli C_{ijmn} are here supposed to be real constants that satisfy the symmetry relations (4.31) and the strong ellipticity condition (4.32); the permittivity constants ε_{ik} form a positive definite 3×3 matrix, and finally the piezoelectric tensor $(e_{imn})_{1\le i,m,n\le 3}$ satisfies the symmetry relation:

$$e_{imn} = e_{inm}, \quad \forall i, m, n = 1, 2, 3.$$

Here for the sake of simplicity we complement this system with Dirichlet boundary conditions. In other words, we consider the problem:

$$\begin{cases} L(\mathbf{u}, \Phi) &= (\mathbf{f}, g)^\top & \text{in } \Omega, \\ \mathbf{u} &= 0 & \text{on } \partial\Omega, \\ \Phi &= 0 & \text{on } \partial\Omega. \end{cases} \tag{4.61}$$

The variational formulation of problem (4.61) is naturally associated with the above physical tensors: We simply take

$$\mathsf{V} = \mathsf{H}_0^1(\Omega), \tag{4.62}$$

$$\mathsf{a}((\mathbf{u},\Phi),(\mathbf{v},\Psi)) = \int_\Omega \Big(\sum_{i,j=1}^{3} \sigma_{ij}(\mathbf{u},\Phi)\,\epsilon_{ij}(\bar{\mathbf{v}}) - \sum_{i=1}^{3} D_i(\mathbf{u},\Phi)\,\partial_i\bar{\Psi}\Big)\,\mathrm{d}\mathbf{x}. \tag{4.63}$$

The strong coerciveness of a on V follows from the ellipticity assumption (4.32), the Poincaré inequality (1.10) and the Korn inequality (4.24) because one readily checks that

$$\mathsf{a}\big((\mathbf{u},\Phi),(\mathbf{u},\Phi)\big) = \int_\Omega \Big(\sum_{i,j,m,n=1}^{3} C_{ijmn}\epsilon_{ij}(\mathbf{u})\epsilon_{mn}(\bar{\mathbf{u}}) + \sum_{i,j=1}^{3} \varepsilon_{ij}\partial_i\Phi\partial_j\Phi\Big)\,\mathrm{d}\mathbf{x}.$$

4.8 Influence of the weak form on the natural boundary conditions

In this last section we want to give examples for the phenomenon already mentioned before that different sesquilinear forms belonging to the same differential operator can lead to different natural boundary conditions. Let us illustrate the phenomenon for the Lamé system.

We have seen in subsection 4.3.b that the sesquilinear form a defined by (4.23) on $\mathsf{V} = \mathsf{H}^1(\Omega)$ leads to the Lamé system with stress free boundary condition, problem (4.26).

Let us now exhibit two other sesquilinear forms that lead to the Lamé system L given in (4.16) but with different boundary conditions. The first one is defined by

$$\mathsf{a}_1(\mathbf{u},\mathbf{v}) = \int_\Omega \big(\mu \sum_{i,j=1}^{n} \partial_i u_j \partial_i \bar{v}_j + (\lambda+\mu)\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\bar{\mathbf{v}}\big)\,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{u},\mathbf{v} \in \mathsf{H}^1(\Omega).$$

Let $\mathbf{u} \in \mathsf{H}^1(\Omega)$ be a solution of problem

$$\mathsf{a}_1(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathsf{H}^1(\Omega),$$

with $\mathbf{f} \in \mathsf{L}^2(\Omega)$. Then by integration by parts, we see that $\mathbf{u}$ is a solution of

$$\left\{ \begin{array}{rcll} L\mathbf{u} &=& \mathbf{f} & \text{in } \Omega, \\ \mu(\partial_n \mathbf{u})_t &=& \mathbf{0} & \text{on } \partial\Omega, \\ \mu(\partial_n\mathbf{u})\cdot\mathbf{n} + (\lambda+\mu)\,\mathrm{div}\,\mathbf{u} &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4.64}$$

We readily see that the above boundary conditions are different from the stress-free boundary conditions $\sigma(\mathbf{u})\mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

Note that for $\mu > 0$ and $\lambda + \mu \geq 0$ the sesquilinear form a_1 is formally positive in the sense of Definition 3.2.4 and hence coercive on $\mathbf{H}^1(\Omega)$ (without making use for Korn's inequality). Therefore by Theorem 3.2.6, the system (4.64) is elliptic on $\overline{\Omega}$.

As a second example, let us take the form (compare with the sesquilinear form (4.34) used for the regularized Maxwell system)

$$\mathsf{a}_2(\mathbf{u}, \mathbf{v}) = \int_\Omega \big(\mu \, \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \bar{\mathbf{v}} + (\lambda + 2\mu) \operatorname{div} \mathbf{u} \operatorname{div} \bar{\mathbf{v}}\big) \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{4.65}$$

Here if $\mathbf{u} \in \mathbf{H}^1(\Omega)$ is a solution of problem

$$\mathsf{a}_2(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \bar{\mathbf{v}} \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

with $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then by integration by parts we see that $\mathbf{u}$ is a solution of

$$\begin{cases} L\mathbf{u} &= \mathbf{f} & \text{in } \Omega, \\ \mathbf{curl}\, \mathbf{u} \times \mathbf{n} &= \mathbf{0} & \text{on } \partial\Omega, \\ \operatorname{div} \mathbf{u} &= 0 & \text{on } \partial\Omega. \end{cases} \tag{4.66}$$

Once again these boundary conditions are different from the stress-free boundary conditions. Note that these new boundary conditions do not depend on λ and μ and that the sesquilinear form a_2 is not coercive on $\mathbf{H}^1(\Omega)$, since the boundary conditions from (4.66) do not cover the Lamé system. This non-coerciveness property can also be checked directly, since for any harmonic function φ in $\mathbb{R}^n$, the function $\mathbf{u} = \nabla\varphi$ (restricted to Ω) satisfies $\mathsf{a}_2(\mathbf{u}, \mathbf{u}) = 0$. This means that the kernel of the associated operator A from $\mathbf{H}^1(\Omega)$ into its dual has infinite dimension, hence a_2 cannot be coercive on $\mathbf{H}^1(\Omega)$ due to Theorem 3.3.1, and the system (4.66) is *not elliptic*.

Let us now compare the three sesquilinear forms if we impose an essential boundary condition in the form of a smaller variational space

$$\mathbf{V} = \mathbf{H}^1_N(\Omega)\,.$$

In the case of the standard sesquilinear form a (4.23) corresponding to the elastic energy for the Lamé system, this describes simply supported boundary conditions as discussed in Section 4.3.c. The associated natural boundary condition is then given by the normal traction $(\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{n} = 0$, see (4.28), which can be written as

$$2\mu(\partial_n \mathbf{u}) \cdot \mathbf{n} + \lambda \operatorname{div} \mathbf{u} = 0\,. \tag{4.67a}$$

In the second case of the sesquilinear form $\mathbf{a}_1$ we get the natural boundary condition from the last line in (4.64)

$$\mu(\partial_n \mathbf{u}) \cdot \mathbf{n} + (\lambda + \mu) \operatorname{div} \mathbf{u} = 0\,. \tag{4.67b}$$

In the third case of the sesquilinear form $\mathbf{a}_2$ we get the natural boundary condition from the last line in (4.66)

$$\operatorname{div} \mathbf{u} = 0 \,. \tag{4.67c}$$

Thus we have the three boundary value problems

$$\begin{cases} L\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u}_t = \mathbf{0} & \text{on } \partial\Omega, \\ \text{One of (4.67a), (4.67b), (4.67c)} & \text{on } \partial\Omega. \end{cases} \tag{4.68}$$

We have already seen that all three boundary value problems are elliptic. It is also not hard to see that in general the three natural boundary conditions will be distinct, so that the three boundary value problems will have different solutions.

Note, however, that in a boundary point near which the boundary is *flat*, we have for $\mathbf{u} \in \mathbf{H}^1_N(\Omega)$

$$\operatorname{div} \mathbf{u} = \partial_n(\mathbf{u} \cdot \mathbf{n}) = (\partial_n \mathbf{u}) \cdot \mathbf{n} \,,$$

and therefore all three natural boundary conditions coincide there. This has the curious consequence that if Ω is a *polyhedron*, then actually all three boundary value problems have the same solution.

Chapter 5

Transmission problems

Introduction

We devote this chapter to the class of *elliptic transmission problems*, which constitutes a natural extension of the class of elliptic boundary value problems. A transmission problem is characterized by the presence inside a given domain of an *interface*, which is the boundary of an internal subdomain. On each side of this interface, we have elliptic equations or systems which may be distinct, and across the interface we have a combination of transmission conditions involving traces on both sides of the interface. This kind of problems occurs in solid mechanics for instance when the body is occupied by different materials. We consider in this chapter fairly general situations, namely

- we may suppose that the size of the system may be different on different parts of the domain,
- we introduce general interface conditions combining boundary and transmission conditions, which "cover" the system.

It turns out that the notion of covering transmission conditions can be bridged to the notion of covering boundary conditions as presented in Chapter 2. Here we present local and global definitions, natural regularity shift results, variational formulations and examples.

Plan of Chapter 5

- **§1** Definition of elliptic transmission problems, with piecewise interior ellipticity and covering interface conditions: They include transmission conditions across the interface and internal boundary conditions.
- **§2** Natural regularity shift in classes of piecewise smooth functions (Sobolev and analytic regularity).
- **§3** Problems set in variational form (which is the most frequent setting for elliptic transmission problems).
- **§4** Various examples, from the case of piecewise smooth data for a standard elliptic problem, to problems arising from physics where different materials interact.

Essentials

For simplicity we consider a typical situation of an interface I between two domains Ω_+ and Ω_-:

$$\mathrm{I} = \partial\Omega_+ \cap \partial\Omega_-$$

in the configuration where Ω_+ and Ω_- are the interior and exterior subdomains, respectively, of

$$\Omega = \Omega_- \cup \mathrm{I} \cup \Omega_+,$$

as illustrated by Figure 5.1. The exterior boundary is $\partial\Omega$ and we denote it by Γ. Note that

$$\partial\Omega_+ = \mathrm{I} \quad \text{and} \quad \partial\Omega_- = \mathrm{I} \cup \Gamma.$$

We assume that I and Γ are smooth. In such a configuration, the transmission problems which we consider consist of

- Two elliptic systems of second order operators L_+ (of size $N_+ \times N_+$) and L_- (of size $N_- \times N_-$) given on Ω_+ and Ω_- respectively; they are associated with the unknown $\mathbf{u}_+$ of size N_+ on Ω_+ and $\mathbf{u}_-$ of size N_- on Ω_-,
- A system of boundary operators $C_\Gamma = (T_\Gamma, D_\Gamma)$ which covers L_- on Γ,
- A system of interface operators $C_\mathrm{I} = (T_\mathrm{I}, D_\mathrm{I})$ which covers $\{L_+, L_-\}$ on I.

Let us explain what we understand by "system of interface operators". We recall that (after local trivialization) C_Γ is an $N_- \times N_-$ system of operators defined on Γ, of order 1 (T_Γ) or 0 (D_Γ), and C_Γ acts on the traces of $\mathbf{u}_-$ on Γ. Likewise, C_I is a system of size $(N_+ + N_-) \times (N_+ + N_-)$, whose coefficients are operators of order 1 (T_I) or 0 (D_I), and C_I acts on the traces of $\mathbf{u}_+$ and $\mathbf{u}_-$ on I. Obviously we can write

$$C_\mathrm{I}(\mathbf{u}_+, \mathbf{u}_-) = C_+\mathbf{u}_+ + C_-\mathbf{u}_-$$

and

$$T_\mathrm{I}(\mathbf{u}_+, \mathbf{u}_-) = T_+\mathbf{u}_+ + T_-\mathbf{u}_-, \quad D_\mathrm{I}(\mathbf{u}_+, \mathbf{u}_-) = D_+\mathbf{u}_+ + D_-\mathbf{u}_-$$

We define what is the covering condition for a system of interface operators in a similar way as we did for boundary conditions in Chapter 2. Instead of the periodic half-space $\mathbb{T}^n_+ = \mathbb{T}^{n-1} \times \mathbb{R}_+$, our model domain is the cylinder $\Omega = \mathbb{T}^{n-1} \times \mathbb{R}$, that is

$$\Omega = \mathbb{T}^n_- \cup \mathrm{I} \cup \mathbb{T}^n_+, \quad \text{with} \quad \mathbb{T}^n_\pm = \mathbb{T}^{n-1} \times \mathbb{R}_\pm \text{ and } \mathrm{I} = \mathbb{T}^{n-1} \times \{0\}.$$

We denote by $\mathbf{x}'$ the variable in $\mathbb{T}^{n-1}$ and by t the variable in $\mathbb{R}$. Model transmission problems are set on this model domain and are written as

$$\begin{cases} L_+\mathbf{u}_+ &= \mathbf{f}_+ & \text{in } \mathbb{T}^n_+, \\ L_-\mathbf{u}_- &= \mathbf{f}_- & \text{in } \mathbb{T}^n_-, \\ T_+\mathbf{u}_+ + T_-\mathbf{u}_- &= \mathbf{g} & \text{on } \mathrm{I}, \\ D_+\mathbf{u}_+ + D_-\mathbf{u}_- &= \mathbf{h} & \text{on } \mathrm{I}. \end{cases} \tag{5.a}$$

As in (2.c), the model operators have constant coefficients. Taking their principal parts and considering their symbols with respect to the variable $\mathbf{x}'$ as in (2.e), we define the family of one-dimensional transmission problems:

$$\begin{cases} L_+^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_+ &= 0 \quad \text{in } \mathbb{R}_+, \\ L_-^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_- &= 0 \quad \text{in } \mathbb{R}_-, \\ T_+^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_+ + T_-^{\mathsf{pr}}(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_- &= \mathbf{G} \quad \text{in } t = 0, \\ D_+(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_+ + D_-(\boldsymbol{\xi}', \mathrm{D}_t)\mathbf{U}_- &= \mathbf{H} \quad \text{in } t = 0, \end{cases} \tag{5.b}$$

for $\boldsymbol{\xi}' \in \mathbb{R}^{n-1}$. This leads to the definition of ellipticity as in Definition 2.A.

Definition 5.A *In the case of a transmission problem* $\{L_+, L_-, C_{\mathrm{I}} = C_+ + C_-\}$ *with constant coefficients, let the second order* $N_\pm \times N_\pm$ *systems* $L_\pm$ *be elliptic. The interface operator* $C_{\mathrm{I}} = C_+ + C_-$ *is said to* complement *or* cover *the operators* (L_+, L_-)*, and the corresponding transmission problem is called* elliptic, *if for any* $\boldsymbol{\xi}' \in \mathbb{R}^{n-1} \setminus \mathbf{0}$ *the boundary value problem* (5.b) *admits, for any* $(\mathbf{G}, \mathbf{H}) \in \mathbb{C}^{N_+ + N_-}$*, a unique solution* $(\mathbf{U}_+, \mathbf{U}_-) \in \mathsf{L}^2(\mathbb{R}_+)^{N_+} \times \mathsf{L}^2(\mathbb{R}_-)^{N_-}$.

If we make the change of variables $t \to -t$ in $\mathbb{R}_-$, we transform problem (5.a) and its symbol (5.b) into a boundary value problem $\{L^\sharp, C^\sharp\}$ of size $(N_+ + N_-) \times (N_+ + N_-)$. Its ellipticity in the classical sense (Definition 2.A) is equivalent to the ellipticity of the transmission problem $\{L_+, L_-, C_{\mathrm{I}} = C_+ + C_-\}$ in the sense of Definition 5.A.

The ellipticity and covering conditions are defined in the variable coefficient case by freezing the operators at each interface point.

The close relation between model transmission problems and elliptic boundary value problems for systems allows to deduce from the results of Chapter 2 a variety of regularity shift results in Sobolev and analytic classes, see Theorems 5.2.1 and 5.2.2. Note that the correct notion of regularity is separate regularity for $\mathbf{u}_+$ and $\mathbf{u}_-$, using the spaces

$$\mathsf{P}\mathbf{H}^k(\Omega) = \big\{\mathbf{u} = (\mathbf{u}_+, \mathbf{u}_-) : \ \mathbf{u}_+ \in \mathsf{H}^k(\Omega_+)^{N_+}, \ \mathbf{u}_- \in \mathsf{H}^k(\Omega_-)^{N_-}\big\}.$$

All examples of transmission problems that we will consider originate from variational formulations. The general framework is, like in Chapter 3, to choose a sesquilinear form a and a variational space $\mathbf{V}$. In a two-domain configuration $\Omega_\pm$, a takes the form

$$\mathsf{a}(\mathbf{u}, \mathbf{v}) = \mathsf{a}_+(\mathbf{u}_+, \mathbf{v}_+) + \mathsf{a}_-(\mathbf{u}_-, \mathbf{v}_-),$$

where $\mathsf{a}_\pm$ is defined on $\Omega_\pm$ like in (3.a), and $\mathbf{V}$ is a subspace of $\mathsf{P}\mathbf{H}^1(\Omega)$ defined by essential boundary and interface conditions on Γ and I, respectively. The associated variational formulation is still written as

$$\textit{Find} \quad \mathbf{u} \in \mathbf{V} \quad \textit{such that} \quad \forall \mathbf{v} \in \mathbf{V}, \quad \mathsf{a}(\mathbf{u}, \mathbf{v}) = \big\langle \mathbf{q}, \mathbf{v} \big\rangle, \tag{5.c}$$

for any element $\mathbf{q} \in \mathbf{V}'$.

Let us give a generic example of definition for $\mathbf{V}$ in the case when the two dimensions $N_\pm$ coincide (we address a much more general case in Section 5.3). We set $N = N_+ = N_-$. For simplicity we consider

- Dirichlet conditions on the external boundary Γ
- Essential interface conditions defined by only one smooth mapping Π^D from the interface I into the space of orthogonal projection operators $\mathbb{C}^N \to \mathbb{C}^N$.

The space $\mathbf{V}$ is defined by

$$\mathbf{V} = \{\mathbf{u} \in \mathrm{P}\mathbf{H}^1(\Omega) : \ \gamma_0\mathbf{u} = 0 \ \text{ on } \ \Gamma \quad \text{and} \quad \Pi^D\gamma_+\mathbf{u}_+ = \Pi^D\gamma_-\mathbf{u}_- \ \text{ on } \ \mathrm{I}\}\,. \tag{5.d}$$

Here $\gamma_\pm$ denote the trace operators from $\mathrm{H}^1(\Omega_\pm)$ to $\mathrm{H}^{\frac12}(\mathrm{I})$.

Lemma 5.B *In the framework above, let $\mathbf{f}_\pm$ be given in $\mathrm{L}^2(\Omega_\pm)^N$, $\mathbf{g}_\mathrm{I}$ and $\mathbf{g}_{\mathrm{I},\pm}$ be given in $\mathrm{H}^{-\frac12}(\mathrm{I})^N$. The projector field Π^T is defined as $\mathbb{I} - \Pi^D$. The expression*

$$\begin{aligned}\langle \mathbf{q}, \mathbf{v}\rangle = \langle \mathbf{f}_+, \mathbf{v}_+\rangle_{\Omega_+} + \langle \mathbf{f}_-, \mathbf{v}_-\rangle_{\Omega_-} &+ \\ \langle \mathbf{g}_{\mathrm{I},+}, \Pi^T\gamma_+\mathbf{v}_+\rangle_\mathrm{I} + \langle \mathbf{g}_{\mathrm{I},-}, \Pi^T\gamma_-\mathbf{v}_-\rangle_\mathrm{I} &+ \langle \mathbf{g}_\mathrm{I}, \Pi^D\gamma_-\mathbf{v}_-\rangle_\mathrm{I}\end{aligned} \tag{5.e}$$

defines an element $\mathbf{q}$ of $\mathbf{V}'$. If $\mathbf{u}$ is a solution of the variational problem (5.c), then $\mathbf{u}$ solves the boundary value problem

$$\left\{\begin{array}{rcll} L_\pm\mathbf{u}_\pm &=& \mathbf{f}_\pm & \text{in } \ \Omega_\pm \\ \gamma_0\mathbf{u}_- &=& \mathbf{0} & \text{on } \ \Gamma, \\ \Pi^D\gamma_+\mathbf{u}_+ - \Pi^D\gamma_-\mathbf{u}_- &=& \mathbf{0} & \text{on } \ \mathrm{I}, \\ \Pi^T B_+\mathbf{u}_+ &=& \Pi^T\mathbf{g}_{\mathrm{I},+} & \text{on } \ \mathrm{I}, \\ \Pi^T B_-\mathbf{u}_- &=& \Pi^T\mathbf{g}_{\mathrm{I},-} & \text{on } \ \mathrm{I}, \\ \Pi^D B_+\mathbf{u}_+ + \Pi^D B_-\mathbf{u}_- &=& \Pi^D\mathbf{g}_\mathrm{I} & \text{on } \ \mathrm{I}. \end{array}\right. \tag{5.f}$$

Here $B_\pm$ are the conormal operators associated with the forms $\mathsf{a}_\pm$ and the unit outward normal fields $\mathbf{n}_\pm(\mathbf{x})$ on the interface I, cf. (3.g) *and* (5.10).

The above lemma is the analogue of Lemma 3.A. The four conditions on I are the interface conditions, the first and last one being transmission conditions, the other two ones being internal boundary conditions. If N_0 denotes the rank of Π^D, we see that we have an essential interface condition of rank N_0, two independent natural internal boundary conditions of rank $N_1 := N - N_0$ and one natural transmission condition of rank N_0.

Among the three interface terms in the right hand side of (5.e), we note that the first two terms are on one side of the interface, whereas the third one lies equally on both sides since for any element $\mathbf{v}$ of $\mathbf{V}$

$$\langle \mathbf{g}_\mathrm{I}, \Pi^D\gamma_-\mathbf{v}_-\rangle_\mathrm{I} = \langle \mathbf{g}_\mathrm{I}, \Pi^D\gamma_+\mathbf{v}_+\rangle_\mathrm{I}\,.$$

Like in Chapter 3, we can prove various regularity shift results when the form a is coercive on $\mathbf{V}$, see Theorems 5.3.7 and 5.3.8.

5.1 Complementing interface conditions

5.1.a Model case

Let us explain the setting we have in mind in the simple case of constant coefficients operators defined on two half-spaces sharing one interface. For consistency with Chapters 1 and 2 we take as model domain with $n \geq 2$

$$\Omega = \mathbb{T}^{n-1} \times \mathbb{R}$$

"divided" into the two periodic half-spaces

$$\mathbb{T}^n_+ = \{\mathbf{x} = (\mathbf{x}', t) : \mathbf{x}' \in \mathbb{T}^{n-1},\ t > 0\} \text{ and } \mathbb{T}^n_- = \{\mathbf{x} = (\mathbf{x}', t) : \mathbf{x}' \in \mathbb{T}^{n-1},\ t < 0\}.$$

The interface is $\partial\mathbb{T}^n_+ \cap \partial\mathbb{T}^n_-$ that coincides with the boundary of the periodic half-space Γ from subsection 2.2.a. Here we prefer to denote it by I.

$$\partial\mathbb{T}^n_+ \cap \partial\mathbb{T}^n_- = \mathrm{I} = \{\mathbf{x} = (\mathbf{x}', 0) : \quad \mathbf{x}' \in \mathbb{T}^{n-1}\}.$$

On $\mathbb{T}^n_+$ (resp. $\mathbb{T}^n_-$), we assume given a $N_+ \times N_+$ (resp. $N_- \times N_-$) system of second order operators L_+ (resp. L_-) elliptic with constant coefficients and without lower order terms. On the interface I, we consider homogeneous $1 \times N_\pm$ systems of operators $C_{k,\pm}, k = 1, \ldots, N_+ + N_-$ with constant coefficients and orders m_k equal to 0 or 1 of the form

$$C_{k,\pm}\mathbf{u} = \sum_{j=1}^{N_\pm} \sum_{|\alpha|=m_k} b^\alpha_{k,\pm,j}\, \partial^\alpha_{\mathbf{x}} u_j, \quad b^\alpha_{k,\pm,j} \in \mathbb{C}.$$

Our model interface problem is the system

$$\left\{ \begin{array}{rcll} L_+\mathbf{u}_+ &=& \mathbf{f}_+ & \text{in } \mathbb{T}^n_+, \\ L_-\mathbf{u}_- &=& \mathbf{f}_- & \text{in } \mathbb{T}^n_-, \\ C_{k,+}\mathbf{u}_+ + C_{k,-}\mathbf{u}_- &=& b_k & \text{on } \mathrm{I}, \quad \forall k = 1, \ldots, N_+ + N_-. \end{array} \right. \tag{5.1}$$

To define *covering interface* conditions, the main idea is to transform this system into an equivalent larger *boundary value* problem, set on the half-space $\mathbb{T}^n_+$. Namely define

$$\forall \mathbf{x}' \in \mathbb{T}^{n-1},\ t > 0, \quad \mathbf{u}^\sharp(\mathbf{x}', t) = \begin{pmatrix} \mathbf{u}_+(\mathbf{x}', t) \\ \mathbf{u}_-(\mathbf{x}', -t) \end{pmatrix}.$$

Set (with $\mathrm{D}_{\mathbf{x}'} = (-i\partial_1, \ldots, -i\partial_{n-1})$ and $\mathrm{D}_t = -i\partial_t$)

$$L^\sharp(\mathrm{D}_{\mathbf{x}}) = \begin{pmatrix} L_+(\mathrm{D}_{\mathbf{x}'}, \mathrm{D}_t) & 0 \\ 0 & L_-(\mathrm{D}_{\mathbf{x}'}, -\mathrm{D}_t) \end{pmatrix},$$

and

$$C^\sharp_k = \big(C_{k,+}(\mathrm{D}_{\mathbf{x}'}, \mathrm{D}_t) \quad C_{k,-}(\mathrm{D}_{\mathbf{x}'}, -\mathrm{D}_t)\big).$$

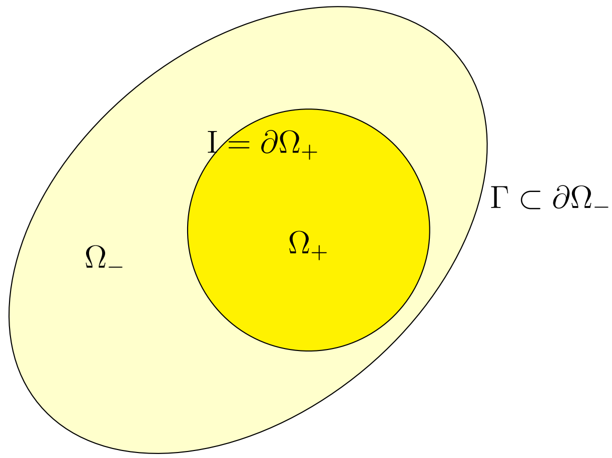


Figure 5.1: Typical smooth interface configuration with two subdomains.

Then we immediately see that (5.1) is equivalent to

$$\begin{cases} L^\sharp \mathbf{u}^\sharp = \mathbf{f}^\sharp & \text{in } \mathbb{T}^n_+, \\ C^\sharp_k \mathbf{u}^\sharp = b_k & \text{on } \Gamma, \quad \forall k = 1, \ldots, N_+ + N_-. \end{cases} \tag{5.2}$$

Note that the ellipticity assumption on $\{L_+, L_-\}$ is equivalent to the ellipticity of the $(N_+ + N_-) \times (N_+ + N_-)$ system $L^\sharp$. Concerning the covering condition we define:

Definition 5.1.1 *Under the above setting, the set $C = \{C_{k,+}, C_{k,-}, k = 1, \ldots, N_+ + N_-\}$ of interface conditions is said to* **cover** *$\{L_+, L_-\}$ over the interface I if the equivalent set of boundary conditions $C^\sharp = \{C^\sharp_k, k = 1, \ldots, N_+ + N_-\}$ covers the system $L^\sharp$ on Γ (cf. Definition 2.2.6).*

In fact, a direct definition in the spirit of Definition 2.2.6 provides an equivalent condition: It is possible to consider the space $\mathfrak{M}_+[L_+, L_-; \boldsymbol{\xi}']$ of stable solutions over $\mathbb{R}_+ \cup \mathbb{R}_-$, and the condition of invertibility of the interface symbol $B(\boldsymbol{\xi}')$ from this space onto $\mathbb{C}^{N_+ + N_-}$.

5.1.b Smooth case

For a bounded smooth domain Ω, we suppose that it is divided into two smooth subdomains Ω_+, Ω_- such that $\overline{\Omega}_+ \subset \Omega$ (interior subdomain) and $\Omega_- = \Omega \setminus \overline{\Omega}_+$ (exterior subdomain) (see Figure 5.1). The interface (or transmission surface) is $\partial\Omega_+ = \partial\Omega_+ \cap \partial\Omega_-$ which we denote by I. For the sake of simplicity, we only consider here the case of two sub-domains, the extension to more general configurations of several subdomains with smooth interfaces is straightforward, see Figure 5.2.

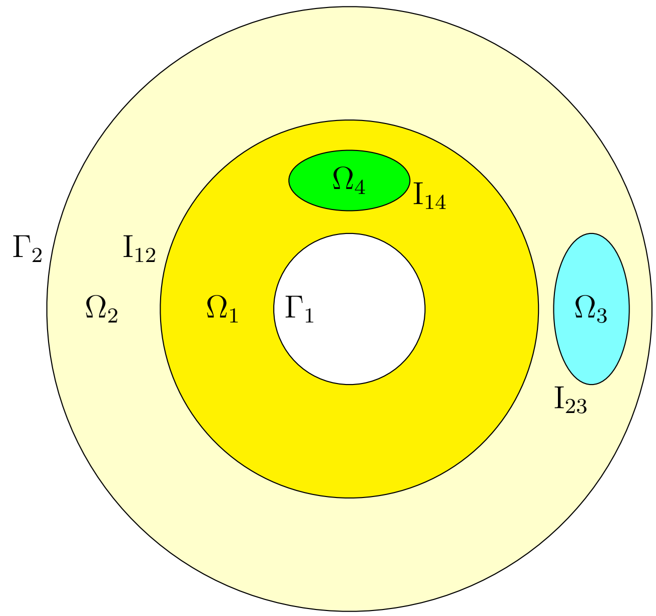


Figure 5.2: Smooth interface configuration with 4 subdomains and 3 interfaces.

On $\Omega_\pm$, we assume given a $N_\pm \times N_\pm$ system of second order operators $L_\pm$ elliptic in $\overline{\Omega}_\pm$. On the interface I, we consider $1 \times N_\pm$ systems of operators $C_{k,\pm}$ of order $m_{k,\pm}$ equal to 0 or 1, for all $k = 1, \dots, N_+ + N_-$. Finally on the exterior boundary $\partial\Omega$ denoted, later on, by Γ, we fix $1 \times N_-$ systems of operators C_k of orders m_k equal to 0 or 1, for all $k = 1, \dots, N_-$.

The transmission problem we have in mind is then

$$\left\{ \begin{array}{rcll} L_\pm \mathbf{u}_\pm &=& \mathbf{f}_\pm & \text{in } \Omega_\pm, \\ C_k \mathbf{u}_- &=& b_{\Gamma,k} & \text{on } \Gamma, \quad \forall k = 1, \dots, N_-, \\ C_{k,+}\mathbf{u}_+ + C_{k,-}\mathbf{u}_- &=& b_{\mathrm{I},k} & \text{on } \mathrm{I}, \quad \forall k = 1, \dots, N_+ + N_-. \end{array} \right. \tag{5.3}$$

The above boundary and transmission conditions can be shortly written as follows:

$$\begin{array}{rcll} C_\Gamma \mathbf{u}_- &=& \mathbf{b}_\Gamma & \text{on } \Gamma, \\ C_{\mathrm{I}}(\mathbf{u}_+, \mathbf{u}_-)^\top &=& \mathbf{b}_{\mathrm{I}} & \text{on } \mathrm{I}, \end{array}$$

where $C_\Gamma = (C_k)_{k=1,\dots,N_-}$ is the $N_- \times N_-$ system of operators whose lines are C_k; $C_{\mathrm{I}} = (C_{k,+} \quad C_{k,-})_{k=1,\dots,N_++N_-}$ is the $(N_+ + N_-) \times (N_+ + N_-)$ system of operators whose lines are $C_{k,+} \quad C_{k,-}$; the right-hand side being defined accordingly.

The *covering* condition at a point of the interface I is now defined in a standard way as in Section 2.2.c by introducing the corresponding tangent interface problem on $\mathbb{T}^n_+ \cup \mathbb{T}^n_-$. Let $\mathbf{x}_0 \in \mathrm{I}$ be a point on the interface. There exists a smooth local map $\phi_{\mathbf{x}_0}$ which

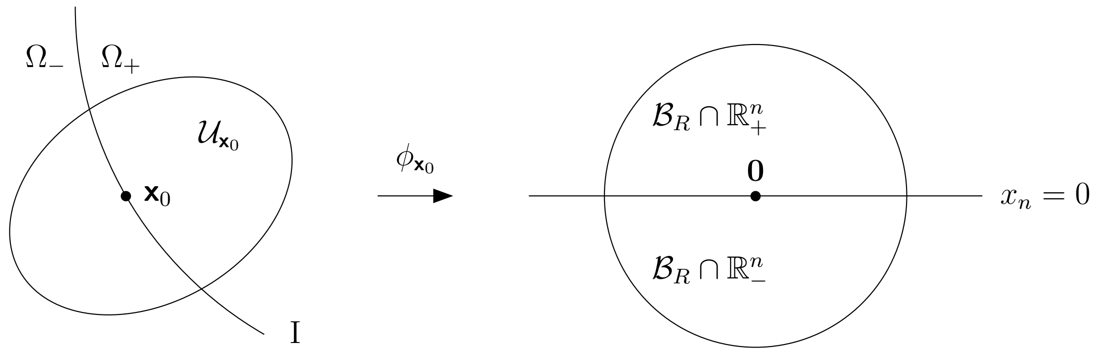


Figure 5.3: Local diffeomorphism $\phi_{\mathbf{x}_0} : \mathcal{U}_{\mathbf{x}_0} \to \mathcal{B}_R$.

transforms a neighborhood $\mathcal{U}$ of $\mathbf{x}_0$ in $\mathbb{R}^n$ into a ball $\mathcal{B}_R$ centered at $\mathbf{0}$ in $\mathbb{R}^n$, so that

$$\phi_{\mathbf{x}_0}(\Omega_+ \cap \mathcal{U}) = \mathcal{B}_R \cap \mathbb{R}^n_+, \quad \phi_{\mathbf{x}_0}(\Omega_- \cap \mathcal{U}) = \mathcal{B}_R \cap \mathbb{R}^n_-.$$

This map allows to transform locally the problem (5.3) into a problem of the form (5.1), except that the operators have variable coefficients and lower order terms. Taking the principal part and freezing the coefficients at $\mathbf{0}$, we obtain the system

$$\begin{cases} \underline{L}_{\mathbf{x}_0,+}\mathbf{u}_+ & = & \mathbf{f}_+ & \text{in } \mathbb{T}^n_+, \\ \underline{L}_{\mathbf{x}_0,-}\mathbf{u}_- & = & \mathbf{f}_- & \text{in } \mathbb{T}^n_-, \\ \underline{C}_{k,\mathbf{x}_0,+}\mathbf{u}_+ + \underline{C}_{k,\mathbf{x}_0,-}\mathbf{u}_- & = & g_k & \text{on } \mathrm{I}, \quad \forall k = 1, \dots, N_+ + N_-. \end{cases} \tag{5.4}$$

Definition 5.1.2 *Consider the transmission problem (5.3).*

- ⋆ *Let $\mathbf{x}_0 \in \mathrm{I}$. The set of interface operators C_I is said to cover the system $\{L_+, L_-\}$ at $\mathbf{x}_0$ if the system $\underline{C}_{\mathbf{x}_0}$ covers $\{\underline{L}_{\mathbf{x}_0,+}, \underline{L}_{\mathbf{x}_0,-}\}$ in the sense of Definition 5.1.1. The system $\{L_+, L_-, C_\mathrm{I}\}$ is then called elliptic at $\mathbf{x}_0$.*
- ⋆ *The set of interface operators C_I is said to cover (or complement) $\{L_+, L_-\}$ on I if C_I covers $\{L_+, L_-\}$ at any point of I.*
- ⋆ *If C_Γ covers L_- on Γ, and C_I covers $\{L_+, L_-\}$ on I, the system $\{L_+, L_-, C_\mathrm{I}, C_\Gamma\}$ is called elliptic on $\overline{\Omega}_+ \cup \overline{\Omega}_-$, and the problem (5.3) is called an **elliptic transmission problem**.*

5.2 Regularity through the interface and to the boundary

As the ellipticity condition at a point of the interface is reduced to the ellipticity at the boundary of a larger system, all desired a priori estimates can be proved using those from

Chapters 1 and 2. The only difference is that we need to use the following piecewise Sobolev spaces: for $k \in \mathbb{N}$, we set

$$\mathsf{P}\mathbf{H}^k(\Omega) = \big\{\mathbf{u} = (\mathbf{u}_+, \mathbf{u}_-) : \ \mathbf{u}_+ \in \mathsf{H}^k(\Omega_+)^{N_+}, \ \mathbf{u}_- \in \mathsf{H}^k(\Omega_-)^{N_-}\big\},$$

equipped with the natural norm:

$$\|\mathbf{u}\|^2_{\mathsf{P}\,k;\,\Omega} = \|\mathbf{u}_+\|^2_{k;\,\Omega_+} + \|\mathbf{u}_-\|^2_{k;\,\Omega_-}.$$

Like for boundary operators, among the interface operators $C_{k,\pm}$, $k = 1, \dots, N_+ + N_-$, we distinguish the operators of order 1, which we denote by $T_{k,\pm}$, $k = 1, \dots, N_3$, and the operators of order 0, written $D_{k,\pm}$, $k = 1, \dots, N_2 = N_+ + N_- - N_3$.

Finally we define the operator $\mathbb{A}$

$$\begin{aligned} \mathbb{A} \ \ : \ \ & \mathsf{P}\mathbf{H}^k(\Omega) \to \mathsf{RP}\mathbf{H}^k(\Omega) \\ : \ \ & \mathbf{u} \to (L_+\mathbf{u}_+, L_-\mathbf{u}_-, T\mathbf{u}_-, D\mathbf{u}_-, T_+\mathbf{u}_+ + T_-\mathbf{u}_-, D_+\mathbf{u}_+ + D_-\mathbf{u}_-), \end{aligned} \tag{5.5}$$

where in a natural way we have set (compare with Notation 2.2.28)

$$\begin{aligned} \mathsf{RP}\mathbf{H}^k(\Omega) = \mathsf{H}^{k-2}(\Omega_+)^{N_+} \times \mathsf{H}^{k-2}(\Omega_-)^{N_-} \\ \times \mathsf{H}^{k-\frac32}(\Gamma)^{N_1} \times \mathsf{H}^{k-\frac12}(\Gamma)^{N_0} \times \mathsf{H}^{k-\frac32}(\mathrm{I})^{N_3} \times \mathsf{H}^{k-\frac12}(\mathrm{I})^{N_2}. \end{aligned}$$

Here follows the statement corresponding to Theorem 2.3.2 for transmission problems on two subdomains:

$$\begin{cases} L_\pm \mathbf{u}_\pm &= \mathbf{f}_\pm \quad \text{in } \ \Omega_\pm, \\ T\mathbf{u}_- &= \mathbf{g} \quad \text{on } \ \Gamma, \\ D\mathbf{u}_- &= \mathbf{h} \quad \text{on } \ \Gamma, \\ T_+\mathbf{u}_+ + T_-\mathbf{u}_- &= \mathbf{g}_\mathrm{I} \quad \text{on } \ \mathrm{I}, \\ D_+\mathbf{u}_+ + D_-\mathbf{u}_- &= \mathbf{h}_\mathrm{I} \quad \text{on } \ \mathrm{I}, \end{cases} \tag{5.6}$$

when $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{g}_\mathrm{I}, \mathbf{h}_\mathrm{I})$ belongs to $\mathsf{RP}\mathbf{H}^k(\Omega)$, or to its local analogue. In the following, we denote as usual by boldface letters the spaces of vector functions, regardless of their dimension when there is no confusion possible

Theorem 5.2.1 *Let k be an integer $k \geq 2$. Let the following assumptions be satisfied:*

- ***a)*** *The domain $\Omega \subset \mathbb{R}^n$ has a $\mathscr{C}^k$ boundary and is divided into two sub-domains Ω_+, Ω_- with a $\mathscr{C}^k$ interface as explained above.*
- ***b)*** *The system $\{L_+, L_-, C_\mathrm{I}, C_\Gamma\}$ is elliptic in $\overline{\Omega}_+ \cup \overline{\Omega}_-$ in the sense of Definition 5.1.2, satisfying regularity assumptions like in Definition 2.3.1.*

Then we have the following local (i) and global (ii) regularity results and estimates:

(i) *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be bounded subdomains of $\mathbb{R}^n$ with $\overline{\mathcal{U}}_1 \subset \mathcal{U}_2$, and let $\Omega_1 = \mathcal{U}_1 \cap \Omega$ and $\Omega_2 = \mathcal{U}_2 \cap \Omega$. The subdivision of Ω induces natural subdivisions of Ω_1 and Ω_2 by $\Omega_{i,\pm} = \mathcal{U}_i \cap \Omega_\pm$. We set $\Gamma_2 = \partial\Omega_2 \cap \Gamma$ and $\mathrm{I}_2 = \partial\Omega_{2,+} \cap \partial\Omega_{2,-}$.*
If $\mathbf{u} \in \mathsf{P}\mathbf{H}^2(\Omega)$ is solution of (5.6) and if

$$\mathbf{f} \in \mathsf{P}\mathbf{H}^{k-2}(\Omega_2)\,, \quad \mathbf{g} \in \mathbf{H}^{k-\frac{3}{2}}(\Gamma_2)\,, \quad \mathbf{h} \in \mathbf{H}^{k-\frac{1}{2}}(\Gamma_2)\,, \\ \mathbf{g}_{\mathrm{I}} \in \mathbf{H}^{k-\frac{3}{2}}(\mathrm{I}_2)\,, \quad \mathbf{h}_{\mathrm{I}} \in \mathbf{H}^{k-\frac{1}{2}}(\mathrm{I}_2)\,,$$

then $\mathbf{u} \in \mathsf{P}\mathbf{H}^k(\Omega_1)$, with the estimate

$$\|\mathbf{u}\|_{\mathsf{P}\,k;\,\Omega_1} \le c\Big(\|\mathbf{f}\|_{\mathsf{P}\,k-2;\,\Omega_2} + \|\mathbf{g}\|_{k-\frac{3}{2};\,\Gamma_2} + \|\mathbf{h}\|_{k-\frac{1}{2};\,\Gamma_2} \\ + \|\mathbf{g}_{\mathrm{I}}\|_{k-\frac{3}{2};\,\mathrm{I}_2} + \|\mathbf{h}_{\mathrm{I}}\|_{k-\frac{1}{2};\,\mathrm{I}_2} + \|\mathbf{u}\|_{\mathsf{P}\,1;\,\Omega_2}\Big).$$

(ii) *The operator $\mathbb{A}$ defined by (5.5) is Fredholm. If moreover the right hand side $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{g}_{\mathrm{I}}, \mathbf{h}_{\mathrm{I}})$ is in $\mathsf{RP}\mathbf{H}^k(\Omega)$, then any solution $\mathbf{u} \in \mathsf{P}\mathbf{H}^2(\Omega)$ of (5.6) belongs to $\mathsf{P}\mathbf{H}^k(\Omega)$ with the estimate*

$$\|\mathbf{u}\|_{\mathsf{P}\,k;\,\Omega} \le c\Big(\|\mathbf{f}\|_{\mathsf{P}\,k-2;\,\Omega} + \|\mathbf{g}\|_{k-\frac{3}{2};\,\Gamma} + \|\mathbf{h}\|_{k-\frac{1}{2};\,\Gamma} \\ + \|\mathbf{g}_{\mathrm{I}}\|_{k-\frac{3}{2};\,\mathrm{I}} + \|\mathbf{h}_{\mathrm{I}}\|_{k-\frac{1}{2};\,\mathrm{I}} + \|\mathbf{u}\|_{\mathsf{P}\,1;\,\Omega}\Big).$$

The analytic version of Theorem 5.2.1 can be stated as follows, cf. Theorem 2.7.1.

Theorem 5.2.2 *Let the assumptions of Theorem 5.2.1 be satisfied. Assume furthermore that*

a) *the boundaries of Ω and of Ω_+ are analytic,*

b) *the coefficients of $L_\pm$ (resp. T and D; $T_\pm$ and $D_\pm$) are analytic on $\overline{\Omega}_\pm$ (resp. on Γ; on I).*

Then there holds

(i) *Under the assumptions of item (i) of Theorem 5.2.1, any solution $\mathbf{u} \in \mathsf{P}\mathbf{H}^2(\Omega)$ of (5.6) satisfies the following analytic type estimates: There exists $A > 0$ independent of k such that*

$$\frac{1}{k!}\,|\mathbf{u}|_{\mathsf{P}\,k;\,\Omega_1} \le A^{k+1}\Big\{\sum_{\ell=0}^{k-2}\frac{1}{\ell!}\Big(|\mathbf{f}|_{\mathsf{P}\,\ell;\,\Omega_2} + \|\mathbf{g}\|_{\ell+\frac{1}{2};\,\Gamma_2} + \|\mathbf{h}\|_{\ell+\frac{3}{2};\,\Gamma_2} \\ + \|\mathbf{g}_{\mathrm{I}}\|_{\ell+\frac{1}{2};\,\mathrm{I}_2} + \|\mathbf{h}_{\mathrm{I}}\|_{\ell+\frac{3}{2};\,\mathrm{I}_2}\Big) + \sum_{\ell=0}^{1}|\mathbf{u}|_{\mathsf{P}\,\ell;\,\Omega_2}\Big\}. \quad (5.7)$$

(ii) *If the right hand side $(\mathbf{f}_+, \mathbf{f}_-, \mathbf{g}, \mathbf{h}, \mathbf{g}_{\mathrm{I}}, \mathbf{h}_{\mathrm{I}})$ belongs to $\mathbf{A}(\Omega_+) \times \mathbf{A}(\Omega_-) \times \mathbf{A}(\Gamma) \times \mathbf{A}(\Gamma) \times \mathbf{A}(\mathrm{I}) \times \mathbf{A}(\mathrm{I})$, then any solution $\mathbf{u} \in \mathsf{P}\mathbf{H}^2(\Omega)$ of problem (5.6) belongs to $\mathsf{P}\mathbf{A}(\Omega)$.*

5.3 Variational formulations

Like in the case of standard boundary value problems, many elliptic transmission problems occurring in modelization admit a coercive variational formulation. This guarantees the existence of a (weak) solution provided a finite number of compatibility conditions are satisfied. Moreover, if the right hand side has the required regularity, variational solutions are piecewise H^2.

For a bounded domain Ω subdivided into Ω_+ and Ω_- as before, we suppose given two positive natural numbers N_+ and N_-, that will correspond to the size of our system in each sub-domain. Now we consider on $\mathsf{P}\mathbf{H}^1(\Omega)$, the sesquilinear form

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \mathsf{a}_+(\mathbf{u}_+,\mathbf{v}_+) + \mathsf{a}_-(\mathbf{u}_-,\mathbf{v}_-), \tag{5.8a}$$

where $\mathsf{a}_\pm$ is defined on $\mathbf{H}^1(\Omega_\pm)$ as (3.1)

$$\mathsf{a}_\pm(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{N_\pm}\sum_{j=1}^{N_\pm}\sum_{|\alpha|\le 1}\sum_{|\gamma|\le 1}\int_{\Omega_\pm} a_{ij,\pm}^{\alpha\gamma}(\mathbf{x})\,\partial_{\mathbf{x}}^{\alpha}u_j(\mathbf{x})\,\partial_{\mathbf{x}}^{\gamma}\overline{v}_i(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{5.8b}$$

with complex valued coefficients $a_{ij,\pm}^{\alpha\gamma}$, smooth up to the boundary of $\Omega_\pm$.

From this expression, we see that the second order $N_\pm \times N_\pm$ system $L_\pm$ defined on $\Omega_\pm$ associated with a is given by (3.3) where the coefficients $a_{ij}^{\alpha\gamma}$ are simply replaced by $a_{ij,\pm}^{\alpha\gamma}$.

As in Chapter 3, we now take the (compare with Definition 3.1.1)

Definition 5.3.1 *A **variational space** $\mathbf{V}$ for the sesquilinear form a defined by (5.8) is a closed subspace of $\mathsf{P}\mathbf{H}^1(\Omega)$ which contains $\mathbf{H}_0^1(\Omega_+) \times \mathbf{H}_0^1(\Omega_-)$.*

Here, due to the jump of the coefficients through the interface, we have more flexibilities for a choice of a variational space for the form a. Obviously, the choice of full Dirichlet conditions

$$\mathbf{u}_+ = 0 \quad \text{and} \quad \mathbf{u}_- = 0 \quad \text{on} \quad \mathrm{I}$$

does not bring anything but two decoupled boundary value problems in Ω_+ and Ω_-. The same occurs if we take no condition at all on the interface: We then have two uncoupled Neumann conditions on the interface.

This means that the most interesting spaces are spaces with some pointwise restrictions on the interface I. If $N_+ = N_-$, one classical choice is $\mathbf{V} = \mathbf{H}_0^1(\Omega)$, which furnishes *the Dirichlet transmission problem* (see also section 5.4)

$$\begin{cases} L_\pm \mathbf{u}_\pm &= \mathbf{f}_\pm & \text{in } \Omega_\pm, \\ \mathbf{u}_- &= \mathbf{0} & \text{on } \Gamma, \\ \mathbf{u}_+ - \mathbf{u}_- &= \mathbf{0} & \text{on } \mathrm{I}, \\ B_+\mathbf{u}_+ + B_-\mathbf{u}_- &= \mathbf{0} & \text{on } \mathrm{I}, \end{cases} \tag{5.9}$$

where the system $B_\pm$ is the conormal system given by

$$B_\pm = \big(B_{ij,\pm}\big)_{1\le i,j\le N_\pm} \quad \text{with} \quad B_{ij,\pm} = \sum_{|\alpha|\le 1}\sum_{|\gamma|=1} n_\pm^\gamma(\mathbf{x}) a_{ij,\pm}^{\alpha\gamma}(\mathbf{x})\,\partial_{\mathbf{x}}^\alpha, \quad \mathbf{x}\in\partial\Omega_\pm. \tag{5.10}$$

Here, for $\gamma = (\gamma_1,\ldots,\gamma_n)$ with $\gamma_k = \delta_{k\ell}$, $n_\pm^\gamma(\mathbf{x})$ is the component ℓ of the unit outward normal $\mathbf{n}_\pm(\mathbf{x})$ to $\partial\Omega_\pm$ at the point $\mathbf{x}$.

Of course more general choices are possible: A canonical choice related to the framework of Chapter 3 is to suppose that we are given

- a smooth mapping Π^D from the boundary Γ to the space of orthogonal projection operators $\mathbb{C}^{N_-}\to\mathbb{C}^{N_-}$,
- two smooth mappings $\Pi^{D,\pm}$ from the interface I to the space of orthogonal projection operators $\mathbb{C}^{N_\pm}\to\mathbb{C}^{N_\pm}$

$$\Pi^D : \Gamma\ni\mathbf{x}\longmapsto\Pi^D(\mathbf{x}) \quad \text{and} \quad \Pi^{D,\pm} : \partial\Omega_\pm\ni\mathbf{x}\longmapsto\Pi^{D,\pm}(\mathbf{x}). \tag{5.11}$$

As in Chapter 3 we set

$$\Pi^{T,\pm} = \mathbb{I} - \Pi^{D,\pm}.$$

Let us assume that $N_+\le N_-$ (the converse situation is treated in a similar manner). We further assume given a smooth mapping R_- from the interface I to the space of linear operators $\mathbb{C}^{N_-}\to\mathbb{C}^{N_+}$

$$R_- : \mathrm{I}\ni\mathbf{x}\longmapsto R_-(\mathbf{x}). \tag{5.12}$$

Let γ_0 and $\gamma_\pm$ be the trace operators

$$\gamma_0 : H^1(\Omega_-)\to H^{\frac12}(\Gamma) \quad \text{and} \quad \gamma_\pm : H^1(\Omega_\pm)\to H^{\frac12}(\mathrm{I}). \tag{5.13}$$

Definition 5.3.2 *Let Π^D, $\Pi^{D,\pm}$ and R_- be fixed as before. Then the associated variational space $\mathbf{V}$ is the subspace of $\mathsf{P}\mathbf{H}^1(\Omega)$ defined as*

$$\begin{aligned}\mathbf{V} = \big\{\mathbf{u}\in\mathsf{P}\mathbf{H}^1(\Omega) \ :\ & \Pi^D\gamma_0\mathbf{u} = 0 \quad \textit{on } \Gamma \qquad \textit{and} \\ & \Pi^{D,+}\gamma_+\mathbf{u}_+ - R_-\Pi^{D,-}\gamma_-\mathbf{u}_- = 0 \quad \textit{on } \mathrm{I}\big\}\end{aligned} \tag{5.14}$$

Note that in the case when $N_+ = N_-$, $\Pi^{D,\pm}\equiv\mathbb{I}$, and $R_- = \mathbb{I}$, the space $\mathbf{V}$ coincides with $\mathbf{H}_0^1(\Omega)$ if $\Pi^D\equiv\mathbb{I}$, and with $\mathbf{H}^1(\Omega)$ if $\Pi^D\equiv 0$.

In the above setting, we consider the following variational formulation

$$\begin{aligned}\forall\mathbf{v}\in\mathbf{V},\quad \mathsf{a}(\mathbf{u},\mathbf{v}) \ =\ & \int_\Omega \mathbf{f}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} + \int_\Gamma \mathbf{g}\cdot\Pi^T\gamma_0\bar{\mathbf{v}}\,\mathrm{d}\sigma \\ & + \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},+}\cdot\Pi^{T,+}\gamma_+\bar{\mathbf{v}}_+\,\mathrm{d}\sigma + \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},-}\cdot\gamma_-\bar{\mathbf{v}}_-\,\mathrm{d}\sigma,\end{aligned} \tag{5.15a}$$

with

$$\mathbf{f}_\pm\in\mathsf{L}^2(\Omega)^{N_\pm}, \quad \mathbf{g}\in\mathsf{H}^{\frac12}(\Gamma)^{N_-}, \quad \text{and} \quad \mathbf{g}_{\mathrm{I},\pm}\in\mathsf{H}^{\frac12}(\mathrm{I})^{N_\pm}. \tag{5.15b}$$

We give a distributional interpretation of the variational problem (5.15):

Lemma 5.3.3 *Let the sesquilinear form be defined by* (5.8) *on the space* $\mathbf{V}$ *from Definition* 5.3.2*. Let* $\mathbf{u} \in \mathbf{V}$ *be a solution of problem* (5.15)*. Then, with* $R_+ = R_-^\top$*,* $\mathbf{u}$ *satisfies*

$$\left\{\begin{array}{rcll} L_\pm \mathbf{u}_\pm &=& \mathbf{f}_\pm & \textit{in } \Omega_\pm, \\ \Pi^D \gamma_0 \mathbf{u}_- &=& \mathbf{0} & \textit{on } \Gamma, \\ \Pi^T B_- \mathbf{u}_- &=& \Pi^T \mathbf{g} & \textit{on } \Gamma, \\ \Pi^{D,+}\gamma_+ \mathbf{u}_+ - R_- \Pi^{D,-}\gamma_- \mathbf{u}_- &=& \mathbf{0} & \textit{on } \mathrm{I}, \\ R_+ \Pi^{D,+} B_+ \mathbf{u}_+ + \Pi^{D,-} B_- \mathbf{u}_- &=& \Pi^{D,-}\mathbf{g}_{\mathrm{I},-} & \textit{on } \mathrm{I}, \\ \Pi^{T,+} B_+ \mathbf{u}_+ &=& \Pi^{T,+}\mathbf{g}_{\mathrm{I},+} & \textit{on } \mathrm{I}, \\ \Pi^{T,-} B_- \mathbf{u}_- &=& \Pi^{T,-}\mathbf{g}_{\mathrm{I},-} & \textit{on } \mathrm{I}. \end{array}\right. \tag{5.16}$$

Remark 5.3.4 The last four equations are the interface conditions. The first two of them are transmission conditions while the last two are boundary conditions. Note also that problem (5.16) is a particular case of problem (5.6) with the dimension N_2 equal to the rank of $\Pi^{D,+}$. △

Proof: In (5.15) taking test functions $\mathbf{v} \in \mathscr{D}(\Omega_+)^{N_+}$ (extended by zero in Ω_-), we directly get

$$L_+ \mathbf{u}_+ = \mathbf{f}_+ \quad \text{in } \Omega_+,$$

in the distributional sense. Exchanging the role of $+$ and $-$, the similar identity is proved in Ω_-. This implies that $\mathbf{u}_\pm$ belongs to the maximal domain of $L_\pm$ (see (3.7)):

$$\mathbf{u}_\pm \in \mathbf{H}^1(\Omega_\pm; L_\pm).$$

At this stage we can apply Green's formula (3.9) in $\Omega_\pm$ and therefore (5.15) implies

$$\forall \mathbf{v} \in \mathbf{V}, \quad \langle B_+ \mathbf{u}_+, \mathbf{v}_+ \rangle_{\partial\Omega_+} + \langle B_- \mathbf{u}_-, \mathbf{v}_- \rangle_{\partial\Omega_-} = \int_\Gamma \mathbf{g} \cdot \Pi^T \gamma_0 \bar{\mathbf{v}} \, \mathrm{d}\sigma + \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},+} \cdot \Pi^{T,+} \gamma_+ \bar{\mathbf{v}}_+ \, \mathrm{d}\sigma + \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},-} \cdot \gamma_- \bar{\mathbf{v}}_- \, \mathrm{d}\sigma.$$

Taking test functions which are zero near the interface I and arguing as in the proof of Lemma 3.1.5, we obtain the requested natural boundary conditions on Γ.

Using these boundary conditions and inserting the operators $\Pi^{D,\pm}$ and $\Pi^{T,\pm}$, the above identity becomes

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{V}, \quad & \langle \Pi^{D,+} B_+ \mathbf{u}_+, \Pi^{D,+}\gamma_+ \mathbf{v}_+ \rangle_{\mathrm{I}} + \langle \Pi^{T,+} B_+ \mathbf{u}_+, \Pi^{T,+}\gamma_+ \mathbf{v}_+ \rangle_{\mathrm{I}} \\ & + \langle \Pi^{D,-} B_- \mathbf{u}_-, \Pi^{D,-}\gamma_- \mathbf{v}_- \rangle_{\mathrm{I}} + \langle \Pi^{T,-} B_- \mathbf{u}_-, \Pi^{T,-}\gamma_- \mathbf{v}_- \rangle_{\mathrm{I}} \\ & = \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},+} \cdot \Pi^{T,+}\gamma_+ \bar{\mathbf{v}}_+ \, \mathrm{d}\sigma + \int_{\mathrm{I}} \mathbf{g}_{\mathrm{I},-} \cdot \gamma_- \bar{\mathbf{v}}_- \, \mathrm{d}\sigma. \end{aligned}$$

Finally using in the first term the essential transmission condition

$$\Pi^{D,+}\gamma_+\mathbf{v}_+ = R_-\Pi^{D,-}\gamma_-\mathbf{v}_- \quad \text{on } \mathrm{I}$$

satisfied by $\mathbf{v}$, and using the relation $R_+ = R_-^\top$, we arrive at

$$\begin{aligned}\forall \mathbf{v}\in\mathbf{V},\quad & \big\langle R_+\Pi^{D,+}B_+\mathbf{u}_+ + \Pi^{D,-}B_-\mathbf{u}_-, \Pi^{D,-}\gamma_-\mathbf{v}_-\big\rangle_\mathrm{I} \\ & + \big\langle \Pi^{T,+}B_+\mathbf{u}_+, \Pi^{T,+}\gamma_+\mathbf{v}_+\big\rangle_\mathrm{I} + \big\langle \Pi^{T,-}B_-\mathbf{u}_-, \Pi^{T,-}\gamma_-\mathbf{v}_-\big\rangle_\mathrm{I} \\ & = \int_\mathrm{I} \mathbf{g}_{\mathrm{I},+}\cdot\Pi^{T,+}\gamma_+\bar{\mathbf{v}}_+\,\mathrm{d}\sigma + \int_\mathrm{I} \mathbf{g}_{\mathrm{I},-}\cdot\gamma_-\bar{\mathbf{v}}_-\,\mathrm{d}\sigma.\end{aligned}$$

Restricting this identity to those $\mathbf{v}\in\mathbf{V}$ such that $\mathbf{v}_- = 0$, we find

$$\big\langle \Pi^{T,+}B_+\mathbf{u}_+, \Pi^{T,+}\gamma_+\mathbf{v}_+\big\rangle_\mathrm{I} = \int_\mathrm{I} \mathbf{g}_{\mathrm{I},+}\cdot\Pi^{T,+}\gamma_+\bar{\mathbf{v}}_+\,\mathrm{d}\sigma,$$

for all $\mathbf{v}_+\in\mathbf{H}^1(\Omega_+)$ such that $\Pi^{D,+}\gamma_+\mathbf{v}_+ = 0$. Arguing like in the proof of Lemma 3.1.5, we prove the equation $\Pi^{T,+}B_+\mathbf{u}_+ = \Pi^{T,+}\mathbf{g}_{\mathrm{I},+}$ on I.

Thus we are left with the identity

$$\begin{aligned}\big\langle R_+\Pi^{D,+}B_+\mathbf{u}_+ + \Pi^{D,-}B_-\mathbf{u}_-, \Pi^{D,-}\gamma_-\mathbf{v}_-\big\rangle_\mathrm{I} + \big\langle \Pi^{T,-}B_-\mathbf{u}_-, \Pi^{T,-}\gamma_-\mathbf{v}_-\big\rangle_\mathrm{I} \\ = \int_\mathrm{I} \mathbf{g}_{\mathrm{I},-}\cdot\gamma_-\bar{\mathbf{v}}_-\,\mathrm{d}\sigma,\end{aligned}$$

which holds for all $\mathbf{v}_-\in\mathbf{H}^1(\Omega_-)$. This identity shows the remaining natural transmission conditions on I. □

The coercivity needs here to be adapted as follows:

Definition 5.3.5 *The form* a *is said to be* coercive *on* $\mathbf{V}$ *if there exist positive constants* c *and* C *such that*

$$\forall \mathbf{u}\in\mathbf{V},\quad \operatorname{Re}\mathsf{a}(\mathbf{u},\mathbf{u}) \ge c\|\mathbf{u}\|^2_{\mathrm{P}\,1;\Omega} - C\|\mathbf{u}\|^2_{\mathrm{P}\,0;\Omega}. \tag{5.17}$$

With this adapted notion of coercivity, the results of Theorems 3.2.6 and 3.3.1 remain valid (with the piecewise H^1 norm instead of the H^1 norm). Moreover the $\mathbf{V}$-coercivity of a implies the ellipticity of the associated transmission problem, namely we can prove as Theorem 3.2.6, the next result.

Theorem 5.3.6 *Let* a *be a coercive sesquilinear form* (5.8) *defined on a subspace* $\mathbf{V}$ *of* $\mathrm{P}\mathbf{H}^1(\Omega)$ *according to Definition* 5.3.2*. The coefficients of the form* $\mathsf{a}_\pm$ *are assumed to belong to* $\mathscr{C}^1(\overline{\Omega}_\pm)$*. If* a *is coercive on* $\mathbf{V}$ *in the sense of Definition* 5.3.5*, then the system* (5.16) *is an elliptic transmission problem.*

Using difference quotient techniques similarly as in Theorem 3.4.1, we can prove piecewise H^2 regularity of variational solutions:

Theorem 5.3.7 *Let* a *be a coercive sesquilinear form* (5.8) *defined on a subspace* V *of* $\mathsf{PH}^1(\Omega)$ *according to Definition* 5.3.2. *The coefficients of the form* $\mathsf{a}_\pm$ *are assumed to belong to* $\mathscr{C}^1(\overline{\Omega}_\pm)$. *Let* $\mathbf{u} \in \mathsf{V}$ *be a variational solution of problem* (5.15). *Then* $\mathbf{u}$ *belongs to* $\mathsf{PH}^2(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{\mathsf{P}\,2;\,\Omega} \le C\Big(\|\mathbf{f}\|_{\mathsf{P}\,0;\,\Omega} + \|\mathbf{g}\|_{\frac12;\,\Gamma} + \|\mathbf{g}_{\mathrm{I},+}\|_{\frac12;\,\mathrm{I}} + \|\mathbf{g}_{\mathrm{I},-}\|_{\frac12;\,\mathrm{I}} + \|\mathbf{u}\|_{\mathsf{P}\,1;\,\Omega}\Big). \tag{5.18}$$

Combining this result with Theorems 5.2.1 or 5.2.2, we get the following shift results.

Theorem 5.3.8 *Let* a *be a coercive sesquilinear form* (5.8) *defined on a subspace* V *of* $\mathsf{PH}^1(\Omega)$ *according to Definition* 5.3.2. *Let* $\mathbf{u} \in \mathsf{V}$ *be a variational solution of problem* (5.15).

(i) Let k *be a non-negative integer. We assume that the boundary of* $\Omega_\pm$ *is of class* $\mathscr{C}^{k+2}$ *and that the coefficients of the form* $\mathsf{a}_\pm$ *are in* $\mathscr{C}^{k+1}(\overline{\Omega}_\pm)$. *If* $\mathbf{f} \in \mathsf{PH}^k(\Omega)$, $\mathbf{g} \in \mathsf{H}^{k+\frac12}(\Gamma)$, $\mathbf{g}_{\mathrm{I},+} \in \mathsf{H}^{k+\frac12}(\mathrm{I})$ *and* $\mathbf{g}_{\mathrm{I},-} \in \mathsf{H}^{k+\frac12}(\mathrm{I})$, *then* $\mathbf{u} \in \mathsf{PH}^{k+2}(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{\mathsf{P}\,k+2;\,\Omega} \le C\Big(\|\mathbf{f}\|_{\mathsf{P}\,k;\,\Omega} + \|\mathbf{g}\|_{k+\frac12;\,\Gamma} + \|\mathbf{g}_{\mathrm{I},+}\|_{k+\frac12;\,\mathrm{I}} + \|\mathbf{g}_{\mathrm{I},-}\|_{k+\frac12;\,\mathrm{I}} + \|\mathbf{u}\|_{\mathsf{P}\,1;\,\Omega}\Big). \tag{5.19}$$

(ii) We assume that $\Omega_\pm$ *is analytic and that the coefficients of the form* $\mathsf{a}_\pm$ *are analytic up to boundary of* $\Omega_\pm$. *If* $\mathbf{f} \in \mathsf{PA}(\Omega)$, $\mathbf{g} \in \mathsf{A}(\Gamma)$, $\mathbf{g}_{\mathrm{I},+} \in \mathsf{A}(\mathrm{I})$ *and* $\mathbf{g}_{\mathrm{I},-} \in \mathsf{A}(\mathrm{I})$, *then* $\mathbf{u}$ *belongs to* $\mathsf{PA}(\Omega)$.

5.4 Examples

5.4.a Piecewise smooth right hand sides

An interesting application of our analysis of elliptic transmission problems is the case of a standard boundary value problem where the coefficients are smooth on the whole domain but the right hand side $\mathbf{f}$ is only piecewise smooth. If the subdomains on which $\mathbf{f}$ is smooth have a regular boundary, we are in the setting of Sections 5.1 and 5.2. It suffices to take $N_+ = N_-$ and as interface conditions the equality of traces of the solution and its normal derivative on both sides of the interface.

We can then apply Theorems 5.2.1 and 5.2.2 to get regularity results in piecewise Sobolev or analytic spaces provided the data are in the corresponding piecewise Sobolev or analytic spaces.

In the remainder of this section, we will illustrate our general setting for some particular systems. For shortness, we mainly give the systems and the transmission conditions (since these are our main concerns here). All chosen examples enter into the variational setting and therefore we sometimes only give the essential transmission conditions.

5.4.b Scalar operators with piecewise smooth coefficients

The prototype of scalar transmission problem is often written in a simplified way in divergence form

$$\mathrm{div}(a\nabla u) = f$$

with a piecewise constant coefficient a which is equal to $a_\pm$ on $\Omega_\pm$. This equation has to be interpreted in the distributional sense, which means that it is associated with the sesquilinear form

$$\mathsf{a}(u,v) = \int_{\Omega_+} a_+ \nabla u \cdot \nabla \overline{v}\,\mathrm{d}\mathbf{x} + \int_{\Omega_-} a_- \nabla u \cdot \nabla \overline{v}\,\mathrm{d}\mathbf{x}. \tag{5.20}$$

The associated transmission problem is (here we omit external boundary conditions on Γ)

$$\begin{cases} a_\pm \Delta u_\pm &= f_\pm & \text{in } \Omega_\pm, \\ u_+ - u_- &= 0 & \text{on } \mathrm{I}, \\ a_+\partial_{n_+} u_+ + a_-\partial_{n_-} u_- &= 0 & \text{on } \mathrm{I}. \end{cases} \tag{5.21}$$

Note that if we choose one normal field, e.g. we set $\mathbf{n} = \mathbf{n}_- = -\mathbf{n}_+$, the last transmission conditions becomes

$$a_+\partial_n u_+ - a_-\partial_n u_- = 0.$$

If $a_\pm$ are smooth functions inside $\Omega_\pm$, the form a is still defined by (5.20), the transmission conditions are the same as in (5.21) and the interior equations become

$$\mathrm{div}\, a_\pm \nabla u_\pm = f_\pm \quad \text{in} \quad \Omega_\pm\,.$$

The natural generalization of this is the case of second order scalar operators in divergence form with piecewise smooth coefficients in matrix form, cf. § 4.2: This means that we take

$$L_\pm = -\sum_{i,j=1}^{n} \partial_i (A_\pm^{ij} \partial_j)\,,$$

where $D_\pm = (A_\pm^{ij})_{1\le i,j\le n}$ is a positive definite $n \times n$ matrix smooth up to the boundary of $\Omega_\pm$. As essential transmission condition, we take the constraint:

$$u_+ = u_- \quad \text{on } \mathrm{I}.$$

For simplicity we impose Dirichlet conditions on the exterior boundary. Then this problem enters in the setting of section 5.3 if we take the sesquilinear form (4.12) defined on $\mathrm{H}_0^1(\Omega)$ with $A = A_+$ on Ω_+ and $A = A_-$ on Ω_-. By the previous considerations, the natural homogeneous transmission condition associated with this problem is

$$\sum_{i,j=1}^{n} n_i A_+^{ij} \partial_j u_+ - \sum_{i,j=1}^{n} n_i A_-^{ij} \partial_j u_- = 0 \quad \text{on } \mathrm{I},$$

with the components n_i of the normal field $\mathbf{n} := \mathbf{n}_+ = -\mathbf{n}_-$.

5.4.c The anisotropic discontinuous elasticity system

We consider as in § 4.4 the anisotropic elasticity system, now with piecewise constant elasticity moduli C_{ijmn}. Such a system represents a body formed by two different elastic materials:

$$L_\pm(\mathbf{x}, \mathrm{D}_x)\mathbf{u}_\pm = \Big(-\sum_{j=1}^{n} \partial_j \sigma_{ij,\pm}(\mathbf{u}_\pm)\Big)_{1\le i\le n},$$

where the stress tensor is here given by Hooke's law in each material

$$\sigma_{ij,\pm}(\mathbf{u}_\pm) = \sum_{m,n=1}^{n} C_{ijmn,\pm}\, \epsilon_{mn}(\mathbf{u}_\pm)\,.$$

As previously, we fix a unit normal field on the interface $\mathbf{n} := \mathbf{n}_+ = -\mathbf{n}_-$.

The standard essential transmission condition is the continuity of the displacement fields through the interface I [54, 80]

$$\mathbf{u}_+ = \mathbf{u}_- \quad \text{on } \mathrm{I}.$$

For this choice, the natural homogeneous transmission condition is the continuity of the vector of the normal traction:

$$\sigma_+(\mathbf{u}_+)\mathbf{n} - \sigma_-(\mathbf{u}_-)\mathbf{n} = \mathbf{0} \quad \text{on } \mathrm{I}\,.$$

Since the unknowns $\mathbf{u}_\pm$ are n-dimensional vectors, we may consider other types of essential transmission conditions. For instance we can consider the continuity of the normal component

$$\mathbf{u}_+ \cdot \mathbf{n} = \mathbf{u}_- \cdot \mathbf{n} \quad \text{on } \mathrm{I},$$

i. e., $\Pi^{D,\pm}\mathbf{u}_\pm = (\mathbf{u}_\pm \cdot \mathbf{n})\mathbf{n}$.

The natural homogeneous interface conditions can be read from equation (5.f): If we write $\mathbf{t}$ for the normal traction, $\mathbf{t}_\pm = \sigma_\pm(\mathbf{u}_\pm)\mathbf{n}$, then we get a transmission condition for its normal component

$$\mathbf{t}_+ \cdot \mathbf{n} - \mathbf{t}_- \cdot \mathbf{n} = 0 \quad \text{on } \mathrm{I}\,,$$

and one-sided boundary conditions for its tangential component

$$\mathbf{t}_+ - (\mathbf{t}_+ \cdot \mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{t}_- - (\mathbf{t}_- \cdot \mathbf{n})\mathbf{n} = \mathbf{0} \quad \text{on } \mathrm{I}.$$

Another choice is the continuity of the tangential component. The determination of the natural interface conditions for this case is left to the reader.

5.4.d A simple vector-scalar coupling problem

We consider here a coupling between the elasticity system in Ω_- with constant coefficient and the Laplace equation in Ω_+. Namely we take on the one hand $N_- = n$ for

$$L_-(\mathrm{D}_x)\mathbf{u}_- = \Big(- \sum_{j=1}^{n} \partial_j \sigma_{ij}(\mathbf{u}_-) \Big)_{1\le i\le n}, \tag{5.22}$$

with

$$\sigma_{ij}(\mathbf{u}_-) = \sum_{m,n=1}^{n} C_{ijmn}\epsilon_{mn}(\mathbf{u}_-),$$

the elasticity moduli C_{ijmn} being real valued, constant and fulfilling the standard symmetry and ellipticity conditions, and on the other hand $N_+ = 1$ for

$$L_+u_+ = -\Delta u_+.$$

We here take as essential boundary condition

$$\mathbf{u}_- = 0 \quad \text{on } \Gamma, \tag{5.23}$$

and as essential transmission condition

$$u_{1,-} = u_+ \quad \text{on } \mathrm{I}. \tag{5.24}$$

Then the obtained transmission problem is

$$\left\{ \begin{array}{rcll} L_-\mathbf{u}_- &=& \mathbf{f}_- & \text{in } \Omega_-, \\ -\Delta u_+ &=& f_+ & \text{in } \Omega_+, \\ \mathbf{u}_- &=& 0 & \text{on } \Gamma, \\ u_+ &=& u_{1,-} & \text{on } \mathrm{I}, \\ R_+B_+u_+ + \Pi^{D,-}B_-\mathbf{u}_- &=& \Pi^{D,-}\mathbf{g}_{\mathrm{I},-} & \text{on } \mathrm{I}, \\ \Pi^{T,-}B_-\mathbf{u}_- &=& \Pi^{T,-}\mathbf{g}_{\mathrm{I},-} & \text{on } \mathrm{I}, \end{array} \right. \tag{5.25}$$

where $B_-\mathbf{u}_- = \sigma(\mathbf{u}_-)\mathbf{n}_-$, $B_+u_+ = \nabla u_+ \cdot \mathbf{n}_+$, $\Pi^{D,-}\mathbf{u} = (u_1, 0, \dots, 0)^\top$, $\Pi^{D,+} = \mathbb{I}$, and $R_-\mathbf{u} = u_1$. Then $\Pi^{T,-}\mathbf{u} = (0, u_2, \dots, u_n)^\top$, $\Pi^{T,+} = 0$, and $R_+u_+ = (u_+, 0, \dots, 0)^\top$. Note that we would obtain the same essential transmission conditions with $\Pi^{D,-} = \mathbb{I}$, keeping the same R_-. But in the latter case, since $\Pi^{T,-} = 0$, the last two interface conditions would be replaced with

$$R_+B_+u_+ + B_-\mathbf{u}_- = \mathbf{g}_{\mathrm{I},-} \quad \text{on } \mathrm{I},$$

which is, in fact, equivalent.

The variational setting of this problem consists in taking

$$\mathbf{V} = \big\{ (\mathbf{u}_-, u_+) \in \mathbf{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega_+) \quad \text{satisfying (5.23), (5.24)} \big\},$$

and

$$\mathsf{a}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) = \int_{\Omega_-} \sum_{i,j=1}^{n} \sigma_{ij}(\mathbf{u}_-)\epsilon_{ij}(\bar{\mathbf{v}}_-)\,\mathrm{d}\mathbf{x} + \int_{\Omega_+} \nabla u_+ \cdot \nabla \bar{v}_+ \,\mathrm{d}\mathbf{x}.$$

Combining the results of our discussion of the stress-free elasticity system in §4.3.b and the Neumann problem in §4.1.b, we see that this sesquilinear form is coercive.

5.4.e A fluid-structure interaction system

This example is taken from [68, 72]. It is a model of the vibrations in harmonic mode of an elastic structure Ω_- containing an incompressible liquid Ω_+. The displacement D of the fluid Ω_+ is supposed to be small, hence it is governed by the linearized Euler equation

$$\rho_+ \partial_{tt}^2 D + \nabla P = 0 \quad \text{in } \Omega_+,$$

where $\rho_+ > 0$ is the volumic mass of the constitutive material of Ω_+ and P is the pressure of the fluid. A time harmonic movement means that we assume that

$$D(x,t) = d(x)\mathrm{e}^{i\omega t} \quad P(x,t) = p(x)\mathrm{e}^{i\omega t},$$

where $\omega \in \mathbb{R}$ is the inverse of the frequency of the oscillation. Therefore the pair (d, p) satisfies

$$-\rho_+ \omega^2 d + \nabla p = 0 \quad \text{in } \Omega_+,$$

or equivalently

$$d = \frac{1}{\rho_+ \omega^2} \nabla p \quad \text{in } \Omega_+.$$

Setting $u_+ = \frac{1}{\rho_+ \omega^2} p$, we then have

$$d = \nabla u_+ \quad \text{in } \Omega_+.$$

The incompressibility assumption means that $\operatorname{div} D = 0$ in Ω_+ and therefore we get

$$\Delta u_+ = 0 \quad \text{in } \Omega_+.$$

Moreover the elastic structure is supposed to be time harmonic (with the same frequency as the one of the fluid), i.e., the displacement $\mathbf{u}_-$ of the body Ω_- satisfies

$$L_-(\mathrm{D}_x)\mathbf{u}_- - \rho_- \omega^2 \mathbf{u}_- = 0 \quad \text{in } \Omega_-,$$

where ρ_- is a fixed positive constant that represents the volumic mass of the constitutive material of Ω_- and $L_-(\mathrm{D}_x)$ is the elasticity system in Ω_- defined by (5.22) in the previous example.

This means that the principal parts of the systems of partial differential operators are the same as the ones from the previous section. The transmission conditions, on the other hand, are different since the slip of the fluid along the elastic structure and the pressure of the fluid on the elastic structure yield respectively

$$\begin{cases} \partial_{n_+} u_+ &= \mathbf{u}_- \cdot \mathbf{n}_+ & \text{on } \mathrm{I}, \\ \sigma(\mathbf{u}_-)\mathbf{n}_- &= \rho_+ \omega^2 u_+ \mathbf{n}_+ & \text{on } \mathrm{I}. \end{cases} \tag{5.26}$$

Note that these four transmission conditions are natural ones since they involve first order derivatives of the unknowns. Finally on the exterior boundary, traction boundary forces are applied, i.e.,

$$\sigma(\mathbf{u}_-)\mathbf{n}_- = \mathbf{g} \quad \text{on } \Gamma,$$

where $\mathbf{g}$ represents the external force acting on the solid body.

In summary the following transmission problem is obtained

$$\begin{cases} L_-(\mathrm{D}_x)\mathbf{u}_- - \rho_- \omega^2 \mathbf{u}_- &= 0 & \text{in } \Omega_- \\ \Delta u_+ &= 0 & \text{in } \Omega_+, \\ \sigma(\mathbf{u}_-)\mathbf{n}_- &= \mathbf{g} & \text{on } \Gamma, \\ \partial_{n_+} u_+ - \mathbf{u}_- \cdot \mathbf{n}_+ &= 0 & \text{on } \mathrm{I}, \\ \sigma(\mathbf{u}_-)\mathbf{n}_- - \rho_+ \omega^2 u_+ \mathbf{n}_+ &= \mathbf{0} & \text{on } \mathrm{I}. \end{cases} \tag{5.27}$$

This system is an elliptic transmission problem, because the principal part of the transmission conditions is reduced to $\frac{\partial u_+}{\partial n_+}$ for the first condition and to $\sigma(\mathbf{u}_-)\mathbf{n}_-$ for the second one. Hence the principal part splits up into the Neumann problem for u_+ and the stress free boundary problem for the elastic component $\mathbf{u}_-$, for which we already checked in Chapter 4 that they form elliptic problems.

The transmission conditions on I are of Robin type, cf. (3.43), hence for the variational setting of problem (5.27) we combine the general framework of section 5.3 and the strategy of §3.5 by adding a boundary term in the sesquilinear form. Thus we take

$$\mathbf{V} = \mathbf{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega_+),$$

and

$$\tilde{\mathsf{a}}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) = \mathsf{a}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) + \mathsf{b}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)),$$

where

$$\begin{aligned} \mathsf{a}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) &= \int_{\Omega_-} \Big(\sum_{i,j=1}^{n} \sigma_{ij}(\mathbf{u}_-)\epsilon_{ij}(\bar{\mathbf{v}}_-) - \rho_- \omega^2 \mathbf{u}_- \cdot \bar{\mathbf{v}}_- \Big) \,\mathrm{d}\mathbf{x} \\ &\quad + \int_{\Omega_+} \nabla u_+ \cdot \nabla \bar{v}_+ \,\mathrm{d}\mathbf{x}, \\ \mathsf{b}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) &= \int_{\mathrm{I}} \big(\mathbf{u}_- \cdot \mathbf{n}_- \bar{v}_+ + \rho_+ \omega^2 u_+ \bar{\mathbf{v}}_- \cdot \mathbf{n}_- \big) \,\mathrm{d}\sigma. \end{aligned}$$

Hence the variational formulation of (5.27) consists in looking for $(\mathbf{u}_-, u_+) \in \mathbf{V}$ solution of

$$\tilde{\mathsf{a}}((\mathbf{u}_-, u_+), (\mathbf{v}_-, v_+)) = \int_{\Gamma} \mathbf{g} \cdot \mathbf{v}_- \,\mathrm{d}\sigma \quad \forall (\mathbf{v}_-, v_+) \in \mathbf{V}. \tag{5.28}$$

Indeed integration by parts shows that if $(\mathbf{u}_-, u_+) \in \mathbf{V}$ is solution of this last problem, then it is a solution of (5.27).

Lemma 5.4.1 *The sesquilinear form* $\tilde{\mathsf{a}}$ *is coercive on* $\mathbf{V}$ *in the following sense: there exist* $C_1, C_2 > 0$ *such that*

$$\forall(\mathbf{u}_-, u_+) \in \mathbf{V}, \quad \operatorname{Re} \tilde{\mathsf{a}}((\mathbf{u}_-, u_+), (\mathbf{u}_-, u_+)) \geq C_1 \|(\mathbf{u}_-, u_+)\|^2_{\mathsf{P}\,1;\Omega} - C_2 \|(\mathbf{u}_-, u_+)\|^2_{\mathsf{P}\,0;\Omega}.$$

Proof: The sesquilinear form a is coercive as we have seen in the previous section. The coercivity of $\tilde{\mathsf{a}}$ follows from the coercivity of a as in the proof of Lemma 3.5.1. □

This coercivity property shows that problem (5.27) enters into our general framework of elliptic transmission problems, and there hold the usual results, such as Fredholm alternative and regularity in Sobolev spaces and analytic classes.

5.4.f The piezoelectric system coupled with the elasticity system

As discussed in section 4.7, the piezoelectric system modelizes sensors or actuators, and in practice this system is coupled with the elasticity system. Here is an example coming from an application to common-rail diesel engines [37]: It is a coupling between the elasticity system in Ω_- and the piezoelectric system in Ω_+. The problem considered by theses authors corresponds (mainly) to the following variational setting:

$$\Big\{(\{\mathbf{u}_+, \Phi_+\}, \mathbf{u}_-) \in \mathrm{H}^1(\Omega_+)^4 \times \mathrm{H}^1(\Omega_-)^3 \quad : \quad \Phi_+ = 0 \quad \text{on } \mathrm{I}, \quad \mathbf{u}_+ = \mathbf{u}_- \quad \text{on } \mathrm{I}, \quad \mathbf{u}_- = 0 \quad \text{on } \Gamma \Big\},$$

$$\mathsf{a}\big((\{\mathbf{u}_+, \Phi_+\}, \mathbf{u}_-), (\{\mathbf{v}_+, \Psi_+\}, \mathbf{v}_-)\big) = \mathsf{a}_+(\{\mathbf{u}_+, \Phi_+\}, \{\mathbf{v}_+, \Psi_+\}) + \mathsf{a}_-(\mathbf{u}_-, \mathbf{v}_-),$$

where a_+ corresponds to the sesquilinear form of the piezoelectric system in Ω_+ described in section 4.7, while a_- is the sesquilinear form associated with the elasticity system in Ω_- defined in section 4.4.

Bibliography

[1] M. ABRAMOWITZ, I. A. STEGUN. *Handbook of Mathematical Functions*. Dover Publications, New York 1972.

[2] R. A. ADAMS. *Sobolev Spaces*. Academic Press, New York 1975.

[3] S. AGMON. *Lectures on elliptic boundary value problems*. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London 1965.

[4] S. AGMON, A. DOUGLIS, L. NIRENBERG. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. *Comm. Pure Appl. Math.* **12** (1959) 623–727.

[5] S. AGMON, A. DOUGLIS, L. NIRENBERG. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.* **17** (1964) 35–92.

[6] M. S. AGRANOVICH. Elliptic boundary problems. In *Partial differential equations, IX*, volume 79 of *Encyclopaedia Math. Sci.*, pages 1–144, 275–281. Springer, Berlin 1997. Translated from the Russian by the author.

[7] C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRAULT. Vector potentials in three-dimensional nonsmooth domains. *Math. Meth. Appl. Sci.* **21** (1998) 823–864.

[8] F. V. ATKINSON. The normal solubility of linear equations in normed spaces. *Mat. Sbornik N.S.* **28(70)** (1951) 3–14.

[9] I. BABUŠKA, B. GUO. Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order. *SIAM J. Math. Anal.* **19**(1) (1988) 172–203.

[10] I. BABUŠKA, B. GUO. Regularity of the solution of elliptic problems with piecewise analytic data. II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions. *SIAM J. Math. Anal.* **20**(4) (1989) 763–781.

[11] I. BABUŠKA, B. GUO. Approximation properties of the h-p version of the finite element method. *Comput. Methods Appl. Mech. Engrg.* **133**(3-4) (1996) 319–346.

[12] H. BARUCQ, B. HANOUZET. étude asymptotique du système de Maxwell avec la condition aux limites absorbante de silver-müller ii. *C. R. Acad. Sci. Paris Sér. I* **316** (1993) 1019–1024.

[13] D. A. BERLINCOURT, D. CURRAN, H. JAFFE. Piezoelectric and piezomagnetic materials and their function in transducers. *Physical Acoustics* **1** (1964) 169–270.

[14] M. BIRMAN, M. SOLOMYAK. L^2-theory of the Maxwell operator in arbitrary domains. *Russ. Math. Surv.* **42 (6)** (1987) 75–96.

[15] J. H. BRAMBLE, S. R. HILBERT. Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation. *SIAM J. Numer. Anal.* **7** (1970) 112–124.

[16] A. BUFFA, M. COSTABEL, M. DAUGE. Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron. *C. R. Acad. Sc. Paris, Série I* **336** (2003) 565–570.

[17] A. BUFFA, M. COSTABEL, M. DAUGE. Algebraic convergence for anisotropic edge elements in polyhedral domains. *Numer. Math.* **101** (2005) 29–65.

[18] J. CHAZARAIN, A. PIRIOU. *Introduction to the theory of linear partial differential equations*. North-Holland Publishing Co., Amsterdam 1982. Translated from the French.

[19] P. CIARLET, JR., C. HAZARD, S. LOHRENGEL. Les équations de Maxwell dans un polyèdre : un résultat de densité. *C. R. Acad. Sc. Paris, Série I Math.* **326**(11) (1998) 1305–1310.

[20] P. G. CIARLET. *The finite element method for elliptic problems*. North-Holland Publishing Co., Amsterdam 1978. Studies in Mathematics and its Applications, Vol. 4.

[21] P. G. CIARLET. *Mathematical elasticity. Volume I: Three-dimensional elasticity.* Studies in Mathematics and its Applications, 20. Amsterdam etc.: North- Holland. xi, 451 p. 1988.

[22] M. COSTABEL. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Math. Methods Appl. Sci.* **12 (4)** (1990) 365–368.

[23] M. COSTABEL. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.* **157 (2)** (1991) 527–541.

[24] M. COSTABEL, M. DAUGE. Développement asymptotique le long d'une arête pour des équations elliptiques d'ordre 2 dans $\mathbb{R}^3$. *C. R. Acad. Sc. Paris, Série I* **312** (1991) 227–232.

[25] M. COSTABEL, M. DAUGE. Edge asymptotics on a skew cylinder. In B.-W. SCHULZE, H. TRIEBEL, editors, *Symposium "Analysis in Domains and on Manifolds with Singularities", Breitenbrunn 1990*, Teubner-Texte zur Mathematik, Vol. 131, pages 28–42. B. G. Teubner, Leipzig 1992.

[26] M. COSTABEL, M. DAUGE. General edge asymptotics of solutions of second order elliptic boundary value problems I. *Proc. Royal Soc. Edinburgh* **123A** (1993) 109–155.

[27] M. COSTABEL, M. DAUGE. General edge asymptotics of solutions of second order elliptic boundary value problems II. *Proc. Royal Soc. Edinburgh* **123A** (1993) 157–184.

[28] M. COSTABEL, M. DAUGE. Stable asymptotics for elliptic systems on plane domains with corners. *Comm. Partial Differential Equations* n° **9 & 10** (1994) 1677–1726.

[29] M. COSTABEL, M. DAUGE. Un résultat de densité pour les équations de Maxwell régularisées dans un domaine lipschitzien. *C. R. Acad. Sc. Paris, Série I* **327** (1998) 849–854.

[30] M. COSTABEL, M. DAUGE. Maxwell and Lamé eigenvalues on polyhedra. *Math. Meth. Appl. Sci.* **22** (1999) 243–258.

[31] M. COSTABEL, M. DAUGE. Singularities of electromagnetic fields in polyhedral domains. *Arch. Rational Mech. Anal.* **151**(3) (2000) 221–276.

[32] M. DAUGE. *Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions*. Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, Berlin 1988.

[33] G. DUVAUT, J.-L. LIONS. *Les inéquations en mécanique et en physique*. Dunod, Paris 1972. Travaux et Recherches Mathématiques, No. 21.

[34] J. N. ERINGEN, G. A. MAUGIN. *Electrodynamics of continua*, volume 1,2. Springer-Verlag, New-York 1990.

[35] G. FICHERA. *Existence theorems in elasticity*, volume VIa/2 of *Handbuch der Physik*. Springer-Verlag, Berlin 1972.

[36] L. GÅRDING. Dirichlet's problem for linear elliptic partial differential equations. *Math. Scand.* **1** (1953) 55–72.

[37] W. GEIS, G. MISHURIS, A.-M. SÄNDIG. Piezoelectricity in multi-layer actuators: Modelling and analysis in two and three-dimensions. Preprint IAANS 2003/23, Universität Stuttgart 2003.

[38] D. GILBARG, N. S. TRUDINGER. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin 2001. Reprint of the 1998 edition.

[39] V. GIRAULT, P. RAVIART. *Finite Element Methods for the Navier–Stokes Equations, Theory and Algorithms*. Springer series in Computational Mathematics, 5. Springer-Verlag, Berlin 1986.

[40] J. GOBERT. Une inégalité fondamentale de la théorie de l'élasticité. *Bull. Soc. Royale Sciences Liège* **3-4** (1962) 182–191.

[41] I. C. GOHBERG, M. G. KREĬN. The basic propositions on defect numbers, root numbers and indices of linear operators. *Amer. Math. Soc. Transl. (2)* **13** (1960) 185–264.

[42] P. GRISVARD. *Boundary Value Problems in Non-Smooth Domains*. Pitman, London 1985.

[43] P. GRISVARD. *Singularities in boundary value problems*, volume 22 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris 1992.

[44] B. GUO. The h-p version of the finite element method for solving boundary value problems in polyhedral domains. In M. COSTABEL, M. DAUGE, S. NICAISE, editors, *Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993)*, pages 101–120. Dekker, New York 1995.

[45] C. HAZARD, M. LENOIR. On the solution of time-harmonic scattering problems for Maxwell's equations. *SIAM J. Math. Anal.* **27 (6)** (1996) 1597–1630.

[46] I. HLAVÁČEK, J. NEČAS. On inequalities of Korn's type. II: Applications to linear elasticity. *Arch. Ration. Mech. Anal.* **36** (1970) 312–334.

[47] L. HÖRMANDER. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin 1983. Distribution theory and Fourier analysis.

[48] L. HÖRMANDER. *The analysis of linear partial differential operators. II*, volume 257 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin 1983. Differential operators with constant coefficients.

[49] L. HÖRMANDER. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin 1985. Pseudodifferential operators.

[50] D. IEŞAN. Plane strain problems in piezoelectricity. *Int. J. Eng. Sci.* **25** (1987) 1511–1523.

[51] T. IKEDA. *Fundamentals of piezoelectricity*. Oxford Univ. Press, Oxford 1996.

[52] V. A. KONDRAT'EV. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16** (1967) 227–313.

[53] V. A. KOZLOV, V. G. MAZ'YA, J. ROSSMANN. *Elliptic boundary value problems in domains with point singularities*. Mathematical Surveys and Monographs, 52. American Mathematical Society, Providence, RI 1997.

[54] D. LEGUILLON, E. SANCHEZ-PALENCIA. *Computation of singular solutions in elliptic problems and elasticity*. RMA 5. Masson, Paris 1987.

[55] L. LICHTENSTEIN. Über die konforme Abbildung ebener analytischer Gebiete mit Ecken. *J. für Math.* **140** (1911) 100–119.

[56] L. LICHTENSTEIN. Zur Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus. Die erste Randwertaufgabe für analytische Gebiete mit Ecken. *Acta Math.* **36** (1913) 345–386.

[57] J.-L. LIONS, E. MAGENES. *Problèmes aux limites non homogènes et applications. Vol. 1*. Travaux et Recherches Mathématiques, No. 17. Dunod, Paris 1968.

[58] J.-L. LIONS, E. MAGENES. *Problèmes aux limites non homogènes et applications. Vol. 2*. Travaux et Recherches Mathématiques, No. 18. Dunod, Paris 1968.

[59] J.-L. LIONS, E. MAGENES. *Problèmes aux limites non homogènes et applications. Vol. 3*. Dunod, Paris 1970. Travaux et Recherches Mathématiques, No. 20.

[60] L. MARIN. The behaviour of piezoelectric materials under mechanical and electrical loadings. Master's thesis, Univ. Kaiserslautern, Germany 1998.

[61] W. MCLEAN. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge 2000.

[62] D. MERCIER, S. NICAISE. Existence, uniqueness, and regularity results for piezoelectric systems. *SIAM J. Math. Anal.* **37**(2) (2005) 651–672.

[63] S. G. MIHLIN. The spectrum of the pencil of operators of elasticity theory. *Uspehi Mat. Nauk* **28**(3(171)) (1973) 43–82.

[64] S. G. MIKHLIN, S. PRÖSSDORF. *Singular integral operators*, volume 68 of *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs]*. Akademie-Verlag, Berlin 1986. Translated from the German by Albrecht Böttcher and Reinhard Lehmann.

[65] R. MINDLIN. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. Appl. Mech.* **18** (1951) 31–38.

[66] C. MIRANDA. *Partial differential equations of elliptic type*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2. Springer-Verlag, New York 1970. Second revised edition. Translated from the Italian by Zane C. Motteler.

[67] P. MONK. *Finite element methods for Maxwell's equations*. Numer. Math. Scientific Comp. Oxford Univ. Press, New York 2003.

[68] H. J.-P. MORAND, R. OHAYON. *Fluid-structure interactions. (Interactions fluides-structures.)*. Recherches en Mathématiques Appliquées. 23. Paris etc.: Masson. xi, 212 p. 1992.

[69] C. B. MORREY, JR. *Multiple integrals in the calculus of variations*. Die Grundlehren der mathematischen Wissenschaften, Band 130. Springer-Verlag New York, Inc., New York 1966.

[70] C. B. MORREY, JR., L. NIRENBERG. On the analyticity of the solutions of linear elliptic systems of partial differential equations. *Comm. Pure Appl. Math.* **10** (1957) 271–290.

[71] N. MUSKHELISHVILI. *Some basic problems of the mathematical theory of elasticity. Third Edition. Translated from the russian by J. R. M. Radok.* P. Noordhoff Ltd. Groningen-Holland 1953.

[72] D. NATROSHVILI, A.-M. SÄNDIG, W. L. WENDLAND. Fluid-structure interaction problems. In *Mathematical aspects of boundary element methods (Palaiseau, 1998)*, volume 414 of *Chapman & Hall/CRC Res. Notes Math.*, pages 252–262. Chapman & Hall/CRC, Boca Raton, FL 2000.

[73] V. E. NAZAIKINSKII, A. Y. SAVIN, B.-W. SCHULZE, B. Y. STERNIN. *Elliptic theory on singular manifolds*, volume 7 of *Differential and Integral Equations and Their Applications*. Chapman & Hall/CRC, Boca Raton, FL 2006.

[74] J.-C. NÉDÉLEC. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980) 315–341.

[75] J.-C. NÉDÉLEC. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York 2001.

[76] J. NEČAS. *Les méthodes directes en théorie des équations elliptiques*. Masson-Academia, Paris-Prague 1967.

[77] J. NEČAS, I. HLAVÁČEK. *Mathematical theory of elastic and elasto-plastic bodies: an introduction.* Studies in Applied Mechanics, 3. Amsterdam - Oxford - New York: Elsevier Scientific Publishing Company. 342 p. 1981.

[78] S. NICAISE. Le laplacien sur les réseaux deux-dimensionnels polygonaux topologiques. *J. Math. Pures Appl. (9)* **67**(2) (1988) 93–113.

[79] S. NICAISE. *Polygonal interface problems*. Methoden und Verfahren der Mathematischen Physik, 39. Verlag Peter D. Lang, Frankfurt-am-Main 1993.

[80] S. NICAISE, A.-M. SÄNDIG. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation. *Math. Models Methods Appl. Sci.* **9**(6) (1999) 855–898.

[81] W. NOVACKI. Mathematical models of phenomenological piezoelecticity. In *Math. Models and Methods in Mechanics*, volume 15 of *Banach Center publications*, pages 593–607. Polish Acad. of Sciences, Warsaw, Poland 1985.

[82] E. REISSNER. On the theory of bending of elastic plates. *J. Math. Phys.* **23** (1944) 184–191.

[83] S. REMPEL, B. W. SCHULZE. *Asymptotics for Elliptic Mixed Boundary Problems*. Akademie-Verlag, Berlin 1989.

[84] A. RÖSSLE, A.-M. SÄNDIG. Corner singularities and regularity-results for the reissner/mindlin plate-model. Preprint SFB 404 2001/04, Univ. Stuttgart 2001.

[85] M. SCHECHTER. General boundary value problems for elliptic partial differential equations. *Bull. Amer. Math. Soc.* **65** (1959) 70–72.

[86] E. SCHROHE, B.-W. SCHULZE. Boundary value problems in Boutet de Monvel's algebra for manifolds with conical singularities. II. In *Boundary value problems, Schrödinger operators, deformation quantization*, pages 70–205. Akademie Verlag, Berlin 1995.

[87] B. W. SCHULZE. *Pseudo-differential operators on manifolds with singularities*. Studies in Mathematics and its Applications, Vol. 24. North-Holland, Amsterdam 1991.

[88] B.-W. SCHULZE. *Pseudo-differential boundary value problems, conical singularities, and asymptotics*, volume 4 of *Mathematical Topics*. Akademie Verlag, Berlin 1994.

[89] B.-W. SCHULZE. *Boundary value problems and singular pseudo-differential operators*. Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., Chichester 1998.

[90] M. TAYLOR. *Pseudodifferential Operators*. University Press, Princeton 1981.

[91] M. E. TAYLOR. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York 1996. Basic theory.

[92] F. TRÈVES. *Introduction to pseudodifferential and Fourier integral operators. Vol. 1*. Plenum Press, New York 1980. Pseudodifferential operators, The University Series in Mathematics.

[93] H. TRIEBEL. *Interpolation theory. Function spaces. Differential operators*. North-Holland Mathematical Library. North-Holland, Amsterdam 1978.

[94] J. T. WLOKA, B. ROWLEY, B. LAWRUK. *Boundary value problems for elliptic systems*. Cambridge University Press, Cambridge 1995.

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