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BOUNDARY INTEGRAL EQUATIONS FOR MIXED BOUNDARY VALUE PROBLEMS IN POLYGONAL DOMAINS AND GALERKIN APPROXIMATION

MARTIN COSTABEL and ERNST STEPHAN

Fachbereich Mathematik, Technische Hochschule Darmstadt, Darmstadt

§ 0. Introduction

The main purpose of this paper is the proof of asymptotic error estimates for the finite element Galerkin approximation of the boundary integral equations for a mixed Dirichlet–Neumann boundary value problem for the Laplacian in a plane polygonal domain. This is a generalization of [58], where the case of a domain with a smooth boundary was treated. This generalization is natural in two respects:

On the one hand, it is well known that the solutions of elliptic boundary value problems have a similar kind of singular behaviour at corner points of the boundary and at points where the boundary is smooth but the boundary conditions change.

On the other hand, a lot of applied problems leading to mixed boundary value problems for the Laplacian (for some examples see the introduction of [58] or [51]) actually involve domains with nonsmooth boundaries.

The method used in [58] is analogous to the Fix method for 2-*D* finite elements [59], [53]. It uses the decomposition of the solutions of the boundary value problem into a singular and a regular part, where the singular part is a finite linear combination of known singular functions which do not depend on the given boundary data but only on the geometry, and the regular part is approximated by regular finite element functions. This decomposition was proved, e.g., by Lehman [34] using conformal mappings for piecewise analytic boundaries and the Dirichlet problem, by Kondratiev [31] using the Mellin transform and the framework of weighted Sobolev spaces for general boundary conditions, by Kellogg [28], [29] using eigenfunction expansions, and it has been generalized by various authors (e.g., [38], [39], [18], [4], [37]). Our standard reference is [20].

The main tool for proving the convergence of a Galerkin procedure is *strong ellipticity* ([52], [22], [55]). For the boundary integral equations this means that one has to prove a Gårding inequality for the integral operator involved. There are two ways known to do this:

For the Dirichlet problem one can exploit the coerciveness of the Dirichlet bilinear form, thus reducing the boundary integral equation of the first kind to the variational problem on the two-dimensional domain. This was used to show the positivity of the integral operator of the single layer potential on smooth curves in [24], [35] and of other integral operators [43], and error estimates for the Galerkin approximation for the boundary integral equation for the Dirichlet problem on polygonal boundaries were treated in [7]. The boundary integral equations for the mixed problem on a smooth boundary are of a special form which allows the reduction of the Gårding inequality for the whole system to the one for the single layer potential. This was used in [57], [58].

On smooth boundaries there exists also the notion of strongly elliptic systems of pseudodifferential operators, which can be used because the boundary integral operators are pseudodifferential operators. The Gårding inequality in this case can be proved by showing the positivity of the symbol which is calculated by means of a local Fourier transformation or expansion into Fourier series. This was used in [56], [52].

For the mixed boundary value problem on domains with corners none of these two methods works. Even for the proof of the continuity of the integral operators in the appropriate Sobolev spaces one has to find different tools.

We use the Mellin transform, and it turns out that this is a very useful tool, which allows us to prove not only the continuity of the integral operators but also the required Gårding inequality and even the decomposition of the solution into regular and singular parts.

The usefulness of the Mellin transform in connection with domains with corners was shown by Kondratiev [31] and generalized to pseudodifferential boundary value problems by Komeč [30]. For the boundary integral equations it was used, e.g., by Eskin [9] for the mixed problem on smooth domains and by Fabes *et al.* [10], [11] for the Dirichlet problem (in L^p spaces) in a sector. The latter authors use an integral equation of the second kind involving the operator of the double layer potential, and they calculate its Mellin symbol. For singular integral equations with piecewise continuous coefficients and curves with corners the Mellin transform was used by Dudučava [8] and similarly in [6] to obtain a calculus of symbols. (Cf. [14].)

The use of the Mellin transform for the investigation of the operators of the single and the double layer potentials in Sobolev spaces seems to be new. Therefore, in spite of the fact that the aim of the paper is the investigation of the numerical approximation scheme, we devote a lot of space to the

derivation of functional analytic properties of the system of integral equations and a calculus of symbols, which then allows us quite easily to find the explicit form of the singular parts of the solution. It turns out that we do not need the local Fourier transformation. Potential theoretic arguments are essentially only needed for the uniqueness proof.

There exist numerical methods for approximately solving mixed boundary value problems on polygonal domains by means of boundary integral equations (see [1], [23], [27], [40], [46]). They are based on the collocation method, and in general no error estimates are available [47]. The error estimates which we obtain are of the same order as those obtained by the 2-*D* finite element method ([5], [12]). But, in contrast to the latter, we get simultaneously error estimates in higher Sobolev spaces and not only in the energy norm (i.e., the norm which corresponds to the weak formulation of the boundary value problem). This allows us, e.g., to get easily L^∞-estimates as well as error estimates for the coefficients of the singular functions, which have a direct and important meaning, e.g., as stress-intensity factors in mechanical problems.

Some problems which we have not yet treated but which are solved for smooth boundaries in [58] are:

Investigation of the perturbations arising from curvature terms if one considers curved polygons;

Investigation of the mapping properties of the integral operators in Sobolev spaces with negative indices. This would give higher orders of convergence by using the Aubin–Nitsche trick [25] and error estimates for the Galerkin collocation method [25]. This method is easily implementable on a computer and gives good numerical results, as was shown in [33] for the case of a smooth boundary. The paper is organized as follows:

In § 1 we collect some facts from potential theory and give the decomposition of the weak solution of the mixed boundary value problem into singular and regular parts.

In § 2 we collect the necessary facts about Sobolev spaces (with and without weights) on polygons and on the Mellin transform in weighted Sobolev spaces. Here we prove a result on the Mellin transform in $H^{1/2}(\boldsymbol{R}_+)$. Then the Mellin symbols of the integrals of the single and double layer potentials are computed and used to prove the continuity of the integral operators in Sobolev spaces. The Gårding inequalities for the system of integral equations corresponding to the boundary value problem are derived: one for the original system in the Hilbert space which corresponds to the energy norm, and a second one for a modified system in the Hilbert space which corresponds to the standard Galerkin procedure for integral equations. Unfortunately, these two Hilbert spaces do not coincide, and this causes a lot of trouble.

In § 3 the unique solvability of the system of integral equations is derived. At this point we need an assumption (V) on the domain, namely that there is no eigensolution of the exterior Dirichlet problem. This can always be achieved by a scale transformation of the domain. Then we show the bijectivity of the integral operators and the equivalence of the system of integral equations with the weak formulation and the distributional formulation of the boundary value problem.

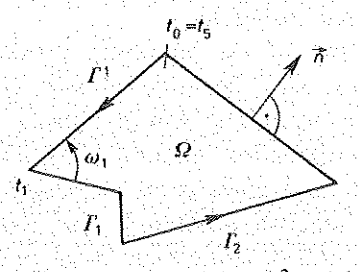
In § 4 we use the calculus of Mellin symbols and calculate the explicit form of the singular functions as well as the compatibility conditions which have to be satisfied by the data and by the solution in various special cases. Here we also obtain the regularity results for the solution of the integral equations which are needed for the equivalence theorem in § 3.

Finally, in § 5 we define the augmented finite element spaces and the Galerkin approximation schemes. Then we show the stability of the Galerkin operator in the energy norm. With the help of the convergence and inverse properties of the augmented finite element spaces we then prove asymptotic error estimates in various norms.

§ 1. Various formulations of the mixed boundary value problem

1.1. We consider the mixed boundary value problem for the Laplacian

$$\text{(P)}\qquad \begin{aligned} \Delta u &= 0 \quad \text{in } \Omega,\\ u &= g_1 \quad \text{on } \Gamma_1,\\ \frac{\partial u}{\partial n} &= g_2 \quad \text{on } \Gamma_2,\end{aligned}$$



where Ω is a bounded simply connected domain in $\boldsymbol{R}^2$ with a polygonal boundary $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \bigcup_{j=1}^{J} \bar{\Gamma}^j$, Γ^j being open straight line segments. By $t_j (t_j = 0, \ldots, J)$, we denote the corner points where Γ_j and Γ_{j+1} meet $(t_J = t_0)$. The interior angle at t_j is denoted by ω_j.

Let D, N, and M be the subsets of $\{1, \ldots, J\}$ for which $t_j \in \Gamma_1$, $t_j \in \Gamma_2$, or $t_j \in \bar{\Gamma}_1 \cap \bar{\Gamma}_2$, respectively. $\partial u/\partial n$ means the normal derivative w.r.t. the outer normal $\vec{n}$, which exists outside the corners. The definition of Sobolev spaces is as usual [20], [36]:

$$H^s(\Omega) = \{u_{|\Omega} |\; u \in H^s(\boldsymbol{R}^2)\} \qquad (s \in \boldsymbol{R});$$

$$\text{(1.1)}\qquad H^s(\Gamma) = \begin{cases} \{u_{|\Gamma} |\; u \in H^{s+1/2}(\boldsymbol{R}^2)\} & (s > 0),\\ L^2(\Gamma) & (s = 0),\\ (H^{-s}(\Gamma))' \text{ (dual space)} & (s < 0);\end{cases}$$

$$H^s(\Gamma_j) = \{u_{|\Gamma_j} |\; u \in H^s(\Gamma)\} \qquad (s \geq 0)\ (j = 1, 2; \text{ similarly for } \Gamma^j),$$

$$\tilde{H}^s(\Gamma_j) = \{u \in H^s(\Gamma_j) |\; \tilde{u} \in H^s(\Gamma)\}\ (s \geq 0)\}.$$

Here $\tilde{u} = \begin{cases} u & \text{on } \Gamma_j\\ 0 & \text{on } \Gamma \setminus \Gamma_j\end{cases}$ means the continuation of u by 0 outside Γ_j.

Finally,

$$H^s(\Gamma_j) = (\tilde{H}^{-s}(\Gamma_j))' \qquad (s < 0),$$
$$\tilde{H}^s(\Gamma_j) = (H^{-s}(\Gamma_j))' \qquad (s < 0).$$

The most general case where (P) can be converted into a variational problem is the following:

$g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in \tilde{H}^{-1/2}(\Gamma_2)$ are given, and we look for $u \in H^1(\Omega)$.

In this case, $\dfrac{\partial u}{\partial n} \in \tilde{H}^{-1/2}(\Gamma_2) \subset H^{-1/2}(\Gamma)$ is defined by Green's formula:

LEMMA 1.1 ([48], p. 6). *Let* $u \in H^1(\Delta, p, \Omega) := \{u \in H^1(\Omega) |\; \Delta u \in L^p(\Omega)\}$ $(p > 1)$, $v \in H^1(\Omega)$. *Then*

$$\text{(1.2)}\qquad \int_\Omega \Delta u \cdot v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx = \left\langle \frac{\partial u}{\partial n}, v|_\Gamma \right\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}.$$

The mapping $u \mapsto \left.\dfrac{\partial u}{\partial n}\right|_\Gamma : H^1(\Delta, p, \Omega) \to H^{-1/2}(\Gamma)$ *is continuous.*

Here $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ means the duality between these spaces.

1.2. In addition to the distribution formulation (P) of the mixed boundary value problem, we need the variational formulation for the same problem and the boundary integral equations which are obtained by inserting the fundamental solution $\dfrac{1}{2\pi} \log|z - \zeta|$ for the Laplacian into Green's 2nd identity. For this, we have the following representation by volume potentials and boundary potentials:

LEMMA 1.2. *Let* $u \in H^1(\Delta, p, \Omega)$; $\Delta u = f \in L^2$. *Then for* $z \in \Omega$

$$\text{(1.3)}\qquad u(z) = \frac{1}{2\pi} \int_\Omega f(x) \log|z - x|\, dx + {}$$
$$+ \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z - \zeta|\, ds_\zeta - \frac{1}{2\pi} \int_\Gamma \frac{\partial u(\zeta)}{\partial n} \log|z - \zeta|\, ds_\zeta.$$

Here ds_ζ is the measure on Γ defined by the arc length and

$$\int_\Gamma (\partial u/\partial n) \log|z - \zeta|\, ds_\zeta$$

is to be understood as $\langle \partial u/\partial n, \log|z - .| \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$.

Proof. For $u, v \in H^2(\Omega)$ we have Green's 2nd identity ([42]):

$$(1.4)\qquad \int_{\Omega_\varepsilon} (u\Delta v - \Delta u \cdot v)\,dx = \int_{\partial\Omega_\varepsilon} \left(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v\right) ds.$$

We use this for $v(x) = \frac{1}{2\pi}\log|z-x|$ and $\Omega_\varepsilon = \Omega \setminus \overline{K_\varepsilon(z)}$, where $K_\varepsilon(z)$ is a small disc with centre z and radius ε which is contained in Ω. Thus $\Gamma_\varepsilon = \partial\Omega_\varepsilon = \Gamma \cup \partial K_\varepsilon(z)$. Obviously, $v \in C^\infty(\overline{\Omega_\varepsilon})$. Now $C^\infty(\overline{\Omega})$ is dense in $H^1(\Delta, p, \Omega)$ ([20], Lemma 1.5.3.9), and if $u_k \to u$ in $H^1(\Delta, p, \Omega)$ then $u_k|_{\Gamma_\varepsilon} \to u|_{\Gamma_\varepsilon}$ in $H^{1/2}(\Gamma_\varepsilon)$ and (e.g., by (1.2)) $(\partial u_k/\partial n)|_{\Gamma_\varepsilon} \to (\partial u/\partial n)|_{\Gamma_\varepsilon}$ in $H^{-1/2}(\Gamma_\varepsilon)$. Therefore, (1.4) holds for $u \in H^1(\Delta, p, \Omega)$. Now, inside $K_\varepsilon(z)$ we have $u \in H^2(K_\varepsilon(z))$ ([36], p. 125, Th. 3.2) and therefore the representation formula

$$(1.5)\qquad u(z) = \int_{K_\varepsilon(z)} \Delta u(x)\, v(z-x)\,dx + \int_{\partial K_\varepsilon(z)} \left(u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} v\right) ds$$

holds. Now we use $\Delta v = 0$ in Ω_ε, $\Delta u = f$, and observe that the normal vectors in (1.4) and (1.5) have opposite directions. Thus we obtain (1.3) by adding (1.4) and (1.5). ∎

In addition to this we shall need the following continuity properties of single and double layer potentials:

LEMMA 1.3. (i) ([32], Th. 219). *Let $g \in C^0(\Gamma)$ and*

$$u(z) := -\frac{1}{\pi}\int_\Gamma g(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta \qquad (z \in \Omega).$$

Then u is harmonic in Ω, continuous in $\overline{\Omega}$, and

$$(1.6)\qquad u(z) = -g(z) - \frac{1}{\pi}\int_\Gamma g(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,ds_\zeta \qquad (z \in \Gamma).$$

(ii) ([13], Satz 1) *Let $g \in L^p(\Gamma)$ $(p > 1)$ and*

$$u(z) = -\frac{1}{\pi}\int_\Gamma g(\zeta)\log|z-\zeta|\,ds \qquad (z \in \overline{\Omega}).$$

Then u is harmonic in Ω and continuous in $\overline{\Omega}$.

1.3. Let us now suppose that $u \in H^1(\Omega)$ is a solution of (P) with $g_1 \in H^{1/2}(\Gamma_1)$ and $g_2 \in H^{-1/2}(\Gamma_2)$. Then, if $v \in V := \{v \in H^1(\Omega) | v|_{\Gamma_1} = 0\}$, by Lemma 1.1 we have

$$(1.7)\qquad \int_\Omega \nabla u \cdot \nabla v\,dx = \langle g_2, v|_{\Gamma_2}\rangle_{H^{-1/2}(\Gamma_2)\times \tilde{H}^{1/2}(\Gamma_2)}.$$

(Note that $v|_{\Gamma_2} \in \tilde{H}^{1/2}(\Gamma_2)$ because $v|_\Gamma \in H^{1/2}(\Gamma)$ and $v|_{\Gamma_1} = 0$.) This means that u is a solution of the following variational problem: Find $u \in H^1(\Omega)$ with $u|_{\Gamma_1} = g_1$ such that for each $v \in V$ (1.7) holds.

The converse is also true: Let u be a solution of this variational problem. Then $\int_\Omega \nabla u \nabla v = 0$ for all $v \in C_0^\infty(\Omega)$ and thus u is harmonic in Ω. By Lemma 1.1, $\partial u/\partial n \in H^{-1/2}(\Gamma)$ and $\partial u/\partial n|_{\Gamma_2} = g_2$. Thus u is a solution of (P).

In order to use Grisvard's regularity results for the weak solution of (P), we want to formulate the variational problem in a different way which is equivalent to it:
Let $\tilde{g}_1 \in H^{1/2}(\Gamma)$ and $\tilde{g}_2 \in H^{-1/2}(\Gamma)$ be, respectively, extensions of g_1 and g_2 to Γ. By the trace theorem (see [20] and Lemma 2.11) there exists an $h \in H^1(\Omega)$ with $h|_\Gamma = \tilde{g}_1$ and $\partial h/\partial n|_\Gamma = \tilde{g}_2$. Define $w \in u - h \in H^1(\Omega)$. Then from (1.7) follows

$$(1.8)\qquad \int_\Omega \nabla w \cdot \nabla v\,dx = -\int_\Omega \nabla h \cdot \nabla v\,dx + \langle g_2, v|_{\Gamma_2}\rangle_{H^{-1/2}(\Gamma_2)\times \tilde{H}^{1/2}(\Gamma_2)}.$$

The right-hand side of (1.8) defines a bounded linear functional on $v \in V$. The variational problem now reads: Find $w \in V$ such that (1.8) holds for each $v \in V$.

By the coercivity of the Dirichlet bilinear form on V and the Lax–Milgram theorem this problem has a unique solution w.

If the data are smoother, e.g., $(g_1, g_2) \in H^{3/2}(\Gamma_1) \times H^{1/2}(\Gamma_2)$; then $h \in H^2(\Omega)$ and by using Green's formula again the right-hand side of (1.8) can be rewritten as

$$(1.9)\qquad \int_\Omega \nabla w \cdot \nabla v\,dx = \int_\Omega \Delta h w\,dx$$

because, on Γ_2, $\partial h/\partial n = g_2$.

(1.9) is the weak formulation of (P) which is of the same form as the one used by Grisvard in the mixed problem for the inhomogeneous Poisson equation with homogeneous boundary values.

1.4. For the solution of the variational problem the following regularity result holds:

THEOREM 1.4. *Let $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times \tilde{H}^{-1/2}(\Gamma_3)$ be such that $g_1|_{\Gamma^j} \in H^s(\Gamma^j)$ for $\Gamma^j \subset \Gamma_1$ and g_1 continuous at t_j for $j \in D$ and $g_2|_{\Gamma^j} \in H^{s-1}(\Gamma^j)$ for $\Gamma^j \subset \Gamma_2$. Let $s > \frac{1}{2}$ and $(s-\frac{1}{2})\omega_j/\pi \notin \mathbf{Z}$ for $j \in N \cup D$ and $(s-\frac{1}{2})\omega_j/\pi + \frac{1}{2} \notin \mathbf{Z}$ for $j \in M$.*

Then the weak solution $u \in H^1(\Omega)$ admits a decomposition

$$(1.10)\qquad u = \sum_{k=1}^{r} c_k u_k + u_0$$

where $u_0 \in H^{s+1/2}(\Omega)$ *and* u_k $(k = 1, \ldots, r)$ *are singular functions not depending on* g_1, g_2, *which are described below. The following a priori estimate holds:*

$$(1.11)\qquad \|u_0\|_{H^{s+1/2}(\Omega)} + \sum_{k=1}^{r} |c_k| \leqslant C\Big(\|g_1\|_{H^s(\Gamma_1)} + \sum_{\Gamma^j \subset \Gamma_2} \|g_2\|_{H^{s-1}(\Gamma^j)}\Big).$$

To each corner point t_j there belongs a set of singular functions $u_{j,l}$, $l = 1, 2, 3, \ldots$:

Define

$$\alpha_{jl} := \begin{cases} l\pi/\omega_j & \text{for } j \in D \cup N, \\ (l-\frac{1}{2})\dfrac{\pi}{\omega_j} & \text{for } j \in M. \end{cases}$$

Then for $\alpha_{jl} \notin N$

$$(1.12)\quad u_{jl}(x) = \begin{cases} \varrho_j^{\alpha_{jl}} \sin \alpha_{jl}\phi_j & \text{for } j \in D \text{ and for } j \in M \text{ with } \Gamma^j \subset \Gamma_2, \\ \varrho_j^{\alpha_{jl}} \cos \alpha_{jl}\phi_j & \text{for } j \in N \text{ and for } j \in M \text{ with } \Gamma^j \subset \Gamma_1, \end{cases}$$

whereas for $\alpha_{jl} \in N$

(1.13) $u_{jl}(x)$

$$= \begin{cases} \varrho_j^{\alpha_{jl}} \log \varrho \sin \alpha_{jl}\phi_j + \phi_j \varrho_j^{\alpha_{jl}} \cos \alpha_{jl}\phi_j & \text{for } j \in D \text{ and for } j \in M \text{ with } \Gamma^j \subset \Gamma_2, \\ \varrho_j^{\alpha_{jl}} \log \varrho \cos \alpha_{jl}\phi - \phi_j \varrho_j^{\alpha_{jl}} \sin \alpha_{jl}\phi_j & \text{for } j \in N \text{ and for } j \in M \text{ with } \Gamma^j \subset \Gamma_1. \end{cases}$$

Here (ϱ_j, ϕ_j) are positively oriented local polar coordinates at the vertex t_j such that $\phi_j = 0$ on Γ^{j+1} and $\phi_j = \omega_j$ on Γ^j.

The functions u_k in (1.10) are constructed from the $u_{j,l}$ by multiplication by a C^∞-cut off function which is 1 near t_j and whose support does not meet $\Gamma^{j'}$ for $j' \notin \{j, j+1\}$.

In (1.10) we find exactly those functions $u_{j,l}$ for which

$$1 \leqslant l < \frac{\omega_j}{\pi}(s - 1/2) \qquad \text{for} \quad j \in D \cup N,$$

$$1 \leqslant l < \frac{\omega_j}{\pi}(s - 1/2) + \tfrac{1}{2} \qquad \text{for} \quad j \in M.$$

Remark 1.5. The general form (1.10) of this theorem, which goes back to Kondratiev [31], is well known (see, e.g., [5], Th. 8.3.1, p. 271, and [29], Th. 1, p. 593 and p. 598 ff.). Proofs for special cases are given by Grisvard [18], Th. 2, $(s = 3/2$, also the $W^{s,p}$-case for $p \neq 2$ is treated there; ([20], Th. 5.1.3.5, p. 5.1.3.4) $(s - 1/2 \in N$, also for oblique derivatives and $p \neq 2$. The explicit form of the $u_{j,l}$ was also derived by Raugel [48].

The general case for noninteger $s \geqslant 3/2$ can be derived from the results of § 4 in connection with the result for integer $s - 1/2$. The singular functions

$u_{j,l}$ in (1.13) are solutions of nonhomogeneous problems with smooth (polynomial) boundary data.

For the present case we want to use a simple consequence of Theorem 1.4: In order to derive the boundary integral equations for the solution of the variational problem we have to satisfy the assumptions of Lemma 1.3:

COROLLARY 1.6. *Let* $s = 3/2 + k$, $k \in N_0$ *and* g_1, g_2 *satisfy the assumptions of Theorem* 1.4. *Then*

$$u|_\Gamma \in C^0(\Gamma) \quad \textit{and} \quad \frac{\partial u}{\partial n}\bigg|_\Gamma \in L^p(\Gamma), \qquad p > 1.$$

Proof. From the representation (1.10) follows $u|_\Gamma \in H^{\tilde\alpha + 1/2}(\Gamma)$ where $\tilde\alpha < \min\limits_{j,l} \alpha_{jl}$. From the definition of α_{jl} follows $\alpha_{jl} > 1/4$, and hence $\tilde\alpha = 1/4$ is possible. Thus $u|_\Gamma \in H^{3/4}(\Gamma) \subset C^0(\Gamma)$ by Sobolev's embedding theorem. For $\partial u/\partial n|_\Gamma$ we use the explicit form of $\partial u_{jl}/\partial n = c\varrho_j^{\alpha_{jl}-1} \in L^p$, $p > 1$ and $u_0 \in H^{s+1/2}(\Omega) \subset H^2(\Omega)$ whence $\partial u_0/\partial n|_\Gamma \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$. ∎

1.5. Let $u \in H^1(\Omega)$ be the variational solution of (P). Then by Lemma 1.2 we have the representation formula

$$(1.15)\quad u(z) = \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z - \zeta|\, ds_\zeta - \frac{1}{2\pi} \int_\Gamma \frac{\partial u(\zeta)}{\partial n} \log|z - \zeta|\, ds_\zeta \qquad (z \in \Omega).$$

Suppose now that (g_1, g_2) satisfy the assumptions of Theorem 1.4 for some $s = 3/2 + k$, $k \in N_0$. Then, by Corollary 1.6, the densities on the right-hand side of (1.15) satisfy the assumptions of Lemma 1.3. By this lemma we can take the limits of (1.15) for $z \in \Gamma$ and get

$$u(z) = \frac{1}{2} u(z) + \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z - \zeta|\, ds_\zeta - \frac{1}{2\pi} \int_\Gamma \frac{\partial u(\zeta)}{\partial n} \log|z - \zeta|\, ds_\zeta$$

or

$$(1.16)\quad u(z) = \frac{1}{\pi} \int_\Gamma u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z - \zeta|\, ds_\zeta - \frac{1}{\pi} \int_\Gamma \frac{\partial u(\zeta)}{\partial n} \log|z - \zeta|\, ds_\zeta \qquad (z \in \Gamma).$$

If we introduce the integral operators of the single layer potential

$$Vg(z) := -\frac{1}{\pi} \int_\Gamma g(\zeta) \log|z - \zeta|\, ds_\zeta \qquad (z \in \Gamma)$$

and of the double layer potential

$$(1.17)\qquad Kg(z) := -\frac{1}{\pi} \int_\Gamma g(\zeta) \frac{\partial}{\partial n_\zeta} \log|z - \zeta|\, ds_\zeta = -\frac{1}{\pi} \int_\Gamma g(\zeta)\, d\theta_\zeta(z),$$

where $\theta_\zeta(z)$ is the angle between $\zeta - z$ and some fixed direction, then (1.16) reads

$$(1.18)\qquad (1+K)(u|_\Gamma) = V\left(\frac{\partial u}{\partial n}\bigg|_\Gamma\right).$$

If we insert into this equation the given data $g_1 = u|_{\Gamma_1}$ and $g_2 = \partial u/\partial n|_{\Gamma_2}$ and denote the unknown boundary data by $v = u|_{\Gamma_2}$ and $\psi = \partial u/\partial n|_{\Gamma_1}$, we get the system of integral eguations

$$(1.19)\qquad \begin{bmatrix} 1+K_{22} & -V_{12} \\ -K_{21} & V_{11} \end{bmatrix} \begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} -K_{12} & V_{22} \\ 1+K_{11} & -V_{21} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

where for $j,\ k = 1,\ 2$

$$K_{jk}\, g(z) = -\frac{1}{\pi}\int_{\Gamma_j} g(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\zeta \qquad (z\in\Gamma_k),$$

$$V_{jk}\, g(z) = -\frac{1}{\pi}\int_{\Gamma_j} g(\zeta)\log|z-\zeta|\, ds_\zeta \qquad (z\in\Gamma_k).$$

This system, which we abbreviate by

$$(1.20)\qquad \mathscr{A}_0\, U = \mathscr{B}_0\, G,$$

is a coupled system of a Fredholm integral equation of the second kind for v on Γ_2 and an integral equation of the first kind for ψ on Γ_1. For the case of a smooth boundary curve, the mapping properties of this system have been analyzed and asymptotic error estimates for the corresponding Galerkin approximation procedure have recently been obtained in [58].

Analogously, from (1.18) one can derive boundary integral equations for different boundary conditions. If instead of the mixed Dirichlet–Neumann problem (P) we have, for example, a mixed Dirichlet–Robin problem

$$u|_{\Gamma_1} = g_1; \quad \left(\frac{\partial u}{\partial n} + du\right)\bigg|_{\Gamma_2} = g_2 \quad (d \text{ a smooth coefficient function}),$$

then instead of (1.18) we get

$$(1.21)\qquad \left[\mathscr{A}_0 + \begin{bmatrix} V_{22}\, d & 0 \\ -V_{21}\, d & 0 \end{bmatrix}\right] U = \mathscr{B}_0\, G.$$

From the results of the next section it will be clear that this differs by a compact perturbation from (1.20). Then from the general results [52] it follows that the optimal error estimates for the approximation of (1.20) imply those for (1.21) (compare also Lemma 5.2). (1.21) was approximately solved by collocation in [40].

§ 2. Mapping properties of the integrak operators

In the case of a smooth boundary it was shown in [58] that (1.20) is a strongly elliptic system of pseudodifferential equations, which means that for $\mathscr{A}_0$ there holds a Gårding inequality in some Hilbert space. This is the essential tool for the proof of convergence of Galerkin's procedure. In this section we want to show that again there is a Gårding inequality in appropriate function spaces. In contrast to the case of a smooth boundary, the integral operators in (1.20) in general are not pseudodifferential operators, so that the Fourier transform is no longer a valuable tool. It has to be replaced by the Mellin transform, which, however, naturally acts in weighted Sobolev spaces and not in the usual Sobolev spaces (cf. [31], [3], [4]). Therefore we need a detailed investigation of the connection between ordinary and weighted Sobolev spaces and the Mellin transform.

2.1. Before considering function spaces on the whole polygonal boundary Γ, we take a closer look at function spaces on a single segment or, more generally, on the semi-axis $\boldsymbol{R}_+$. The weighted Sobolev spaces on $\boldsymbol{R}_+$ which were introduced by Kondratiev [31] for integer m, have the norm

$$\|u\|^2_{\overset{0}{W}{}^m_\alpha(\boldsymbol{R}_+)} = \int_0^\infty \left(x^\alpha |D^m u(x)|^2 + x^{\alpha-2m}|u(x)|^2\right) dx.$$

The general definition and a lot of properties of $\overset{0}{W}{}^s_\alpha(\boldsymbol{R}_+)$ for $s\in\boldsymbol{R}$ can be found in [4]. We need only the case $\alpha = 0$, where we have for $s = m+\sigma$, $m\in\boldsymbol{N}_0$, $\sigma\in(0,\ 1)$ the norm

$$(2.1)\qquad \|u\|^2_{\overset{0}{W}{}^s_0(\boldsymbol{R}_+)} = |u|^2_{H^s(\boldsymbol{R}_+)} + \|x^{-s}u\|^2_{L^2(\boldsymbol{R}_+)}$$

with

$$|u|^2_{H^s(\boldsymbol{R}_+)} = \int_0^\infty\int_0^\infty |D^m u(x) - D^m u(y)|^2 |x-y|^{-(1+2\sigma)}\, dx\, dy.$$

This space coincides with $P^{s,2}$ in [18]. For $s<0$ we define $\overset{0}{W}{}^s_0$ by duality:

$$\overset{0}{W}{}^s_0(\boldsymbol{R}_+) := \left(\overset{0}{W}{}^{-s}_0(\boldsymbol{R}_+)\right)'.$$

It follows from [4] that these spaces coincide for all $s\in\boldsymbol{R}$ with the spaces $H^s_{1/2}(\boldsymbol{R}_+)$ which are defined via the Mellin transform (compare (2.4) below). The definition is as follows:

For $\phi\in C_0^\infty(0,\infty)$ the Mellin transform $\hat{\phi}$ is defined by

$$(2.2)\qquad \hat{\phi}(\lambda) := \int_{-\infty}^{\infty} e^{-i\lambda t}\phi(e^{-t})\, dt = \int_0^\infty x^{i\lambda-1}\phi(x)\, dx.$$

Thus the Mellin transform is a composition of the Euler transform $x \mapsto e^{-t}$, which maps $\boldsymbol{R}_+$ onto $\boldsymbol{R}$, and the Fourier transform. For more general distributions on $\boldsymbol{R}_+$ the Mellin transform is generalized by using the usual generalizations of the Fourier transform. The inversion formula is

$$\phi(e^{-t}) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = \text{const}} e^{i\lambda t} \hat{\phi}(\lambda)\, d\lambda. \tag{2.3}$$

EXAMPLES 2.1. (i) $\phi \in C_0^\infty(0, \infty) \Leftrightarrow \hat{\phi} \in \mathscr{Z}$, i.e. $\hat{\phi}$ is an entire function of exponential type which is rapidly decreasing for $\operatorname{Re}\lambda \to \pm\infty$.

(ii) $\chi \in C_0^\infty[0, \infty)$ with $\operatorname{supp}(1-\chi) \subset (0, \infty) \Leftrightarrow \hat{\chi}(\lambda) = \hat{\phi}(\lambda)/\lambda$ with $\hat{\phi} \in \mathscr{Z}$ ($\operatorname{Im}\lambda < 0$).

(iii) $u(x) = x^\alpha \chi(x)$ with χ as in (ii) $\Leftrightarrow \hat{u}(\lambda) = \hat{\phi}(\lambda)/\lambda - i\alpha$ with $\hat{\phi} \in \mathscr{Z}$ ($\operatorname{Im}\lambda < \alpha$).

(iv) $u(x) = x^\alpha (\log x)^l \chi(x)$ with χ as in (ii) $\Leftrightarrow \hat{u}(\lambda) = \hat{\phi}(\lambda)/(\lambda - i\alpha)^{l+1}$ with $\hat{\phi} \in \mathscr{Z}$ ($\operatorname{Im}\lambda < \alpha$, $l \in \boldsymbol{N}$).

LEMMA 2.1 (Parseval equation for the Mellin transform [4], p. 367, 373). *For $\phi \in C_0^\infty(0, \infty)$ we have the equivalence of norms*

$$\|\phi\|^2_{\overset{0}{W}{}^s(\boldsymbol{R}_+)} \sim \int_{\operatorname{Im}\lambda = s - 1/2} (1+|\lambda|^2)^s |\hat{\phi}(\lambda)|^2\, d\lambda \qquad (s \in \boldsymbol{R}), \tag{2.4}$$

and $\overset{0}{W}{}^s(\boldsymbol{R}_+)$ is the completion of $C_0^\infty(0, \infty)$ in this norm.

Since we are interested in the Sobolev spaces without weight, we shall need a similar characterization of the Sobolev space $H^{1/2}(\boldsymbol{R}_+)$ by means of the Mellin transform.

The norm in $H^{1/2}(\boldsymbol{R}_+)$ is given by

$$\|u\|^2_{H^{1/2}(\boldsymbol{R}_+)} = \|u\|^2_{L^2} + |u|^2_{H^{1/2}(\boldsymbol{R}_+)}$$

with the seminorm defined in (2.1).

LEMMA 2.3. *For $u \in C_0^\infty(0, \infty)$ holds*

(2.5)
$$|u|^2_{H^{1/2}(\boldsymbol{R}_+)} = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda=0} (\pi\lambda \coth \pi\lambda - 1)|\hat{u}(\lambda)|^2\, d\lambda \sim \int_{\operatorname{Im}\lambda=0} \frac{|\lambda|^2}{1+|\lambda|} |\hat{u}(\lambda)|^2\, d\lambda.$$

Here again $\sim$ means equivalence of norms.

Proof. By (2.1)

$$|u|^2_{H^{1/2}(\boldsymbol{R}_+)} = \int_0^\infty \int_0^\infty |u(x) - u(y)|^2 |x-y|^{-2}\, dx dy.$$

We substitute $g(t) = u(e^{-t})$ and get

$$|u|^2_{H^{1/2}(\boldsymbol{R}_+)} = \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|g(t) - g(\tau)|^2}{|e^{-t} - e^{-\tau}|^2} e^{-(t+\tau)}\, dt d\tau$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|g(t) - g(\tau)|^2}{4[\sinh((t-\tau)/2)]^2}\, dt d\tau = \tfrac{1}{4} \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{|g(t+h) - g(t)|^2}{\sinh^2(h/2)}\, dh dt.$$

Now $\hat{u} = Fg$, the Fourier transform of g. By the Parseval equation for the Fourier transform we obtain

$$\int_{-\infty}^\infty |g(t+h) - g(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^\infty |(e^{+i\lambda h} - 1)\hat{u}(\lambda)|^2\, d\lambda.$$

Thus

$$|u|^2_{H^{1/2}(\boldsymbol{R}_+)} = \frac{1}{2\pi} \int_{-\infty}^\infty |\hat{u}(\lambda)|^2 \left(\tfrac{1}{4} \int_{-\infty}^\infty \frac{|e^{i\lambda h} - 1|^2}{\sinh^2(h/2)}\, dh \right) d\lambda.$$

With

$$\tfrac{1}{4} \int_{-\infty}^\infty \frac{|e^{i\lambda h} - 1|^2}{\sinh^2(h/2)}\, dh = \int_{-\infty}^\infty \frac{\sin^2(\lambda h/2)}{\sinh^2(h/2)}\, dh = \int_{-\infty}^\infty \frac{1 - \cos\lambda h}{\cosh h - 1}\, dh = \pi\lambda \coth \pi\lambda - 1$$

by [49], 3.583.2, we obtain (2.5). ∎

COROLLARY 2.4. *There exists a $C > 0$ such that for $u \in C_0^\infty(0, \infty)$*

$$\|u\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)} \leqslant C \int_{\operatorname{Im}\lambda=0} \frac{1+|\lambda|}{|\lambda|^2} |\hat{u}(\lambda - i)|^2\, d\lambda. \tag{2.6}$$

if the integral exists.

Proof. By definition $\tilde{H}^{-1/2}(\boldsymbol{R}_+) = H^{1/2}(\boldsymbol{R}_+)'$. Hence

$$\|u\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)} = \sup_{\psi \in C_0^\infty(0,\infty)} \frac{|\langle u, \psi \rangle_{L^2(\boldsymbol{R}_+)}|^2}{\|\psi\|^2_{H^{1/2}(\boldsymbol{R}_+)}} \leqslant \sup_\psi \frac{|\langle u, \psi \rangle|^2}{|\psi|^2_{1/2}}.$$

Now by (2.5) and the Cauchy–Schwarz inequality we have

$$|\langle u, \psi \rangle|^2 = \frac{1}{2\pi} \left| \int_{\operatorname{Im}\lambda = -1/2} \hat{u}(\lambda) \overline{\hat{\psi}(\bar{\lambda})}\, d\lambda \right|^2 = \frac{1}{2\pi} \left| \int_{\operatorname{Im}\lambda=0} \hat{u}(\lambda - i) \overline{\hat{\psi}(\lambda)}\, d\lambda \right|^2$$

$$\leq \frac{1}{2\pi}\int_{\mathrm{Im}\,\lambda=0}\frac{1+|\lambda|}{|\lambda|^2}|\hat u(\lambda-i)|^2\,d\lambda \int_{\mathrm{Im}\,\lambda=0}\frac{|\lambda|^2}{1+|\lambda|}|\hat\psi(\lambda)|^2\,d\lambda$$

$$\leq C|\psi|^2_{1/2}\int_{\mathrm{Im}\,\lambda=0}\frac{1+|\lambda|}{|\lambda|^2}|\hat u(\lambda-i)|^2\,d\lambda,$$

where the path of integration has been shifted according to the analyticity and rapid decay of $\hat u$ and $\hat\psi$.

The weighted Sobolev spaces $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$, $s\in\boldsymbol{R}$, have the interpolation property

$$[\overset{0}{W}{}^{s_0}_0(\boldsymbol{R}_+), \overset{0}{W}{}^{s_1}_0(\boldsymbol{R}_+)]_\theta = \overset{0}{W}{}^s_0(\boldsymbol{R}_+) \tag{2.7}$$

$$(s_0, s_1\in\boldsymbol{R},\ \theta\in[0,1],\ s=(1-\theta)_{s_0}+\theta s_1).$$

Here $[\cdot,\cdot]_\theta$ is the complex or the real $(\theta, 2)$ interpolation functor. The proof of (2.7) follows from [54], Theorem 3.4.2, p. 275.

Remark 2.5. It is easy to see that $\{\overset{0}{W}{}^s_\alpha \mid \alpha\in\boldsymbol{R}\}$ for fixed s and $\{\overset{0}{W}{}^s_{s+2\alpha} \mid s\in\boldsymbol{R}\}$ for fixed α are interpolation scales, but the corresponding (semi-) groups do not commute, so that (2.6) is indeed nontrivial.

On the semi-axis $\boldsymbol{R}_+$ we have the following relation between the weighted Sobolev spaces $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$ and the Sobolev spaces $\tilde H^s(\boldsymbol{R}_+)$ of $H^s(\boldsymbol{R}_+)$-functions possessing an extension by 0 on $\boldsymbol{R}_-$ in $H^s(\boldsymbol{R})$. (See Definition 1.1.)

LEMMA 2.6. *Let* $\chi\in C_0^\infty[0,\infty)$. *Then the mapping* $u\mapsto\chi u$ *is continuous from* $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$ *in* $\tilde H^s(\boldsymbol{R}_+)$ *and from* $\tilde H^s(\boldsymbol{R}_+)$ *in* $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$ *for* $s\geq 0$. *This means that the norms of* $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$ *and* $\tilde H^s(\boldsymbol{R}_+)$ *are equivalent on compact intervals.*

For $s\leq 0$ *the mapping* $u\mapsto\chi u$ *is continuous from* $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$ *in* $H^s(\boldsymbol{R}_+)$ *and from* $H^s(\boldsymbol{R}_+)$ *in* $\overset{0}{W}{}^s_0(\boldsymbol{R}_+)$.

The proof of the lemma follows directly from [54], (4.3.2/7), or from Thm. 118, p. 69 in [36] by interpolation and duality.

Now we want to describe the Sobolev spaces $H^s(\Gamma)$ and $H^s(\Gamma_k)$ as defined in (1.1). To this end use a partition of unity $(\chi_1,\dots,\chi_J)$ with the following properties:

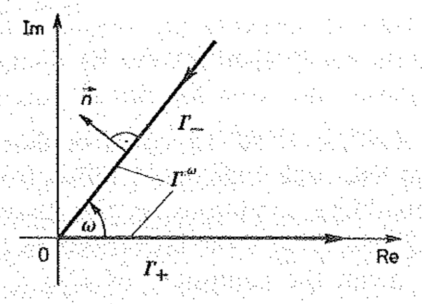
(2.8) χ_j is the restriction of a $C_0^\infty(\boldsymbol{R}^2)$-function to Γ,
$\chi_j\equiv 1$ in a neighbourhood of the vertex t_j, and
$\operatorname{supp}\chi_j\subset\Gamma^j\cup\{t_j\}\cup\Gamma^{j+1}$.

For every function u on Γ we then have

$$u=\sum_{j=1}^J\chi_j\cdot u, \tag{2.9}$$

so that $\chi_j\cdot u$ is the "local representation" of u at t_j. Each $\chi_j\cdot u$ has its support on the angle $\Gamma^j\cup\{t_j\}\cup\Gamma^{j+1}$. By means of an affine transformation of variables this angle can be considered as a part of an infinite angle Γ^{ω_j} with sides $\Gamma_-=e^{i\omega_j}\cdot\boldsymbol{R}_+$ corresponding to Γ^j and $\Gamma_+=\boldsymbol{R}_+$ corresponding to Γ^{j+1}.

Thus, $\chi_j u$ can be considered in a natural way as a function on Γ^{ω_j}, and thus also as a pair $((\chi_j u)_-, (\chi_j u)_+)$ of functions on $\boldsymbol{R}_+$. We will use these natural identifications without further mention if no confusion is possible.



Besides the decomposition (2.9) we can also consider the restriction of u to each Γ^j and in this way obtain a natural identification of functions on Γ and J-tuples of functions on the segments Γ^j, $j=1,\dots,J$. With these identifications we have the following description of $H^s(\Gamma)$.

LEMMA 2.7 (Grisvard [16], [20], § 1, 5, 2, [15]).

(i) *For* $|s|<\frac12$: $H^s(\Gamma)=\prod_{j=1}^J H^s(\Gamma^j)$.

(ii) *For* $s\in[\frac12,\frac32)$: $H^s(\Gamma)=\{u\in\prod_{j=1}^J H^s(\Gamma^j) \mid (\chi_j u)_+\in H^s(\boldsymbol{R}_+)$ and $(\chi_j u)_+-(\chi_j u)_-\in\tilde H^s(\boldsymbol{R}_+)\ (j=1,\dots,J)\}$.

$$\|u\|^2_{H^s(\Gamma)}\sim\sum_{j=1}^J\{\|(\chi_j u)_+\|^2_{H^s(\boldsymbol{R}_+)}+\|(\chi_j u)_-\|^2_{H^s(\boldsymbol{R}_+)}+\|(\chi_j u)_+-(\chi_j u)_-\|^2_{\tilde H^s(\boldsymbol{R}_+)}\}$$

(iii) *For* $s>\frac12$: $H^s(\Gamma)=\{u\in\prod_{j=1}^J H^s(\Gamma^j) \mid (\chi_j u)_+\in H^s(\boldsymbol{R}_+)$ and $(\chi_j u)_+(0)=(\chi_j u)_-(0)$ for $\omega_j\neq\pi$ and $(\chi_j u)_+-(\chi_j u)_-\in\tilde H^s(\boldsymbol{R}_+)$ for $\omega_j=\pi\}$.

COROLLARY 2.8. *Let* $\tau\mapsto z(\tau)$, $\tau\in[0,L]$ *be the parametrization of* Γ_1 *by the arc length. Then* $z^*\colon H^s(\Gamma_1)\to H^s[0,L]$ *defined by* $z^*f=f\circ z$ *is an isomorphism for* $|s|<3/2$. $H^s(\Gamma^\omega)$ *does not depend on* $\omega\in(0,2\pi)$ *for* $|s|<3/2$.

Proof. If $[0, L] = \bigcup_{j=1}^{n} \overline{J_j}$; i.e. $\Gamma^j = z(J_j)$, then for $0 \leqslant s < 3/2$ $H^s[0, L] \subset \prod_{j=1}^{n} H^s(J_j)$ is the subspace which is characterized by compatibility conditions equivalent to those stated in (i) and (ii) of the lemma:
no conditions for $0 \leqslant s < 1/2$;
continuity on all of $[0, L]$ for $1/2 < s < 3/2$; and for $s = 1/2$:

$$\int_0^{\varepsilon} |f_k(\tau_k+\tau) - f_{k+1}(\tau_k-\tau)|^2 \frac{d\tau}{\tau} < \infty, \quad \text{where} \quad f|_{J_j} = f_j; j = k, k+1;$$

$$\tau_k = \overline{J_k} \cap \overline{J_{k+1}}; \text{ and } \varepsilon > 0 \text{ is small enough.}$$

The condition for $s = 1/2$ just means $(\chi_k u)_+ - (\chi_k u)_- \in \tilde{H}^{1/2}(\boldsymbol{R}_+)$ for $f = z^* u$.

For $-3/2 < s < 0$ the result then follows by duality, because the result for $0 \leqslant s < 3/2$ is obviously also true for the $\tilde{H}^s$-spaces ∎

Remark 2.9. For $s \geqslant 3/2$, $H^s[0, L]$ contains more compatibility conditions than $H^s(\Gamma_1)$ if at least one angle in Γ_1 differs from π. For $\omega_j = \pi$ and $s \geqslant 3/2$ there are compatibility conditions on the derivatives, too, cf. (iii) of the lemma. For $s > 1/2$ and $\omega_j \neq \pi$ the conditions in (ii) or equivalently in (iii) of the lemma just mean the continuity of u at t_j.

COROLLARY 2.10. *The spaces $H^s(\Gamma_1)$ and $\tilde{H}^s(\Gamma_1)$ for $|s| < 3/2$ have the interpolation property, i.e.,*

$[H^{s_0}(\Gamma_1), H^{s_1}(\Gamma_1)]_\theta = H^s(\Gamma_1)$;

$[\tilde{H}^{s_0}(\Gamma_1), \tilde{H}^{s_1}(\Gamma_1)]_\theta = \tilde{H}^s(\Gamma_1)$ $(\theta \in [0, 1]; s_0, s_1 \in (-\frac{3}{2}, \frac{3}{2}); s = \theta s_1 + (1-\theta) s_0)$.

Proof. The spaces $H^s[0, L]$ and $\tilde{H}^s[0, L]$ have this property. ∎

LEMMA 2.11 (Grisvard [20], § 1.5.2, [16], [17], [19], or [48]).

(i) *The mapping* $u \mapsto \left.\frac{\partial u}{\partial n}\right|_\Gamma$ *is surjective from* $H^{s+1/2}(\Omega)$ *onto* $\prod_{j=1}^{J} H^{s-1}(\Gamma^j)$ *for* $s > 1$ *if* $\omega_j \neq \pi$ $(j = 1, \ldots, J)$.

(ii) *The mapping* $u \mapsto \left(u|_\Gamma, \left.\frac{\partial u}{\partial n}\right|_\Gamma\right)$ *is surjective from* $H^{s+1/2}(\Omega)$ *onto the subspace of* $H^s(\Gamma^j) \times \prod_{j=1}^{J} H^{s-1}(\Gamma^j)$ *which is characterized by the conditions*

$$\left.\begin{array}{lll} \left(\chi_j \dfrac{\partial u}{\partial n}\right)_-(0) = -\dfrac{d}{dx}(\chi_j u)_+(0) & \textit{and} & \\ \left(\chi_j \dfrac{\partial u}{\partial n}\right)_+(0) = -\dfrac{d}{dx}(\chi_j u)_-(0) & \textit{if} \quad \omega_j = \dfrac{\pi}{2}, & \textit{and} \\ \left(\chi_j \dfrac{\partial u}{\partial n}\right)_\pm(0) = \dfrac{d}{dx}(\chi_j u)_\pm(0) & \textit{if} \quad \omega_j = \dfrac{3\pi}{2} & \end{array}\right\} \textit{for } s > 3/2,$$

and

$$\left.\begin{array}{ll} \left(\chi_j \dfrac{\partial u}{\partial n}\right)_\pm + \dfrac{d}{dx}(\chi_j u)_\pm \in \tilde{H}^{s-1}(\boldsymbol{R}_+), & \omega_j = \dfrac{\pi}{2} \\ \left(\chi_j \dfrac{\partial u}{\partial n}\right)_\pm - \dfrac{d}{dx}(\chi_j u)_\pm \in \tilde{H}^{s-1}(\boldsymbol{R}_+), & \omega_j = \dfrac{3\pi}{2} \end{array}\right\} \textit{for } \frac{3}{2} \leqslant s < \frac{5}{2}. \tag{2.10}$$

(For $1 < s < 3/2$ there are no conditions, and for $3/2 < s < 5/2$ the two sets of conditions coincide.)

LEMMA 2.12. *For* $0 \leqslant s < 3/2$:

(i) $H^s(\Gamma^\omega) = \{u = (u_-, u_+) \in H^s(\boldsymbol{R}_+)^2 |\ u_- - u_+ \in \tilde{H}^s(\boldsymbol{R}_+)\}$;

(ii) $H^{-s}(\Gamma^\omega) = \{u = (u_-, u_+) \in H^{-s}(\boldsymbol{R}_+)^2 |\ u_- + u_+ \in \tilde{H}^{-s}(\boldsymbol{R}_+)\}$.

Furthermore, the mapping

$$D\colon (u_-, u_+) \mapsto (u_- + u_+, u_- - u_+) \tag{2.11}$$

is an isomorphism

$$D\colon H^s(\Gamma^\omega) \to H^s(\boldsymbol{R}_+) \times \tilde{H}^s(\boldsymbol{R}_+) \textit{ and}$$
$$D\colon H^{-s}(\Gamma^\omega) \to \tilde{H}^{-s}(\boldsymbol{R}_+) \times H^{-s}(\boldsymbol{R}_+).$$

Proof. (i) is a simple consequence of Lemma 2.7 (ii), cf. [16]. This means that $D\colon H^s(\Gamma^\omega) \to H^s(\boldsymbol{R}_+) \times \tilde{H}^s(\boldsymbol{R}_+)$ is an isomorphism. If we note that $H^{-s}(\Gamma^\omega) = \tilde{H}^{-s}(\Gamma^\omega) = [H^s(\Gamma^\omega)]'$ and $D' = D$, the result (ii) follows by taking adjoints, whence

$$\|\phi\|^2_{H^{-s}(\Gamma^\omega)} \sim \|\phi_+ + \phi_-\|^2_{\tilde{H}^{-s}(\boldsymbol{R}_+)} + \|\phi_- - \phi_+\|^2_{H^{-s}(\boldsymbol{R}_+)}. \ ∎ \tag{2.12}$$

2.2. In order to show the mapping properties of the integral operators in system (1.20) on Γ we first investigate the operators on the reference angle Γ^ω, to which the general case will then be reduced by localization.

The natural identification of functions on Γ^ω with pairs of functions on $\boldsymbol{R}_+$ induces a natural identification of integral operators on Γ^ω with (2×2)-matrices on integral operators on $\boldsymbol{R}_+$. We denote this correspondence by $\hat{=}$.

In this way for the operator V of the single layer potential we have

$$V \hat{=} \begin{bmatrix} V_{--} & V_{+-} \\ V_{-+} & V_{++} \end{bmatrix}$$

where for $\phi \in C_0^\infty[0, \infty)$

$$V_{++}\phi(x) = -\frac{1}{\pi}\int_0^\infty \log|x-y|\,\phi(y)\,dy$$

$$= -\frac{1}{\pi}\int_0^\infty \log y\phi(y)\,dy - \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{x}{y}\right|\phi(y)\,dy$$

$$=: l\phi + V_0\phi(x);$$

(2.13) $$V_{-+}\phi(x) = -\frac{1}{\pi}\int_0^\infty \log|x-ye^{i\omega}|\,\phi(y)\,dy$$

$$= l\phi - \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{x}{y}e^{-i\omega}\right|\phi(y)\,dy =: l\phi + V_\omega\phi(x);$$

$$V_{+-} = V_{-+};\quad V_{--} = V_{++}.$$

For the operator K of the double layer potential we have

$$K \hat{=} \begin{bmatrix} K_{--} & K_{+-}\\ K_{-+} & K_{++}\end{bmatrix} \quad \text{with} \quad K_{--}=K_{++}=0$$

because of the geometric interpretation of the double layer potential by the change of the angle (cf. (1.17)).

(2.14) $$K_{+-}\phi(x) = \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{1}{xe^{i\omega}-y}\right)\phi(y)\,dy =: K_\omega\phi(x),$$
$$K_{-+} = K_{+-} = K_\omega.$$

In short, we have on Γ^ω

(2.15) $$V = \begin{bmatrix} l & l\\ l & l\end{bmatrix} + \begin{bmatrix} V_0 & V_\omega \\ V_\omega & V_0\end{bmatrix},\qquad K = \begin{bmatrix}0 & K_\omega\\ K_\omega & 0\end{bmatrix}.$$

The decomposition of V is made in such a way that the kernels of the operators V_0, V_ω are homogeneous functions of degree 0, so that the Mellin transform will convert them into multiplication operators (up to a shift). For the double layer potential K this Mellin transform was also calculated by Fabes, Jodeit, and Lewis in [11].

Lemma 2.13. (i) *Let* $\phi\in C_0^\infty[0,\infty)$. *Then*

(2.16) $$\left.\begin{aligned}\widehat{V_0\phi}(\lambda) &= \hat{V}_0(\lambda)\hat\phi(\lambda-i) := \frac{\cosh\pi\lambda}{\lambda\sinh\pi\lambda}\hat\phi(\lambda-i)\\ \widehat{V_\omega\phi}(\lambda) &= \hat{V}_\omega(\lambda)\hat\phi(\lambda-i) := \frac{\cosh(\pi-\omega)\lambda}{\lambda\sinh\pi\lambda}\hat\phi(\lambda-i)\end{aligned}\right\}\quad (\mathrm{Im}\,\lambda\in(0,1)).$$

(ii) *Let* $\phi\in C_0^\infty(0,\infty)$. *Then*

(2.17) $$\widehat{K_\omega\phi}(\lambda) = \hat{K}_\omega(\lambda)\hat\phi(\lambda) := -\frac{\sinh(\pi-\omega)\lambda}{\sinh\pi\lambda}\hat\phi(\lambda)\quad (\mathrm{Im}\,\lambda\in(-1,1)).$$

Proof. We have to use the formula (see, e.g., [9], [50])

(2.18) $$\int_{-\infty}^\infty \frac{e^{-i\lambda\tau}\,d\tau}{e^{-\tau\pm i\omega}-1} = i\pi\frac{e^{\pm\lambda(\omega-\pi)}}{\sinh\pi\lambda}\quad (\omega\in(0,2\pi),\ \mathrm{Im}\,\lambda\in(-1,0)).$$

We use the definition of the Mellin transform via the Fourier transform and then the convolution theorem for the Fourier transform.

For K_ω we obtain

$$K_\omega\phi(e^{-t}) = \frac{1}{\pi}\int_{-\infty}^\infty \mathrm{Im}\left(\frac{1}{e^{-t}e^{i\omega}-e^{-\tau}}\right)\phi(e^{-\tau})e^{-\tau}\,d\tau$$
$$= \frac{1}{2\pi i}\int_{-\infty}^\infty\left(\frac{1}{e^{-(t-\tau)+i\omega}-1} - \frac{1}{e^{-(t-\tau)-i\omega}-1}\right)\phi(e^{-\tau})\,d\tau$$
$$= \int_{-\infty}^\infty f(t-\tau)\phi(e^{-\tau})\,d\tau,$$

where (by (2.18))

$$Ff(\lambda) = \int_{-\infty}^\infty e^{-i\lambda t}f(t)\,dt = \frac{1}{2\pi i}\left(i\pi\frac{e^{\lambda(\omega-\pi)}}{\sinh\pi\lambda} - i\pi\frac{e^{-\lambda(\omega-\pi)}}{\sinh\pi\lambda}\right)$$
$$= -\frac{\sinh(\pi-\omega)\lambda}{\sinh\pi\lambda} = \hat{K}_\omega(\lambda)\quad (\mathrm{Im}\,\lambda\in(-1,0)).$$

From $\mathrm{Im}\big(1/(e^{s+i\omega}-1)\big) = \mathrm{Im}\big(1/(e^{-s+i\omega}-1)\big)$ it follows that f is an even function, and thus $Ff(\lambda) = \hat{K}_\omega(\lambda)$ also for $\mathrm{Im}\,\lambda\in(0,1)$. By the convolution theorem we have $\widehat{K_\omega\phi}(\lambda) = Ff(\lambda)\cdot\hat\phi(\lambda)$, whence (2.17). For V_ω we write

$$V_\omega\phi(e^{-t}) = -\frac{1}{\pi}\int_{-\infty}^\infty \log|1-e^{\tau-t-i\omega}|\,\phi(e^{-\tau})e^{-\tau}\,d\tau = \int_{-\infty}^\infty f(t-\tau)g(\tau)\,d\tau.$$

Here

$$Fg(\lambda) = \int_{-\infty}^\infty e^{-i\lambda\tau}e^{-\tau}\phi(e^{-\tau})\,d\tau = \hat\phi(\lambda-i),$$

and by integration by parts and the use of (2.18)

$$Ff(\lambda) = \int_{-\infty}^\infty e^{-i\lambda t}\cdot\left(-\frac{1}{\pi}\right)\mathrm{Re}\big(\log(1-e^{-t-i\omega})\big)\,dt$$
$$= -\int_{-\infty}^\infty \frac{1}{i\pi\lambda}e^{-i\lambda t}\cdot\mathrm{Re}\frac{e^{-t-i\omega}}{1-e^{-t-i\omega}}\,dt$$
$$= \frac{\cosh(\lambda-\omega)\lambda}{\lambda\sinh\pi\lambda} = \hat{V}_\omega(\lambda)\quad\text{for}\quad \mathrm{Im}\,\lambda\in(-1,0).$$

This gives (2.16) if we notice that V_0 is a special case of V_ω. ∎

COROLLARY 2.14.

$$V_0 \text{ and } V_\omega\colon \overset{0}{W}{}_0^s(\mathbf{R}_+) \to \overset{0}{W}{}_0^{s+1}(\mathbf{R}_+) \quad \text{for} \quad |s| < \tfrac{1}{2};$$

$$K_\omega\colon \overset{0}{W}{}_0^s(\mathbf{R}_+) \to \overset{0}{W}{}_0^s(\mathbf{R}_+) \quad \text{for} \quad s \in (-\tfrac{1}{2}, \tfrac{3}{2})$$

are continuous.

Proof. This follows immediately from the Parseval equation (2.4) and the fact that $|\hat{V}_\omega(\lambda)| \sim 1/(1+|\lambda|)$ on each line $\operatorname{Im}\lambda = h \in (-1, 0)$ and $\hat{K}_\omega(\lambda)$ bounded on $\operatorname{Im}\lambda = h \in (-1, 1)$. For example,

$$\begin{aligned}
\|V_\omega \phi\|^2_{\overset{0}{W}{}_0^{s+1}(\mathbf{R}_+)} &\sim \int_{\operatorname{Im}\lambda = s+1/2} (1+|\lambda|^2)^{s+1} |\hat{V}_\omega(\lambda)|^2 |\hat{\phi}(\lambda - i)|^2 \, d\lambda \\
&\leqslant C \int_{\operatorname{Im}\lambda = s+1/2} (1+|\lambda|^2)^s |\hat{\phi}(\lambda - i|^2 \, d\lambda \\
&\sim \int_{\operatorname{Im}\lambda = s-1/2} (1+|\lambda|^2)^s |\hat{\phi}(\lambda)|^2 \, d\lambda \\
&\sim \|\phi\|^2_{\overset{0}{W}{}_0^s(\mathbf{R}_+)}. \quad ∎
\end{aligned}$$

Now we can prove the continuity of the operators V and K in the Sobolev spaces without weights. The exact distinction between H^s and $\tilde{H}^s$ is crucial for the proof of the compatibility conditions for the image of the operators on Γ.

LEMMA 2.15. *Let* $\chi \in C_0^\infty[0, \infty)$ *with* $\operatorname{supp}(1-\chi) \subset (0, \infty)$. *Then the mappings*

$$\begin{aligned}
&u \mapsto \chi(l + V_\omega)\chi u\colon \tilde{H}^s(\mathbf{R}_+) \to H^{s+1}(\mathbf{R}_+) && \text{for} \quad s \in (-3/2, 1/2);\\
&u \mapsto \chi(l + V_0)\chi u\colon \tilde{H}^s(\mathbf{R}_+) \to H^{s+1}(\mathbf{R}_+) && \text{for} \quad s \in (-3/2, 1/2);\\
&u \mapsto \chi(V_0 - V_\omega)\chi\colon H^s(\mathbf{R}_+) \to \tilde{H}^{s+1}(\mathbf{R}_+) && \text{for} \quad s \in (-3/2, 1/2);\\
&u \mapsto \chi K_\omega \chi u \begin{cases} : \tilde{H}^s(\mathbf{R}_+) \to \tilde{H}^s(\mathbf{R}_+) \\ : H^s(\mathbf{R}_+) \to H^s(\mathbf{R}_+) \end{cases} && \text{for} \quad s \in (-1/2, 3/2);
\end{aligned}$$

are continuous.

Proof. Let $A\colon u \mapsto \chi(l + V_\omega)\chi u$ be defined on $L^2(\mathbf{R}_+)$. We first show the continuity of A from $\tilde{H}^s(\mathbf{R}_+) \to H^{s+1}(\mathbf{R}_+)$ for $s \in [0, 1/2)$: $u \mapsto l(\chi u)$ is by (2.13) a continuous linear functional on $H^s(\mathbf{R}_+)$ for all $s > -1/2$. (But $\chi(x) \cdot \log x \notin H^{1/2}(\mathbf{R}_+)$!). So $u \mapsto \chi \cdot l(\chi u)$: $H^s(\mathbf{R}_+) \to H^t(\mathbf{R}_+)$ is continuous for $s > -1/2$ and all t. By Corollary 2.14 and the local equivalence of norms in $\tilde{H}^s$ and $\overset{0}{W}{}_0^s$ (Lemma 2.6) $\chi V_\omega \chi$: $\tilde{H}^s(\mathbf{R}_+) \to \tilde{H}^{s+1}(\mathbf{R}_+)$ is continuous for $s \in [0, 1/2)$. Thus $A\colon \tilde{H}^s(\mathbf{R}_+) \to H^{s+1}(\mathbf{R}_+)$ is continuous for $s \in [0, 1/2)$.

Now A is a self-adjoint operator in $L^2(\mathbf{R}_+)$. The adjoint operator A' is continuous from $\tilde{H}^{-s-1}(\mathbf{R}_+)$ into $H^{-s}(\mathbf{R}_+)$ $(0 \leqslant s < 1/2)$ and coincides on the dense subspace $L^2(\mathbf{R}_+)$ with A. Therefore we can use interpolation and obtain for every $0 \leqslant s < 1/2$ and $\theta \in [0, 1]$:

A: $[\tilde{H}^s(\mathbf{R}_+), \tilde{H}^{-s-1}(\mathbf{R}_+)]_\theta \to [H^{s+1}(\mathbf{R}_+), H^{-s}(\mathbf{R}_+)]_\theta$ is continuous. By the interpolation property of the spaces $H^s(\mathbf{R}_+)$ and $\tilde{H}^s(\mathbf{R}_+)$ we obtain A: $\tilde{H}^r(\mathbf{R}_+) \to H^{r+1}(\mathbf{R})$ continuous for all $r \in [-s-1, s]$, hence for all $r \in \bigcup_{0 \leqslant s < 1/2} -[s-1, s] = (-3/2, 1/2)$.

Let A_0: $u \mapsto \chi(l + V_0)\chi u$. Then we have $A_0 - A$: $u \mapsto \chi(V_0 - V_\omega)\chi u$ and $A_0 - A$ is self-adjoint.

As $\chi V_\omega \chi$: $\tilde{H}^r(\mathbf{R}_+) \to \tilde{H}^{r+1}(\mathbf{R}_+)$ is continuous for $r \in [0, 1/2)$, we get $A_0 - A$: $\tilde{H}^r(\mathbf{R}_+) = H^r(\mathbf{R}_+) \to \tilde{H}^{r+1}(\mathbf{R}_+)$ cont. for $r \in [0, 1/2)$. By taking adjoints we obtain

$$A_0 - A\colon H^{-r-1}(\mathbf{R}_+) \to \tilde{H}^{-r}(\mathbf{R}_+) \quad \text{cont. for } r \in [0, 1/2).$$

By interpolation we obtain as above

$$A_0 - A\colon H^s(\mathbf{R}_+) \to \tilde{H}^{s+1}(\mathbf{R}_+) \quad \text{cont. for } s \in [-r-1, r]$$

and all $r \in [0, 1/2)$, and thus for all $s \in (-3/2, 1/2)$.

Now let B: $u \mapsto \chi K_\omega \chi u$ be defined on $L^2(\mathbf{R}_+)$. From the equivalence of norms and Corollary 2.10 we know that B: $\tilde{H}^s(\mathbf{R}_+) \to \tilde{H}^s(\mathbf{R}_+)$ is continuous for $s \in (-1/2, 3/2)$. For $s \in (-1/2, 1/2)$ the spaces $H^s(\mathbf{R}_+)$ and $\tilde{H}^s(\mathbf{R}_+)$ are identical. Therefore, if we prove that B: $H^s(\mathbf{R}_+) \to H^s(\mathbf{R}_+)$ is continuous for $s \in (1/2, 3/2)$, then the result for the whole range will follow by interpolation. Thus let $s \in (1/2, 3/2)$. Choose $\phi \in C_0^\infty[0, \infty)$ with $\operatorname{supp}(1-\phi) \subset (0, \infty)$. Then for $u \in H^s(\mathbf{R}_+)$ we have the decomposition

$$u = u(0) \cdot \phi + u_0 \quad \text{with} \quad u_0 \in \tilde{H}^s(\mathbf{R}_+). \tag{2.19}$$

where $u \mapsto u(0)\phi$: $H^2(\mathbf{R}_+) \to C_0^\infty[0, \infty)$ is continuous. Then

$$Bu = u(0) \cdot B\phi + Bu_0.$$

We know that $u \mapsto Bu_0$: $H^s(\mathbf{R}_+) \to \tilde{H}^s(\mathbf{R}_+)$, and hence into $H^s(\mathbf{R}_+)$, is continuous. It remains to show that $B\phi \in H^s(\mathbf{R}_+)$, where $B\phi = \chi K_\omega \chi \phi$. Now $\chi\phi \equiv 1$ on some interval $[0, \varepsilon]$. Let χ_0 be the characteristic function of this interval. Then $\chi\phi = \chi_0 + (\chi\phi - \chi_0)$. The function $\chi\phi - \chi_0$ vanishes on $[0, \varepsilon]$. Therefore $K_\omega(\chi\phi - \chi_0) \in C^\infty[0, \infty)$. Furthermore

$$K_\omega \chi_0(x) = \frac{1}{\pi} \int_0^\varepsilon \operatorname{Im}\left(\frac{1}{xe^{i\omega} - \xi}\right) d\xi = -\frac{1}{\pi} \arg\left(\frac{xe^{i\omega} - \varepsilon}{xe^{i\omega}}\right).$$

For fixed ω this is continuous on $[0, \infty)$. The derivative

$$\frac{d}{dx} K_\omega \chi_0(x) = -\frac{1}{\pi} \operatorname{Im}\left(\frac{e^{i\omega}}{xe^{i\omega} - \varepsilon} - \frac{1}{x}\right) = -\frac{1}{\pi} \operatorname{Im}\left(\frac{e^{i\omega}}{xe^{i\omega} - \varepsilon}\right)$$

is analytic in a neighbourhood of $[0, \infty)$. Thus $K_\omega \chi_0 \in C^\infty[0, \infty)$ and the proof is complete. ∎

Remark 2.16. (i) The range of the indices s for which the operators in the lemma are continuous cannot be extended in general. More information concerning this question will follow from the results in § 4.

(ii) The number $K_\omega \chi u(0)$ depends only on $u(0)$ and ω, as can be seen from the following: Let $u_1, u_2 \in H^s(\mathbf{R}_+)$ $(s > 1/2)$ with $u_1(0) = u_2(0)$. Then $u_1 - u_2 \in \tilde{H}^s(\mathbf{R}_+)$ and thus $K_\omega \chi(u_1 - u_2) \in \mathring{W}^s_0(\mathbf{R}_+)$. This means especially $K_\omega \chi(u_1 - u_2)(0) = 0$, whence $K_\omega \chi u_1 = K_\omega \chi u_2$. In § 4 it follows by the calculus of residues that

$$K_\omega \chi u(0) = \frac{\omega - \pi}{\pi} u(0).$$

Having now the necessary information about the localized operators, we are in a position to prove the global continuity of our system of integral operators:

THEOREM 2.17. (i) *Let $\mathscr{A}_0$ be defined as in* (1.20); *then*

$$\mathscr{A}_0\colon H^s(\Gamma_2) \times \tilde{H}^{s-1}(\Gamma_1) \to H^s(\Gamma_2) \times H^s(\Gamma_1)$$

is continuous for $s \in (-1/2, 3/2)$.

(ii) *Let $\mathscr{B}_0$ be defined as in* (1.20); *then*

$$\mathscr{B}_0\colon H^s(\Gamma_1) \times \tilde{H}^{s-1}(\Gamma_2) \to H^s(\Gamma_2) \times H^s(\Gamma_1)$$

is continuous for $s \in (-1/2, 3/2)$.

Proof. From definition (1.19) it is clear that the result concerning $\mathscr{B}_0$ follows from that on $\mathscr{A}_0$ by interchanging Γ_1 and Γ_2. The proof for $\mathscr{A}_0$ proceeds as follows:

We use a partition of unity $\{\chi_j | \, j = 1, \dots, J\}$ with the properties (2.8) and decompose

$$\mathscr{A}_0 U = \sum_{j,k=1}^{J} \chi_j \mathscr{A}_0 \chi_k U. \tag{2.20}$$

By Lemma 2.7 we have to show that for fixed J and each $k = 1, \dots, J$ the operator $\chi_j \mathscr{A}_0 \chi_k$ maps the Sobolev spaces on Γ^{ω_k} onto the corresponding spaces on Γ^{ω_j} continuously. For this purpose we show that $(\chi_j \mathscr{A}_0 \chi_k U)_\pm$ is contained on the proper Sobolev spaces in $\mathbf{R}_+$ and the compatibility conditions, as specified in Lemma 2.7, are satisfied. The continuity then follows trivially.

We have to distinguish three cases:

(α) $|j-k| > 1$;

(β) $|j-k| = 1$;

(γ) $j = k$.

(α) Here $\operatorname{supp} \chi_j \cap \operatorname{supp} \chi_k = \emptyset$. Now all integral operators in $\mathscr{A}_0$ have kernels $k(z, \zeta)$ which are C^∞ for $z \neq \zeta$. This implies that $\chi_j \mathscr{A}_0 \chi_k U$ belongs to $C^\infty(\Gamma)$ and thus is contained in all Sobolev spaces.

(β) Here $\operatorname{supp} \chi_j \cap \operatorname{supp} \chi_k =: S \subset \Gamma^k \cup \Gamma^{k+1}$. Now outside S we have $\chi_j \mathscr{A}_0 \chi_k U \in C^\infty$ (as in case (α)). From case (γ) we will see that $\mathscr{A}_0 \chi_k U$ is contained in appropriate Sobolev spaces on all of $\Gamma^k \cup \Gamma^{k+1}$, whence on a neighbourhood of S. So $\chi_j \mathscr{A}_0 \chi_k U$ is contained in the correct space.

(γ) For the investigation of $\chi_k \mathscr{A}_0 \chi_k U$ we have to distinguish three cases: "Dirichlet", "Neumann", and "mixed" corners.

(D) This means $k \in D$ or $\Gamma^k \cup \Gamma^{k+1} \subset \Gamma_1$. So $\operatorname{supp} \chi_k \subset \Gamma_1$ and thus

$$\chi_k \mathscr{A}_0 \chi_k = \chi_k \begin{bmatrix} 1 + K_{22} & -V_{12} \\ -K_{21} & V_{11} \end{bmatrix} \chi_k = \begin{bmatrix} 0 & 0 \\ 0 & \chi_k V_{11} \chi_k \end{bmatrix}$$

reduces to the operator $\chi_k V \chi_k$ on Γ^{ω_k}. By (2.15) this operator corresponds to the (2×2)-matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \chi_k l \chi_k + \chi_k \begin{bmatrix} V_0 & V_{\omega_k} \\ V_{\omega_k} & V_0 \end{bmatrix} \chi_k \quad \text{on } \mathbf{R}_+.$$

For further reference, we denote this local correspondence by

$$\mathscr{A}_0 \mathrel{\hat{=}}_{(D)} \begin{bmatrix} l & l \\ l & l \end{bmatrix} + \begin{bmatrix} V_0 & V_{\omega_k} \\ V_{\omega_k} & V_0 \end{bmatrix}. \tag{2.21}$$

Similarly $\chi_k U = \begin{bmatrix} 0 \\ \chi_k \psi \end{bmatrix} \mathrel{\hat{=}} \begin{bmatrix} (\chi_k \psi)_- \\ (\chi_k \psi)_+ \end{bmatrix}$. Therefore we have to show that $\chi_k V \chi_k$: $H^{s-1}(\Gamma^\omega) \to H^s(\Gamma^\omega)$ is continuous for $s \in (-1/2, 3/2)$. By Lemma 2.12, this is equivalent to

$$D \chi_k V \chi_k D^{-1}\colon \tilde{H}^{s-1}(\mathbf{R}_+) \times H^{s-1}(\mathbf{R}_+) \to H^s(\mathbf{R}_+) \times \tilde{H}^s(\mathbf{R}_+)$$

continuous for $s \in (-1/2, 3/2)$. Note that for $|s| < 1/2$ $H^s \equiv \tilde{H}^s$.

D has the matrix representation

$$D \mathrel{\hat{=}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

whence

$$D \chi_k V \chi_k D^{-1} \mathrel{\hat{=}} \chi_k \begin{bmatrix} 2l + V_0 + V_{\omega_k} & 0 \\ 0 & V_0 - V_{\omega_k} \end{bmatrix} \chi_k. \tag{2.22}$$

Lemma 2.11 gives $\chi_k(2l + V_0 + V_{\omega_k})\chi_k$: $\tilde{H}^s(\mathbf{R}_+) \to H^{s+1}(\mathbf{R}_+)$ cont. for $s \in (-3/2, 1/2)$ and $\chi_k(V_0 - V_{\omega_k})\chi_k$: $H^s(\mathbf{R}_+) \to \tilde{H}^{s+1}(\mathbf{R}_+)$ cont. for $s \in (-3/2, 1/2)$. This is just the desired result.

(N) This means $k \in N$ or $\Gamma^k \cup \Gamma^{k+1} \subset \Gamma_2$. We have supp $\chi_k \subset \Gamma_2$ and thus

$$\chi_k \mathscr{A}_0 \chi_k = \chi_k \begin{bmatrix} 1+K_{22} & 0 \\ 0 & 0 \end{bmatrix} \chi_k \quad \text{reduces to} \quad \chi_k(1+K)\chi_k \text{ on } \Gamma^{\omega_k}.$$

By (2.13) this corresponds to

$$\chi_k \begin{bmatrix} 1 & K_{\omega_k} \\ K_{\omega_k} & 1 \end{bmatrix} \chi_k,$$

i.e.,

$$\mathscr{A}_0 \stackrel{\wedge}{=}_{(\mathrm{N})} \begin{bmatrix} 1 & K_{\omega_k} \\ K_{\omega_k} & 1 \end{bmatrix}. \tag{2.23}$$

We have $\chi_k U = \begin{bmatrix} \chi_k v \\ 0 \end{bmatrix} \stackrel{\wedge}{=} \begin{bmatrix} (\chi_k v)_- \\ (\chi_k v)_+ \end{bmatrix}$, and so we have to show that

$$\chi_k(1+K)\chi_k \colon H^s(\Gamma^{\omega_k}) \to H^s(\Gamma^{\omega_k}) \text{ continuous for } s \in (-1/2, 3/2),$$

or

$$D\chi_k(1+K)\chi_k D^{-1} \colon H^s(\boldsymbol{R}_+) \times \tilde{H}^s(\boldsymbol{R}_+) \to H^s(\boldsymbol{R}_+) \times \tilde{H}^s(\boldsymbol{R}_+)$$
$$\text{continuous for } s \in (-\tfrac{1}{2}, \tfrac{3}{2}).$$

$$D\chi_k(1+K)\chi_k D^{-1} \stackrel{\wedge}{=} \chi_k \begin{bmatrix} 1+K_{\omega_k} & 0 \\ 0 & 1-K_{\omega_k} \end{bmatrix} \chi_k. \tag{2.24}$$

Lemma 2.11 gives the continuity of $\chi_k(1+K_{\omega_k})\chi_k$ in $H^s(\boldsymbol{R}_+)$ and of $\chi_k(1-K_{\omega_k})\chi_k$ in $\tilde{H}^s(\boldsymbol{R}_+)$ as desired.

(M) Here we again have to distinguish two cases, namely

$$(\mathrm{M}_1) \qquad \Gamma^k \subset \Gamma_1, \quad \Gamma^{k+1} \subset \Gamma_2,$$

$$(\mathrm{M}_2) \qquad \Gamma^k \subset \Gamma_2, \quad \Gamma^{k+1} \subset \Gamma_1.$$

We shall investigate only the first case (M_1), because the second case results from the first one by a single renumbering.

By our natural correspondence with (2×2)-matrices on Γ^{ω_k} we have

$$\chi_k \begin{bmatrix} 1+K_{22} & -V_{12} \\ -K_{21} & V_{11} \end{bmatrix} \chi_k \stackrel{\wedge}{=} \chi_k \begin{bmatrix} V_{--} & -K_{+-} \\ -V_{-+} & 1+K_{++} \end{bmatrix} = \chi_k \begin{bmatrix} l+V_0 & -K_{\omega_k} \\ -l-V_{\omega_k} & 1 \end{bmatrix} \chi_k;$$

$$\mathscr{A}_0 \stackrel{\wedge}{=}_{(\mathrm{M}_1)} \begin{bmatrix} l+V_0 & -K_{\omega_k} \\ -l-V_{\omega_k} & 1 \end{bmatrix}. \tag{2.25}$$

We have to show that

$$\chi_k \begin{bmatrix} l+V_0 & -K_{\omega_k} \\ -l-V_{\omega_k} & 1 \end{bmatrix} \chi_k \colon \tilde{H}^{s-1}(\boldsymbol{R}_+) \times \tilde{H}^s(\boldsymbol{R}_+) \to H^s(\boldsymbol{R}_+)^2 \quad (s \in (-\tfrac{1}{2}, \tfrac{3}{2})),$$

is continuous. Due to Lemma 2.15, this is true. ∎

Remark 2.18. The proof shows that for $k \in D$ the compatibility conditions for $\mathscr{A}_0 U$ at t_k are always satisfied, although ψ need not be continuous. For $k \in N$, the compatibility conditions for $\mathscr{A}_0 U$ at t_k hold if and only if those for v are satisfied.

2.3. Because the boundary value problem (P) is strongly elliptic, one could expect that the equivalent system of integral equations (1.20) is also strongly elliptic in some sense. For smooth curves it was shown in [58] that this is the case in the sense of elliptic pseudodifferential operators on Γ. This was then used to prove convergence for the Galerkin approximation procedure.

In the case of a polygon Γ it turns out that the operator $\mathscr{A}_0$ of (1.20) is strogly elliptic in the space $H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ of the "energy norm", which corresponds to the space of the variational formulation (1.9) of the problem (P). In the space $L^2(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$, which belongs to the standard Galerkin procedure with boundary elements, the operator $\mathscr{A}_0$, however, is in sharp contrast to the case of a smooth boundary, not even continuous. (The operator K_{21} does not map $L^2(\Gamma_2)$ continuously into $H^{1/2}(\Gamma_1)$.)

The way out this dilemma is to replace system (1.20) by an equivalent system of integral equations which is obtained from (1.20) by some kind of the Gauss elimenation procedure. The new operator $\mathscr{A}$ on the left-hand side of this equation will then indeed be continuous on $L^2(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ and strongly elliptic, i.e., will satisfy a Gårding inequality.

The modified system which we shall use for the Galerkin procedure is obtained from (1.20) by multiplication from the left by $\begin{bmatrix} 1 & 0 \\ K_{21} & 1 \end{bmatrix}$. This gives

$$\mathscr{A}U = \mathscr{B}G \tag{2.26}$$

with

$$\mathscr{A} = \begin{bmatrix} 1+K_{22} & -V_{12} \\ K_{21}K_{22} & V_{11}-K_{21}V_{12} \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} -K_{12} & V_{22} \\ 1+K_{11}-K_{21}K_{12} & -V_{21}+K_{21}V_{22} \end{bmatrix}. \tag{2.27}$$

This new system is equivalent to the original one as soon as K_{21} is continuous, because $\begin{bmatrix} 1 & 0 \\ K_{21} & 1 \end{bmatrix}$ will then be an isomorphism. This is the case, e.g., in $H^s(\Gamma_2) \times H^s(\Gamma_1)$ ($s \in -1/2, 3/2$) by Theorem 2.17. Instead of $\begin{bmatrix} 1 & 0 \\ K_{21} & 1 \end{bmatrix}$ one could also use $\begin{bmatrix} 1 & 0 \\ \chi_M K_{21} & 1 \end{bmatrix}$, where $\chi_M \in C_0^\infty(\Gamma)$ is any function identical to 1 near t_j for $j \in M$. This would mean that $\mathscr{A}_0$ is modified only in a neighbourhood of the "mixed" corners. If $\omega_j = \pi$ for all $j \in M$, then the modification is not necessary, i.e., we can take $\mathscr{A} = \mathscr{A}_0$.

THEOREM 2.19. *The operator*

$$\mathscr{A}\colon L^2(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1) \to L^2(\Gamma_2)\times H^{1/2}(\Gamma_1)$$

is continuous and satisfies a Gårding inequality, i.e., there exists a $\gamma > 0$ *such that for all* U

$$\langle \mathscr{A}U, U\rangle_{L_2(\Gamma_2)} \geqslant \gamma \|U\|^2_{L^2(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1)} - k(U, U) \tag{2.28}$$

where

$$\langle \mathscr{A}U, U\rangle_{L^2(\Gamma)} = \langle (1+K_{22})v - V_{12}\psi, v\rangle_{L^2(\Gamma_2)\times L^2(\Gamma_2)} + \\ + \langle (K_{21}K_{22})v + (V_{11}-K_{21}V_{12})\psi, \psi\rangle_{H^{1/2}(\Gamma)\times \tilde{H}^{-1/2}(\Gamma_1)}$$

and

$$\|U\|^2_{L^2(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1)} = \|v\|^2_{L^2(\Gamma_2)} + \|\psi\|^2_{\tilde{H}^{-1/2}(\Gamma_1)}.$$

k *is a compact bilinear form on* $L^2(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1)$ *with*

$$k(U, U) = \langle TU, U\rangle_{L^2(\Gamma)}, \text{ where } T\colon H^s(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1) \to H^s(\Gamma_2)\times \\ \times H^{1/2}(\Gamma_1) \text{ is compact for } 0 \leqslant s \leqslant 1/2. \tag{2.29}$$

Proof. From Theorem 2.17 we know the continuity properties of the operators $1+K_{22}\colon L^2(\Gamma_2)\to L^2(\Gamma_2)$;

$$C := V_{11} - K_{21}V_{12}\colon \tilde{H}^{-1/2}(\Gamma_1) \to H^{1/2}(\Gamma_1); \tag{2.30}$$

and $V_{12}\colon \tilde{H}^{-1/2}(\Gamma_1)\to L^2(\Gamma_2)$, where the latter is compact by the compactness of the embedding $H^{1/2}(\Gamma_2) \hookrightarrow L^2(\Gamma_2)$.

Thus for the continuity of $\mathscr{A}$ we only have to show that

$$K_{21}K_{22}\colon L^2(\Gamma_2) \to H^{1/2}(\Gamma_1) \quad \text{is continuous.} \tag{2.31}$$

For this purpose we choose a partition of unity $\{\chi_j\}$ as in (2.8) and write

$$K_{21}K_{22} = \sum_{j,k} K_{21}\chi_j K_{22}\chi_k.$$

For the individual terms in this sum we have several cases to consider:

If $j\in D$ then $K_{21}\chi_j = 0$.

If $j\in N$ then $K_{21}\chi_j\colon L^2(\Gamma_2)\to C^\infty(\Gamma_1)$ continuously (cf. (α) in the proof of Theorem 2.17.).

If $j\in M$ and $|k-j|>1$ then $\chi_j K_{22}\chi_k\colon L^2(\Gamma_2)\to C^\infty(\Gamma_2)$ cont. and finally:

If $k=j\in M$ then $\chi_j K_{22}\chi_k = 0$ (see (2.14)).

If $|k-j|=1$ then also $\chi_j K_{22}\chi_k\colon L^2(\Gamma_2)\to C^\infty(\Gamma_2)$, as can be seen by a combination of the preceding two cases.

What we have actually shown is that even $K_{21}K_{22}\colon L^2(\Gamma_2)\to C^\infty(\Gamma_1)$ is continuous, so that $K_{21}K_{22}\colon L^2(\Gamma_2)\to H^{1/2}(\Gamma_1)$ is compact.

Also, $V_{12}\colon \tilde{H}^{-1/2}(\Gamma_1)\to L^2(\Gamma_2)$ is seen to be compact. Hence for the proof of inequality (2.28) it is sufficient to show the two inequalities

$$\langle (1+K_{22})v, v\rangle_{L^2(\Gamma_2)\times L^2(\Gamma_2)} \geqslant \gamma\|v\|^2_{L^2(\Gamma_2)} - k_1(v, v) \tag{2.32}$$

with k_1 compact on $L^2(\Gamma_2)$, and

$$\langle C\psi, \psi\rangle_{H^{1/2}(\Gamma_1)\times \tilde{H}^{-1/2}(\Gamma_1)} \geqslant \gamma\|\psi\|^2_{\tilde{H}^{-1/2}(\Gamma_1)} - k_2(\psi, \psi) \tag{2.33}$$

with k_2 compact on $\tilde{H}^{-1/2}(\Gamma_1)$.

Again, we use a partition of unity as in (2.8) to reduce the global inequalities to local ones, i.e., inequalities (2.29), (2.30) for the individual terms $\chi_k v$, $\chi_k\psi$ $(k = 1, \ldots, J)$ instead of v, ψ. Lemma 2.20 below will justify this reduction.

For inequality (2.32) we have two cases: $k\in N$ and $k\in M$. For $k\in M$ we have $\langle K_{22}\chi_k v, \chi_k v\rangle = 0$, so

$$\langle (1+K_{22})\chi_k v, \chi_k v\rangle_{L^2} = \|\chi_k v\|^2_{L^2}.$$

For $k\in N$ we take the natural identification of $\chi_k v$ with a function on the reference angle Γ^ω, $\omega=\omega_k$ and use form (2.15) of K_{22}. We want to show that the operator K_ω is a contraction:

$$q = \|K_\omega\|_{L^2(\mathbf{R}_+)} < 1, \tag{2.34}$$

in order to obtain

$$\begin{aligned}\langle (1+K_{22})\chi_k v, \chi_k v\rangle &= \|\chi_k v\|^2 + \langle K_\omega(\chi_k v)_-, (\chi_k v)_+\rangle + \langle K_\omega(\chi_k v)_+, (\chi_k v)_-\rangle \\ &\geqslant \|\chi_k v\|^2 - q\,\{\|(\chi_k v)_-\|\cdot\|(\chi_k v)_+\| + \|(\chi_k v)_+\|\cdot\|(\chi_k v)_-\|\} \\ &\qquad (\leqslant \|(\chi_k v)_-\|^2 + \|(\chi_k v)_+\|^2 = \|\chi_k v\|^2) \\ &\geqslant (1-q)\|\chi_k v\|^2.\end{aligned}$$

For (2.34) we take $\phi\in C_0^\infty(0,\infty)$ and use the Mellin transform (2.17) of $K_\omega\phi$ together with Parseval's equation (2.4)

$$\|K_\omega\phi\|^2_{L^2(\mathbf{R}_+)} = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=-1/2} \left|\frac{\sinh(\pi-\omega)\lambda}{\sinh \pi\lambda}\right|^2 |\hat{\phi}(\lambda)^2|\,d\lambda \leqslant q^2\|\varphi\|^2_{L^2(\mathbf{R}_+)}$$

with $q := \sup\limits_{\operatorname{Im}\lambda=-1/2}\left|\dfrac{\sinh(\pi-\omega)\lambda}{\sinh \pi\lambda}\right|$. We have with $\lambda = \sigma + i\tau$, $\tau = -\dfrac{1}{2}$

$$\frac{|\sinh(\pi-\omega)\lambda|^2}{|\sinh \pi\lambda|^2} = \frac{\cosh^2\sigma(\pi-\omega) - \cos^2\tau(\pi-\omega)}{\cosh^2\sigma\pi - \underbrace{\cos^2\tau\pi}_{=0}} = \frac{\cosh^2\sigma(\pi-\omega) - \cos^2\dfrac{\pi-\omega}{2}}{\cosh^2\sigma\pi}.$$

Now this is maximal for $\sigma = 0$, and thus

(2.35) $$q = \left|\sin\frac{\pi-\omega}{2}\right| < 1.$$

This proves (2.34) and finally (2.32).

For (2.33) we again have two cases: $k\in D$ and $k \in M$. For $k \in M$ we will show

(2.36) $$(C\varphi, \varphi)_{L^2(\mathbf{R}_+)} \geqslant \gamma \|\phi\|^2_{\tilde{H}^{-1/2}(\mathbf{R}_+)}$$

for all

(2.37) $$\phi \in C_0^\infty(0, \infty) \quad \text{with} \quad \hat{\phi}(-i) = 0 = \frac{d}{d\lambda}\hat{\phi}(-i).$$

By Lemma 2.22 the set of such functions in dense in $\tilde{H}^{-1/2}(\mathbf{R}_+)$, and so (2.36) will hold for all $\phi \in \tilde{H}^{-1/2}(\mathbf{R}_+)$, especially for $\chi_k\phi$. The operator C from (2.30) now has the from

$$C = (l+V_0) - K_\omega(l+V_\omega).$$

Now

(2.38) $$0 = \frac{d}{d\lambda}\hat{\phi}(-i) = \int_{-\infty}^{\infty} (-it)e^{-i\lambda t}\phi(e^{-t})\,dt|_{\lambda=-i}$$
$$= -i\int_0^\infty \log x\cdot\phi(x)\,dx,$$

so that $l\phi = 0$. Thus $C\phi = (V_0 - K_\omega V_\omega)\phi$ and we can use the Mellin transform and Parseval's equation to obtain

(2.39) $$(C\phi, \phi) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=-1/2} \hat{C}(\lambda)\hat{\phi}(\lambda-i)\overline{\hat{\phi}(\lambda)}\,d\lambda$$
$$= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=0} \hat{C}(y)|\hat{\phi}(\lambda-i)|^2\,d\lambda,$$

where we have deformed the path of integration due to the analyticity and rapid decay of the integrand. Here from (2.14), (2.15) we get

(2.40) $$\hat{C}(\lambda) = \frac{\sinh(2\pi-\omega)\lambda\cosh\omega\lambda}{2\sinh^2\pi\lambda} \sim \frac{1+|\lambda|}{|\lambda|^2} \quad (\lambda\in\mathbf{R}),$$

By inserting (2.40) into (2.39) and using (2.6) we obtain (2.36) for some $\gamma > 0$.

For $k \in D$ we again use the density arguments for Lemma 2.22, so that we may assume that

$$\chi_k \hat{=} (\phi_-, \phi_+), \quad \text{where } \phi_\pm \text{ satisfy (2.37).}$$

Here the local form of C is

$$C \hat{=}_{(D)} \begin{bmatrix} l & l \\ l & l \end{bmatrix} + \begin{bmatrix} V_0 & V_\omega \\ V_\omega & V_0 \end{bmatrix}$$

and with the operator $D = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ we have by (2.22)

$$DCD^{-1} \hat{=}_{(D)} \begin{bmatrix} 2l & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V_0+V_\omega & 0 \\ 0 & V_0-V_\omega \end{bmatrix}.$$

Therefore we get

$$\langle C\phi, \phi\rangle_{H^{1/2}(\Gamma^\omega)\times\tilde{H}^{-1/2}(\Gamma^\omega)}$$
$$= \left\langle D^{-1}\begin{bmatrix} V_0+V_\omega & 0 \\ 0 & V_0-V_\omega \end{bmatrix} D\phi, \phi \right\rangle_{H^{1/2}(\Gamma^\omega)\times\tilde{H}^{-1/2}(\Gamma^\omega)}$$
$$= \frac{1}{2}\left\langle \begin{bmatrix} V_0+V_\omega & 0 \\ 0 & V_0-V_\omega \end{bmatrix} D\phi, D\phi \right\rangle_{H^{1/2}(\Gamma^\omega)\times\tilde{H}^{-1/2}(\Gamma^\omega)}$$
$$= \frac{1}{2}\{\langle (V_0+V_\omega)(\phi_-+\phi_+), \phi_-+\phi_+\rangle_{H^{1/2}(\mathbf{R}_+)\times\tilde{H}^{-1/2}(\mathbf{R}_+)} +$$
$$+ \langle (V_0-V_\omega)(\phi_--\phi_+), \phi_--\phi_+\rangle_{\tilde{H}^{1/2}(\mathbf{R}_+)\times H^{-1/2}(\mathbf{R}_+)}\},$$

where we have used (2.11) in Lemma 2.12. By the Mellin transformation we have

$$\langle C\phi, \phi\rangle_{H^{1/2}(\Gamma^\omega)\times\tilde{H}^{-1/2}(\Gamma^\omega)} = c\int_{\operatorname{Im}\lambda=0} [\mu_1(\lambda)|\hat{\phi}_+(\lambda-i)+\hat{\phi}_-(\lambda-i)|^2 +$$
$$+ \mu_2(\lambda)|\hat{\phi}_-(\lambda-i)-\hat{\phi}_+(\lambda-i)|^2]\,d\lambda,$$

where $\mu_{1,2}(\lambda) = \hat{V}_0(\lambda) \pm \hat{V}_\omega(\lambda)$, hence

$$\mu_1(\lambda) = \frac{\cosh\pi\lambda + \cosh(\pi-\omega)\lambda}{\lambda\sinh\pi\lambda} \sim \frac{1+|\lambda|}{|\lambda|^2} \quad (\lambda\in\mathbf{R}),$$

$$\mu_2(\lambda) = \frac{\cosh\pi\lambda - \cosh(\pi-\omega)\lambda}{\lambda\sinh\pi\lambda} \sim \frac{1}{1+|\lambda|} \quad (\lambda\in\mathbf{R}).$$

Therefore

$$\langle C\phi, \phi\rangle \sim \int_{\operatorname{Im}\lambda=0} \frac{1+|\lambda|}{|\lambda|^2}|\hat{\phi}_+(\lambda-i)+\hat{\phi}_-(\lambda-i)|^2\,d\lambda +$$
$$+ \int_{\operatorname{Im}\lambda=-1} \frac{1}{1+|\lambda|}|\hat{\phi}_-(\lambda)-\hat{\phi}_+(\lambda)|^2\,d\lambda.$$

By Corollary 2.4, the first member on the right-hand side is

$$\geqslant \gamma \|\phi_+ + \phi_-\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)},$$

and by Lemma 2.2 the second integral is

$$\sim \|\phi - \phi_+\|^2_{\overset{0}{W}{}^{-1/2}_{0}(\boldsymbol{R}_+)} = \|\phi_- - \phi_+\|^2_{H^{-1/2}(\boldsymbol{R}_+)}$$

by the local equivalence of norms. Thus finally

$$\langle C\phi, \phi\rangle \geqslant \gamma\{\|\phi_+ + \phi_-\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)} + \|\phi_- - \phi_+\|^2_{H^{-1/2}(\boldsymbol{R}_+)}\}$$
$$= \gamma \|D\phi\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)\times H^{-1/2}(\boldsymbol{R}_+)} \sim \|\phi\|^2_{\tilde{H}^{-1/2}(\Gamma^\infty)}$$

by (2.12). ∎

LEMMA 2.20. (i) *Let $|s| < 3/2$ and $A\colon H^{-s}(\Gamma) \to H^s(\Gamma)$ be a bounded operator with*

(a) $\langle A\phi, \phi\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} \geqslant \gamma_j \|\phi\|_{H^{-s}(\Gamma)} + k_j(\phi, \phi)$ *for all* $\phi \in C_0^\infty(S_j)$ *with* S_j *a compact subset of* $\Gamma^j \cup \{t_j\} \cup \Gamma^{j+1}$ $(j = 1, \ldots, J)$, *where* $\gamma_j > 0$ *and the compact bilinear form* k_j *depends only on* S_j.

(b) *For* $\phi_1, \phi_2 \in C^\infty(\Gamma)$ *with* $\phi_1 \cdot \phi_2 = 0$ *the operator* $\phi_1 A \phi_2$ *is compact, and for* $\phi, \psi \in C^\infty(\Gamma)$ *with* $\operatorname{supp}\psi \subset \Gamma^j$ *for some* j *the operator* $\psi(A\phi - \phi A)\psi$ *is compact.*

Then there exists a $\gamma > 0$ *and a compact bilinear form* k *on* $H^{-s}(\Gamma)$ *with*

$$\langle Au, u\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} \geqslant \gamma \|u\|^2_{H^{-s}(\Gamma)} + k(u, u) \quad \text{for all } u \in H^{-s}(\Gamma).$$

(ii) *The operators* $C\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *and* $K\colon L^2(\Gamma) \to L^2(\Gamma)$ *have the property* (b) *above.*

Proof. (i) Choose a partition of unity $\{\chi_j\}$ as in (2.8) and define $S_j = \operatorname{supp}\chi_j$. We may assume that $\tilde{\chi}_j := \sqrt{\chi_j} \in C^\infty$. Now if $k \in H^{-s}(\Gamma)$ then $\chi_j u \in \tilde{H}^{-s}(S_j)$. Obviously, the inequality in (a) holds by continuity for all $\phi \in \tilde{H}^{-s}(S_j)$, whence for $\chi_j u$. Thus for all $j = 1, \ldots, J$:

$$\langle A\chi_j u, \chi_j u\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} \geqslant \gamma_j \|\chi_j u\|^2_{H^s(\Gamma)} + k_j(\chi_j u, \chi_j u). \tag{2.41}$$

Let $j \neq k$. For $|j-k| > 1$ the operator $\chi_j A \chi_k$ is compact, and so $\langle A\chi_k u, \chi_j u\rangle_{H^s(\Gamma\times H^{-s}(\Gamma)}$ is a compact bilinear form. So let $|j-k| = 1$, for simplicity $k = j+1$. Choose $\psi \in C_0^\infty(\Gamma^{j+1})$ with $\psi = 1$ on $\operatorname{supp}(\chi_j \cdot \chi_k)$.

Then the four operators

$$\chi_j\psi(A\tilde{\chi}_k - \tilde{\chi}_k A)\psi\tilde{\chi}_k, \qquad \chi_j(1-\psi)(A\tilde{\chi}_k - \tilde{\chi}_k A)\psi\tilde{\chi}_k,$$
$$\chi_j\psi(A\tilde{\chi}_k - \tilde{\chi}_k A)(1-\psi)\tilde{\chi}_k, \qquad \chi_j(1-\psi)(A\tilde{\chi}_k - \tilde{\chi}_k A)\cdot(1-\psi)\tilde{\chi}_k$$

are all compact by assumption (b). (The equation $(1-\psi)\chi_j\chi_k = 0$ implies the compactness of the last three operators.) By summation we find that $\chi_j A \chi_k - \chi_j \tilde{\chi}_k A \tilde{\chi}_k$ is compact. Similarly, $\chi_j \tilde{\chi}_k A \tilde{\chi}_k - \tilde{\chi}_j \tilde{\chi}_k A \tilde{\chi}_j \tilde{\chi}_k$, and therefore $\chi_j A \chi_k - \tilde{\chi}_j \tilde{\chi}_k A \tilde{\chi}_j \tilde{\chi}_k$, are compact operators.

Thus we obtain

$$\langle A\chi_k u, \chi_j u\rangle = \langle \chi_j A\chi_k u, u\rangle = \langle A\tilde{\chi}_j\tilde{\chi}_k u, \tilde{\chi}_j\tilde{\chi}_k u\rangle + k_j'(u, u) \tag{2.42}$$
$$\geqslant \gamma_j \|\tilde{\chi}_j\tilde{\chi}_k u\|^2_{\tilde{H}^{-s}(\Gamma)} + k_j(\tilde{\chi}_j\tilde{\chi}_k u, \tilde{\chi}_j\tilde{\chi}_k u) + k_j'(u, u),$$

because $\operatorname{supp}\tilde{\chi}_j\tilde{\chi}_k \subset \operatorname{supp}\chi_j = S_j$. Adding (2.41) and (2.42), we obtain

$$\langle Au, u\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} \geqslant \gamma\{\sum_j \|\chi_j u\|^2_{H^{-s}(\Gamma)} + \sum_{|j-k|=1} \|\tilde{\chi}_j\tilde{\chi}_k u\|^2_{H^{-s}(\Gamma)}\} + k(u, u)$$
$$\geqslant \gamma \sum_{j=1}^{J} \|\chi_j u\|^2_{H^{-s}(\Gamma)} + k(u, u)$$
$$\geqslant \gamma' \|u\|^2_{H^{-s}(\Gamma)} + k(u, u),$$

because we have

$$\|u\|^2_{H^{-s}(\Gamma)} \leqslant \sum_{j,k=1}^{J} |(\chi_j u, \chi_k u)|_{H^{-s}(\Gamma)} \leqslant \sum_{j,k} \tfrac{1}{2}(\|\chi_j u\|^2_{H^{-s}(\Gamma)} + \|\chi_k u\|^2_{H^{-s}(\Gamma)})$$
$$= J \cdot \sum_{j=1}^{J} \|\chi_j u\|^2_{H^{-s}(\Gamma)}.$$

(ii) Let ϕ_1, ϕ_2, ϕ, and ψ be as in assumption (b). Then for $A = K$ or $A = C$ the operator $\phi_1 A \phi_2$ has a C^∞-kernel and is therefore compact, whereas in $\psi(A\phi - \phi A)\psi$ the kernel is different from 0 only on the straight line segment Γ^j, so that we can use the corresponding compactness result (which is well known) for pseudodifferential operators on $\boldsymbol{R}$. ∎

Remark 2.21. The form (2.29) of the compact perturbations is obvious from the above constructions. Most of them are even of order $-\infty$, and the others (namely those of the form $\psi(A\phi - \phi A)\psi$ for $A = C$) are at least of order -1.

LEMMA 2.22.

$$M := \left\{u \in C_0^\infty(0, \infty) \,\middle|\, \hat{u}(-i) = 0 = \frac{d}{d\lambda}\hat{u}(-i)\right\} \quad \textit{is dense in } \tilde{H}^{-1/2}(\boldsymbol{R}_+).$$

Proof. By (2.38) we have for $u \in C_0^\infty(0, \infty)$

$$\frac{d}{d\lambda}\hat{u}(-i) = -\int_0^\infty \log x u(x)\,dx \quad \text{and} \quad \hat{u}(-i) = \int_0^\infty u(x)\,dx.$$

Now neither $\log x$ nor ϕ with $\phi \equiv 1$ is contained in $H^{1/2}(\boldsymbol{R}_+)$, and so the linear functionals $l\colon u \mapsto \int_0^\infty \log x u(x)\,dx$ and $l_1\colon u \mapsto \int_0^\infty u(x)\,dx$ are unbounded in the norm of $\tilde{H}^{-1/2}(\boldsymbol{R}_+)$ on the dense subspace $C_0^\infty(0, \infty)$. Therefore M_1

$= \{u \in C_0^\infty(0, \infty) | lu = 0\}$ is dense in $C_0^\infty(0, \infty)$ and l_1 is unbounded on M_1, and thus $M = \{u \in M_1 | \ l_1 u = 0\}$ is dense in M_1, whence in $C_0^\infty(0, \infty)$ and so in $\tilde{H}^{-1/2}(\boldsymbol{R}_+)$. ∎

Remark 2.23. This density was used in a similar context in [41], § XI, 6.4, in weighted L^2-spaces.

THEOREM 2.24. *The operator*

$$\mathscr{A}_0\colon H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1) \to H^{1/2}(\Gamma_2) \times H^{1/2}(\Gamma_1)$$

satisfies a Gårding inequality, i.e., there is a $\gamma > 0$ *and a compact bilinear form* k *on* $H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ *with*

$$\langle \mathscr{A}_0 U, U \rangle_{\mathscr{H}^{1/2}} \geqslant \gamma \|U\|^2_{H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)} - k(U, U) \tag{2.43}$$

for all $U \in H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$, *where* $\langle \cdot, \cdot \rangle_{\mathscr{H}^{1/2}}$ *is a scalar product belonging to a norm which is equivalent to the norm* $\|\cdot\|^2_{H^{1/2}(\Gamma_2)} + \|\cdot\|^2_{L^2(\Gamma_1)}$; *in short:*

$$\langle \mathscr{A}_0 U, U \rangle_{\mathscr{H}^{1/2}} \sim \langle (1+K_{22}) v - V_{12} \psi, v \rangle_{H^{1/2}(\Gamma_2) \times H^{1/2}(\Gamma_2)} + \langle -K_{21} v + V_{11} \psi, \psi \rangle_{H^{1/2}(\Gamma_1) \times \tilde{H}^{-1/2}(\Gamma_1)}.$$

Proof. By using Lemma 2.20, it is sufficient to prove only the localized versions of (2.43).

If $k \in D$ then we have $C \mathrel{\hat{=}}_{(D)} V$, and so we can use the corresponding result of Theorem 2.19

For $k \in N$ we have $\mathscr{A}_0 \mathrel{\hat{=}}_{(N)} \begin{bmatrix} 1 & K_\omega \\ K_\omega & 1 \end{bmatrix}$, and for $\phi = \chi_k v$ with $\phi_\pm \in C_0^\infty(0, \infty)$ and $\hat{\phi}'_\pm(-i) = \hat{\phi}_\pm(-i) = 0$

$$\begin{aligned} &\langle (1+K_{22}) \phi, \phi \rangle_{H^{1/2}(\Gamma^\omega) \times H^{1/2}(\Gamma^\omega)} \\ &\quad \sim \langle D(1+K_{22}) \phi, D\phi \rangle_{[H^{1/2}(\boldsymbol{R}_+) \times \tilde{H}^{1/2}(\boldsymbol{R}_+)]^2} \\ &\quad = \langle (1+K_\omega)(\phi_+ + \phi_-), \phi_+ + \phi_- \rangle_{H^{1/2}(\boldsymbol{R}_+) \times H^{1/2}(\boldsymbol{R})} + \\ &\quad + \langle (1-K_\omega)(\phi_- - \phi_+), \phi_- - \phi_+ \rangle_{\tilde{H}^{1/2}(\boldsymbol{R}_+) \times \tilde{H}^{1/2}(\boldsymbol{R}_+)}. \end{aligned} \tag{2.44}$$

Here we have used (2.24) and Lemma 2.12.

For the first member on the right-hand side we use the decomposition $\langle \cdot, \cdot \rangle_{H^{1/2} \times H^{1/2}} = \langle \cdot, \cdot \rangle_{L^2 \times L^2} + \langle \cdot, \cdot \rangle_{1/2}$, where by Lemma 2.3

$$\langle \phi, \psi \rangle_{1/2} = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = 0} (\pi\lambda \coth \pi\lambda - 1) \hat{\phi}(\lambda) \overline{\hat{\psi}(\lambda)}\, d\lambda.$$

For $\langle (1+K_\omega)(\phi_+ + \phi_-), \phi_+ + \phi_- \rangle_{L^2(\boldsymbol{R}_+) \times L^2(\boldsymbol{R}_+)}$ we use Theorem 2.19, and for $\langle (1+K_\omega)(\phi_+ + \phi_-), \phi_+ + \phi_- \rangle_{1/2}$ as well as for

$$\begin{aligned} &\langle (1-K_\omega)(\phi_- - \phi_+), \phi_- - \phi_+ \rangle_{\tilde{H}^{1/2}(\boldsymbol{R}_+) \times \tilde{H}^{1/2}(\boldsymbol{R}_+)} \\ &\quad \sim \int_{\operatorname{Im}\lambda = 0} (1+|\lambda|^2)^{1/2} (1 - \hat{K}_\omega(\lambda)) |\phi_-(\lambda) - \phi_+(\lambda)|^2\, d\lambda \end{aligned}$$

we use the fact (cf. (2.35)) that $1 + \hat{K}_\omega(\lambda) \sim 1 - \hat{K}_\omega(\lambda) \sim 1$ ($\lambda \in \boldsymbol{R}$) in order to obtain finally

$$\begin{aligned} \langle (1+K_{22}) \phi, \phi \rangle_{H^{1/2}(\Gamma^\omega) \times H^{1/2}(\Gamma^\omega)} &\sim \langle (1+K_\omega)(\phi_+ + \phi_-), \phi_+ + \phi_- \rangle_{L^2(\boldsymbol{R}_+)^2} + \\ &\quad + \int_{\operatorname{Im}\lambda = 0} (\pi\lambda \coth \pi\lambda - 1) |\hat{\phi}_+(\lambda) + \hat{\phi}_-(\lambda)|^2\, d\lambda + \\ &\quad + \int_{\operatorname{Im}\lambda = 0} (1+|\lambda|^2)^{1/2} |\hat{\phi}_-(\lambda) - \hat{\phi}_+(\lambda)|^2\, d\lambda \\ &\geqslant \tilde{\gamma} \{ \|\phi_+ + \phi_-\|^2_{L^2(\boldsymbol{R}_+)} + |\phi_+ + \phi_-|^2_{1/2} + \|\phi_+ - \phi_-\|^2_{\tilde{H}^{1/2}(\boldsymbol{R}_+)} \} + k(\phi, \phi) \\ &\geqslant \gamma \|\phi\|^2_{H^{1/2}(\Gamma^\omega)} + k(\phi, \phi). \end{aligned}$$

Let $k \in M_1$. The local form of $\mathscr{A}_0$ is given in (2.25):

$$\mathscr{A}_0 \mathrel{\hat{=}}_{(M_1)} \begin{bmatrix} l + V_0 & -K_\omega \\ -l - V_\omega & 1 \end{bmatrix}.$$

We choose again $\phi = (\phi_-, \phi_+)$ as above and obtain ($\chi_k \psi \mathrel{\hat{=}} \phi_-$, $\chi_k v \mathrel{\hat{=}} \phi_+$)

$$\begin{aligned} \langle \mathscr{A}_0 \chi_k U, \chi_k U \rangle &\mathrel{\hat{=}}_{(M_1)} \langle V_0 \phi_- - K_\omega \phi_+, \phi_- \rangle_{H^{1/2}(\boldsymbol{R}_+) \times \tilde{H}^{-1/2}(\boldsymbol{R}_+)} \\ &\quad + \langle -V_\omega \phi_- + \phi_+, \phi_+ \rangle_{H^{1/2}(\boldsymbol{R}_+) \times H^{1/2}(\boldsymbol{R}_+)} \\ &\sim \int_{\operatorname{Im}\lambda = 0} [\hat{V}_0(\lambda) \hat{\phi}_-(\lambda - i) - \hat{K}_\omega(\lambda) \hat{\phi}_+(\lambda)] \overline{\hat{\phi}_-(\lambda - i)}\, d\lambda \\ &\quad + \int_{\operatorname{Im}\lambda = 0} \sigma(\lambda) (-\hat{V}_\omega(\lambda) \hat{\phi}_-(\lambda - i) + \hat{\phi}_+(\lambda)) \overline{\hat{\phi}_+(\lambda)}\, d\lambda \\ &\quad + \langle -V_\omega \phi_- + \phi_+, \phi_+ \rangle_{L^2(\boldsymbol{R}_+) \times L^2(\boldsymbol{R}_+)}. \end{aligned} \tag{2.45}$$

The last term of (2.45) is $\geqslant \|\phi_+\|^2_{L^2(\boldsymbol{R}_+)} - \langle V_\omega \phi_-, \phi_+ \rangle_{L^2(\boldsymbol{R}_+) \times L^2(\boldsymbol{R}_+)}$.

Now $\langle V_\omega \phi_-, \phi_+ \rangle_{L^2(\boldsymbol{R}_+) \times L^2(\boldsymbol{R}_+)}$ is a compact bilinear form on $\tilde{H}^{-1/2}(\boldsymbol{R}_+) \times H^{1/2}(\boldsymbol{R}_+)$.

In the second integral in (2.45) we replace $\sigma(\lambda) = \pi\lambda \coth \pi\lambda - 1$ by the equivalent function $\tilde{\sigma}(\lambda) = \dfrac{\lambda \sinh(\pi - \omega)\lambda}{\cosh(\pi - \omega)\lambda}$. Then we have $\hat{K}_\omega(\lambda) = -\tilde{\sigma}(\lambda) \cdot \hat{V}_\omega(\lambda)$, whence the two integrals in (2.45) have the real part

$$\int_{\operatorname{Im}\lambda = 0} \hat{V}_0(\lambda) |\hat{\phi}_-(\lambda - i)|^2\, d\lambda + \int_{\operatorname{Im}\lambda = 0} \tilde{\sigma}(\lambda) |\hat{\phi}_+(\lambda)|^2\, d\lambda \geqslant \gamma \{ \|\phi_-\|^2_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)} + |\phi_+|^2_{1/2} \}.$$

So for the whole right-hand side in (2.45) we have the desired estimation from below by

$$\gamma \{ \|\phi_-\|_{\tilde{H}^{-1/2}(\boldsymbol{R}_+)} + \|\phi_+\|^2_{H^{1/2}(\boldsymbol{R}_+)} \} + k(\phi, \phi). \ ∎$$

§ 3. The equivalence between the integral equations and the boundary value problem

3.1. In 1.5 we derived the boundary integral equations under the assumption that the given data (g_1, g_2) have some smoothness properties, e.g.,

$$g_1|_{\Gamma^j} \in H^{3/2}(\Gamma^j) \quad (\Gamma^j \subset \Gamma_1); \quad g_2|_{\Gamma^j} \in H^{1/2}(\Gamma^j) \quad (\Gamma^j \subset \Gamma_2).$$

Here we want to show that the weak solution $u \in H^1(\Omega)$ with $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times \tilde{H}^{-1/2}(\Gamma_2)$ already satisfies the integral equations (1.20). We will then show that every solution $(v, \psi) \in H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ of the *homogeneous* integral equations defines a solution of (P) and thus of the variational problem (1.8) and therefore vanishes. So for the data $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times \tilde{H}^{-1/2}(\Gamma_2)$ the three formulations of the mixed boundary value problem are entirely equivalent.

LEMMA 3.1. *Let $u \in H^1(\Omega)$ with $\Delta u = 0$. Then the boundary values $u|_\Gamma \in H^{1/2}(\Gamma)$ and $\partial u/\partial n|_\Gamma \in H^{-1/2}(\Gamma)$ satisfy the integral equation* (1.18)

$$(1+K)u|_\Gamma = V\frac{\partial u}{\partial n}\Big|_\Gamma. \tag{3.1}$$

Proof. We use the density of $C^\infty(\bar{\Omega})$ in $H^1(\Delta, 2, \Omega)$ ([19]) together with the continuity properties of

$$K\colon H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \quad \text{and} \quad V\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

which follow from Theorem 2.17.

We note first that for $w \in C^\infty(\bar{\Omega})$ we have the integral equation

$$(1+K)w|_\Gamma = 2[W(\Delta w)]|_\Gamma + V\frac{\partial w}{\partial n}\Big|_\Gamma \tag{3.2}$$

with

$$Wh(z) = \frac{1}{2\pi}\int_\Omega h(x)\log|z-x|\,dx.$$

This follows from the representation formula (1.3) together with Lemma 1.3. Note that $W(\Delta w)$ is everywhere continuous, because W is a continuous map

$$W\colon L^2(\Omega) \to H^2_{\text{loc}}(\boldsymbol{R}^2). \tag{3.3}$$

(It is a pseudodifferential operator of order -2; see [9], p. 36.) Now let $u \in H^1(\Omega)$ be given with $\Delta u = 0$. Choose a sequence $\{w_n\} \subset C^\infty(\bar{\Omega})$ with $w_n \to u$ in $H^1(\Omega)$ and $\Delta w_n \to \Delta u = 0$ in $L^2(\Omega)$. Then $w_n|_\Gamma \to u|_\Gamma$ in $H^{1/2}(\Omega)$ and $\partial w_n/\partial n|_\Gamma \to \partial u/\partial n|_\Gamma$ in $H^{-1/2}(\Gamma)$ by Lemma 1.1. Inserting this into (3.2), we obtain (3.1) in the limit. ∎

Remark 3.2. It follows from the lemma stating that each $u \in H^1(\Omega)$ which satisfies (P), i.e., $\Delta u = 0$ in Ω, $u|_{\Gamma_1} = g_1$, $\partial u/\partial n|_{\Gamma_2} = g_2$, is a solution of the system of integral equations (1.19).

For the proof of the injectivity of the integral operator $\mathscr{A}_0$ of system (1.20) we need the regularity result (Corollary 4.9) for the solution as well as the following regularity for the potential defined by the solution of the integral equations:

LEMMA 3.3. *Let $v \in H^{1/2+\varepsilon}(\Gamma)$, $\psi \in H^{-1/2+\varepsilon}(\Gamma)$ for some $0 < \varepsilon < 1/2$ and define u in Ω by*

$$u(z) = \frac{1}{2\pi}\int_\Gamma \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,v(\zeta)\,ds_\zeta - \frac{1}{2\pi}\int_\Gamma \log|z-\zeta|\,\psi(\zeta)\,ds_\zeta. \tag{3.4}$$

Then $u \in H^{1+\varepsilon}(\Omega)$ and $\Delta u = 0$.

Proof. We consider Ω as an intersection of half-planes with boundaries which are straight lines containing the segments Γ^j. We can decompose the potentials in (3.4) into a sum of contributions of the individual segments Γ^j. These contribution can be considered as potentials on the half-plane of densities on the line which are the extensions by zero of the restrictions $v|_{\Gamma^j}$ or $\psi|_{\Gamma^j}$.

In order to have the extension of $v|_{\Gamma^j}$ by 0 in $H^{1/2+\varepsilon}$, we first have to subtract $\sum_{k=1}^{J} v(t_k)\cdot\chi_k$, where $\{\chi_k\}$ are cut-off functions as in (2.8). So we have to consider two cases for v:

(α) $v \in H^{1/2+\varepsilon}(\boldsymbol{R})$, and we have to show that

$$u_0(z) = \frac{1}{2\pi}\int_{\boldsymbol{R}} \frac{\partial}{\partial n_\xi}\log|z-\xi|\,v(\xi)\,d\xi \tag{3.5}$$

is in $H^{1+\varepsilon}_{\text{loc}}(\boldsymbol{R}^2_+)$ $(\boldsymbol{R}^2_+ = \{z \in \boldsymbol{C}|\ \operatorname{Im} z \geqslant 0\})$.

(β) $v = \chi_k$ in Γ^ω, and we have to show that

$$u_1(z) = \frac{1}{2\pi}\int_{\Gamma^\omega} \frac{\partial}{\partial n_\zeta}\log|z-\zeta|\,\chi_k(\zeta)\,ds_\zeta = \frac{1}{2\pi}\int_{\Gamma^\omega} \chi_k(\zeta)\,d\theta_z(\zeta) \tag{3.6}$$

is in $H^{1+\varepsilon}(\Omega_0)$, where Ω_0 is some neighbourhood of 0. For case (α) we have $c/((x-\xi)^2+y^2)$ as the kernel of the integral operator $(z = x+iy)$. The Fourier transform of this convolution kernel is $c'\xi_2/(\xi_1^2+\xi_2^2)$ ([9]; (7.42)). This is a rational symbol of order -1.

Therefore it has the transmission property ([9], 10.1). Thus by Lemma 10.1 and Lemma 8.1 of [9] for $v \in H^s(\boldsymbol{R})$ with compact support and $\tilde{\Omega}$ compact in $\boldsymbol{R}^2_+$ we have $u_0 \in H^{s+1}(\tilde{\Omega})$ for all $s \in \boldsymbol{R}$.

For case (β) we show that u_1 is actually real analytic in a neighbourhood of 0. We decompose the integral in (3.6) into two terms: since $\chi_k = 1$ in a neighbourhood of the corner, we can write

(3.7) $$u_1(z) = \frac{1}{2\pi}\int_{\Gamma_0^\omega} d\theta_z(\zeta) + \frac{1}{2\pi}\int_{\Gamma^\omega/\Gamma_0^\omega} \chi_k(\zeta)\, d\theta_z(\zeta),$$

where $\Gamma_0^\omega = e^{i\omega}[0, \delta) \cup (0, \delta)$.

The second integral in (3.7) is analytic near 0, because the kernel is analytic there. For the first integral we use the geometric interpretation of the double layer potential. This gives

$$\frac{1}{2\pi}\theta = \frac{1}{2\pi}\arg\left(\frac{\delta - z}{\delta e^{i\omega} - z}\right),$$

which is analytic near 0. The functions $\psi|_{\Gamma^j} \in H^{-1/2+\varepsilon}(\Gamma^j)$ can immediately be extended by 0 on the straight line passing through Γ^j. So we only have to show that for $\psi \in H^{-1/2+\varepsilon}(\boldsymbol{R})$ with compact support

(3.8) $$u_2(z) := -\frac{1}{2\pi}\int_{\boldsymbol{R}} \log|z-\xi|\,\psi(\xi)\, d\xi$$

is locally in $H^{1+\varepsilon}(\boldsymbol{R}^2)$. For this again we use Eskin's results:

The Fourier transform of the kernel of the operator in (3.8) is $c/(\xi_1^2+\xi_2^2)$, which is of degree -2, so that (8.18) of [9] gives the desired result. ∎

We are now able to generalize Satz 1 of [13] on the continuity of the single layer potential (Lemma 1.3. (ii)).

LEMMA 3.4. *Let $\psi \in H^{-1/2+\varepsilon}(\Gamma)$, $\varepsilon > 0$ and define*

$$u(z) = -\frac{1}{\pi}\int_\Gamma \log|z-\zeta|\,\psi(\zeta)\, ds_\zeta \qquad (z \in \Omega).$$

Then u is continuous in $\boldsymbol{R}^2$ and $u|_\Gamma = V\psi$.

Proof. By Lemma 3.3 and its proof we have $u \in H_{\mathrm{loc}}^{1+\varepsilon}(\boldsymbol{R})^2$, so that $u \in C^0(\boldsymbol{R}^2)$ *and* $\psi \mapsto u|_\Gamma$: $H^{-1/+\varepsilon}(\Gamma) \to C^0(\Gamma)$ is continuous. For the dense subset $H^{-1/2+\varepsilon}(\Gamma) \cap L^2(\Gamma)$ we have Gaier's result $u|_\Gamma = V\psi$, which then holds for all of $H^{-1/2+\varepsilon}(\Gamma)$. ∎

3.2. For the proof of the injectivity of the operator $\mathscr{A}_0$ we need the following property of Γ:

(V) *If $\psi \in H^{-1/2}(\Gamma)$ and $V\psi = 0$ then $\psi = 0$.*

From the regularity results of § 4 (Corollary 4.9) applied to the equation $V\psi = 0$ it follows that (V) holds if V is injective in $L^1(\Gamma)$.

By Gaier [13], Satz 10 and Satz 11, V is injective in $L^1(\Gamma)$ if and only if

(3.9) $$\operatorname{cap}(\Gamma) \neq 1,$$

where $\operatorname{cap}(\Gamma)$ is the capacity (or conformal radius) of Γ. After these preparations we can reduce the question of the injectivity of $\mathscr{A}_0$ to that of V:

THEOREM 3.5. *Let $U = (v, \psi) \in H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ be a solution of the homogeneous integral equations* (1.20)

$$\mathscr{A}_0 U = 0.$$

Let (V) *be satisfied. Then $v = 0$ and $\psi = 0$.*

Proof. If (v, ψ) is a solution of the homogeneous integral equations, then, by Corollary 4.9, $v \in \tilde{H}^{1/2+\varepsilon}(\Gamma_2)$ and $\psi \in H^{-1/2+\varepsilon}(\Gamma_1) \cap L^p(\Gamma_1)$.

Let $\tilde{v}$ and $\tilde{\psi}$ be the extensions by 0 of v and ψ, respectively, on Γ. Define u in Ω by (3.4):

(3.10) $$u(z) = \frac{1}{2\pi}\int_\Gamma \tilde{v}(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\zeta - \frac{1}{2\pi}\int_\Gamma \tilde{\psi}(\zeta)\log|z-\zeta|\, ds_\zeta.$$

Then, by Lemma 3.3, $u \in H^1(\Delta, 2, \Omega)$. Therefore, by Lemma 1.2, the representation formula

(3.11) $$u(z) = \frac{1}{2\pi}\int_\Gamma u(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta|\, ds_\zeta - \frac{1}{2\pi}\int_\Gamma \frac{\partial u(\zeta)}{\partial n}\log|z-\zeta|\, ds_\zeta.$$

holds. Furthermore, $\tilde{v}$ and $\tilde{\psi}$ satisfy the assumptions of Lemma 1.3 (i) and Lemma 1.3 (ii) (or Lemma 3.4), respectively. Therefore we obtain from (3.10)

(3.12) $$u|_\Gamma = -\tfrac{1}{2}(K-1)\tilde{v} + \tfrac{1}{2}V\tilde{\psi}.$$

If we insert the integral equation

$$0 = \mathscr{A}_0\begin{bmatrix} v \\ \psi \end{bmatrix} \Leftrightarrow (1+K)\tilde{v} = V\tilde{\psi}$$

into (3.12) we obtain

(3.13) $$u|_\Gamma = \tilde{v}.$$

We insert this into (3.11), subtract (3.10), from (3.11), go to the boundary with Lemma 3.4, and obtain $V(\tilde{\psi} - \partial u/\partial n|_\Gamma) = 0$. By assumption (V) this gives

(3.14) $$\left.\frac{\partial u}{\partial n}\right|_\Gamma = \tilde{\psi}.$$

Now (3.13) and (3.14) mean in particular that $u|_{\Gamma_1} = 0$ and $\partial u/\partial n|_{\Gamma_2} = 0$, i.e., u is a solution of the homogeneous mixed boundary value problem (P). Since we know $u \in H^1(\Omega)$, u is also a solution of the variational problem (1.8) with vanishing right-hand side. Therefore $u=0$ in Ω, and this implies by (3.13) $v=0$ and by (3.14) $\psi=0$. ∎

Remark 3.6. If $\operatorname{cap}(\Gamma)=1$, then condition (V) is violated [13]. It is known (see [21]) that then integral equations (1.18) are not uniquely solvable. The argument is as follows:

By [13] $\psi_0=1$ is a solution of $V\psi_0=0$. Therefore

$$(3.15)\qquad \mathscr{A}_0\begin{bmatrix}0\\ \psi_0|_{\Gamma_1}\end{bmatrix}=\begin{bmatrix}-V_{12}\ \psi_0|_{\Gamma_1}\\ V_{11}\ \psi_0|_{\Gamma_1}\end{bmatrix}=\begin{bmatrix}V_{22}\ \psi_0|_{\Gamma_2}\\ -V_{21}\ \psi_0|_{\Gamma_2}\end{bmatrix}=\mathscr{B}_0\begin{bmatrix}0\\ \psi_0|_{\Gamma_2}\end{bmatrix}.$$

On the other hand, the mixed boundary value problem

$$\Delta u=0 \text{ in } \Omega,\quad u|_{\Gamma_1}=0\quad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2}=\psi_0|_{\Gamma_2}$$

has a solution in $H^1(\Omega)$ with $v=u|_{\Gamma_2}\neq 0$. Then with $\psi=\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1}$ the integral equation

$$(3.16)\qquad \mathscr{A}_0\begin{bmatrix}v\\ \psi\end{bmatrix}=\mathscr{B}_0\begin{bmatrix}0\\ \psi_0|_{\Gamma_2}\end{bmatrix}$$

is satisfied. By substracting (3.15) from (3.16) we get

$$\mathscr{A}_0\begin{bmatrix}v\\ \psi-\psi_0|_{\Gamma_1}\end{bmatrix}=0.$$

Remark 3.7. If (V) is satisfied, then not only

$$\mathscr{A}_0\colon H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_1)$$

is injective, but also

$$\mathscr{A}\colon H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_1),$$

as defined in (2.27), is injective.

We now use Gårding's inequality to prove surjectivity:

THEOREM 3.8. *The operators*

$$(3.17)\qquad \mathscr{A}\colon L^2(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)\to L^2(\Gamma_2)\times H^{1/2}(\Gamma_1),$$

$$(3.18)\qquad \mathscr{A}\colon H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_1),$$

$$(3.19)\qquad \mathscr{A}_0\colon H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_1),$$

are bijective.

Proof. We know that $\mathscr{A}$ in (3.18) is injective. We show that $\mathscr{A}$ in (3.17) is injective:

Let $(v,\psi)\in L^2(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)$ be such that $\mathscr{A}\begin{bmatrix}v\\ \psi\end{bmatrix}=0$. Then in particular $(1+K_{22})v=V_{12}\psi$. Since $V_{12}\psi\in H^{1/2}(\Gamma_2)$, we can use Corollary 4.5 and get $v\in H^{1/2}(\Gamma_2)$. Thus (v,ψ) is in the kernel of $\mathscr{A}$ in (3.18) and therefore $v=0$ and $\psi=0$.

The Gårding inequality of Theorem 2.19 (2.28) means that $\mathscr{A}$ in (3.17) differs by a compact perturbation from a positive definite (strongly coercive) operator and hence is a Fredholm operator of index 0. It is injective, whence surjective. Thus $\mathscr{A}$ in (3.17) is bijective.

To show the surjectivity of $\mathscr{A}$ in (3.18) (and equivalently of $\mathscr{A}_0$ in (3.19)), we assume that we are given $(h_2,h_1)\in H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_1)$. Then we have by (3.17) $(v,\psi)\in L^2(\Gamma_1)\times\tilde{H}^{-1/2}(\Gamma_1)$ with $\mathscr{A}\begin{bmatrix}v\\ \psi\end{bmatrix}=\begin{bmatrix}h_2\\ h_1\end{bmatrix}$, and in particular

$$(1+K_{22})v=V_{12}\psi+h_2\in H^{1/2}(\Gamma_2).$$

As above, we obtain $v\in H^{1/2}(\Gamma_2)$, which proves the surjectivity of $\mathscr{A}$ in (3.18). ∎

THEOREM 3.9. *Let assumption* (V) *be satisfied, and let* $(g_1,g_2)\in H^{1/2}(\Gamma_1)\times\tilde{H}^{-1/2}(\Gamma_2)$ *be given. Then problem* (P) *for* $u\in H^1(\Omega)$, *the variational formulation* (1.8) *under the same hypothesis, and the integral equations* (1.20) *for* $(v,\psi)\in H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)$ *each have exactly one solution, and they are equivalent, i.e.,* $v=u|_{\Gamma_2}$, $\psi=\partial u/\partial n|_{\Gamma_2}$, *or, conversely,* u *in* Ω *is given by* (3.10) *where*

$$\tilde{v}=\begin{cases}v & \text{on } \Gamma_2,\\ g_1 & \text{on } \Gamma_1,\end{cases}\quad\text{and}\quad \tilde{\psi}=\begin{cases}\psi & \text{on } \Gamma_1,\\ g_2 & \text{on } \Gamma_2.\end{cases}$$

Proof. We know the uniqueness of the solution of all three problems, and, by Lemma 3.1, every solution of (P) gives a solution of the integral equations. This solution is unique, and this in turn proves that the solution of the integral equations defines a solution of problem (P). The equivalence of (P) and (1.8) for $u\in H^1(\Omega)$ was discussed above. ∎

§ 4. Regularity of the solutions of the integral equations

4.1. In this section we proceed along the lines of Kondratiev [31] and use the Mellin transform together with the Cauchy integral theorem for analytic functions in order to obtain an expansion of the solution U of system (1.20) of integral equations in terms of singularity functions similar to the one (Theorem 1.4) which is known for the solution of the variational formulation of the boundary value problem.

As a by-product we get the regularity results which were used in § 3 for the proof ot the bijectivity of the integral operators. The regularity of the solution U of equation (1.20)

$$\mathscr{A}_0 U=\mathscr{B}_0 G$$

is a local property. Therefore we use a partition of unity to reduce the problem to a local one on the reference angle Γ^{ω}, where the three cases (D), (N), and (M) have to be distinguished. Thus we first have to investigate operators on $\boldsymbol{R}_+$. For this purpose we use the following lemma, which displays the connection between an expansion in terms of singularity functions and meromorphic Mellin transforms. It goes back to Kondratiev [31] and has been used in different forms, e.g., by Mazja and Plamenevskii [39].

LEMMA 4.1. *Suppose that*

$$(4.1)\qquad f(x)=\Big(\sum_{k=1}^{n}\sum_{l=0}^{l_k} c_{kl}x^{\alpha_k}\log^l x\Big)\cdot\chi(x)+f_0(x)\qquad (x\in\boldsymbol{R}_+)$$

where $f_0\in C_0^\infty(0,\infty)$, $\chi\in C_0^\infty[0,\infty)$ *with* $\operatorname{supp}(1-\chi)\subset(0,\infty)$, $\alpha_1<\alpha_2<\ldots<\alpha_n$. *Then*

(i) *The Mellin transform* $\hat f(\lambda)$ *exists and is analytic for* $\operatorname{Im}\lambda<\alpha_1$, *and it has a meromorphic extension on* $\boldsymbol{C}$ *with poles at* $\lambda=i\alpha_k$ $(k=1,\ldots,n)$ *of order* l_k+1.

(ii) *In the strip* $\{\lambda\in\boldsymbol{C}\mid \operatorname{Im}\lambda\in(\alpha_1,\alpha_{j+1})\}$, $\hat f$ *is the Mellin transform of* f_j *defined by*

$$(4.2)\qquad f_j(x)=f(x)-\sum_{k=1}^{j}\sum_{l=0}^{l_k}c_{kl}x^{\alpha_k}\log^l x.$$

(iii) *If we define*

$$(4.3)\qquad \gamma_r^{\lambda_0}(\hat f):=\operatorname*{Res}_{\lambda=\lambda_0}\frac{\hat f(\lambda)}{(\lambda-\lambda_0)^r}\qquad (r\in\boldsymbol{Z}),$$

then we have

$$(4.4)\qquad \gamma_r^{\lambda_0}(\hat f)=\frac{1}{(l_k+r)!}\Big(\frac{d}{d\lambda}\Big)^{l_k+r}[\hat f(\lambda)\cdot(\lambda-\lambda_0)^{l_k+1}]|_{\lambda=\lambda_0}$$
$$=\frac{1}{2\pi i}\Big(\int_{\operatorname{Im}\lambda=h_1}-\int_{\operatorname{Im}\lambda=h_2}\Big)\frac{\hat f(\lambda)}{(\lambda-\lambda_0)^r}\,d\lambda,$$

where $\lambda_0=i\alpha_k$, $l_k+r\geqslant 0$, $\alpha_{k-1}<h_1<\alpha_k<h_2<\alpha_{k+1}$; $\gamma_r^{\lambda_0}(\hat f)=0$ *if* $l_k+r<0$ *or if* $\hat f$ *is regual at* λ_0;

$$(4.5)\qquad \gamma_{-l}^{i\alpha_k}(\hat f)=-i^{l+1}l!\,c_{kl}\qquad (0\leqslant l\leqslant l_k,\ k=1,\ldots,n).$$

(iv) *Let*

$$(4.6)\qquad f_j^0(x)=f(x)-\Big(\sum_{k=1}^{j}\sum_{l=0}^{l_k}c_{kl}x^{\alpha_k}\log^l x\Big)\cdot\chi(x)\qquad (j=0,\ldots,n).$$

Then if $\alpha_1>-1$ *we have an estimate*

$$(4.7)\qquad |\gamma_r^{i\alpha_k}(\hat f)|\leqslant C\Big(\sum_{j=1}^{k'}\sum_{l=0}^{l_j}|c_{jl}|+\|f_{k'}^0\|_{\tilde H^s(\boldsymbol{R}_+)}\Big)\qquad (k'\geqslant k;\ s>\alpha_{k'}+\tfrac12).$$

Proof. (i) follows from properties of the Mellin transform which were described in Examples 2.1 at the beginning of § 2. As mentioned there, for $\psi(x)=x^\alpha\log^l x\cdot\chi(x)$ we have

$$\hat\psi(\lambda)=\frac{\hat\phi(\lambda)}{(\lambda-i\alpha)^{l+1}},\qquad\text{where}\qquad \hat\phi\in\boldsymbol{Z}.$$

This means especially that $\hat f(\lambda)$ is rapidly decreasing for $\operatorname{Im}\lambda$ fixed and $|\operatorname{Re}\lambda|\to\infty$. Therefore the inverse Mellin transform

$$(4.8)\qquad f_{(h)}(x):=\frac{1}{2\pi}\int_{\operatorname{Im}\lambda=h}e^{i\lambda t}\hat f(\lambda)\,d\lambda\qquad (x=e^{-t}\in\boldsymbol{R}_+)$$

exists for $h\notin\{\alpha_1,\ldots,\alpha_n\}$, and the path of integration may be shifted if we take into account the residues of $e^{i\lambda t}\hat f(\lambda)$.

Thus we get $f_{(h)}=f$ for $h<\alpha_1$ and

$$f_{(h_2)}(x)-f_{(h_1)}(x)=-i\sum_{\operatorname{Im}\lambda\in(h_1,h_2)}\operatorname{Res}\{\hat f(\lambda)e^{i\lambda t}\}\qquad (h_1<h_2).$$

Therefore, $f_{(h)}$ does not depend on h in each interval $h\in(\alpha_k,\alpha_{k+1})$, and if we denote $f_{(h)}$ in this case by f_k^*, then we can use the well-known formula (4.4) for $\gamma_0^{i\alpha_k}(e^{i\lambda t}\hat f(\lambda))$ to obtain by the Leibniz formula

$$(4.9)\qquad f_k^*(x)-f_{k-1}^*(x)=-i\operatorname*{Res}_{\lambda=i\alpha_k}\{e^{i\lambda t}\hat f(\lambda)\}=-i\gamma_0^{i\alpha_k}(e^{i\lambda t}\hat f(\lambda))$$
$$=-i\frac{1}{(l_k)!}\Big(\frac{d}{d\lambda}\Big)^{l_k}[e^{i\lambda t}\hat f(\lambda)(\lambda-i\alpha_k)^{l_k+1}]|_{\lambda=i\alpha_k}$$
$$=-i\sum_{l=0}^{l_k}\frac{1}{l!}(-i)^l\gamma_{-l}^{i\alpha_k}(\hat f)\,x^{\alpha_k}\cdot\log^l x$$
$$=-\sum_{l=0}^{l_k}c_{kl}^*x^{\alpha_k}\log^l x.$$

In order to obtain (4.5) and also (4.2), and thus (ii) and (iii), we have to show that

$$(4.10)\qquad c_{kl}^*=c_{kl}\quad\text{for}\quad 0\leqslant l\leqslant l_k,\ 1\leqslant k\leqslant n.$$

This is done by induction on k. We indicate only the first step: By the definition of f_1 we have

$$f_1(x)-f_1^*(x)=f_1(x)-f(x)-(f_1^*(x)-f(x))$$
$$=\sum_{l=0}^{l}(c_{1l}^*-c_{1l})x^{\alpha_1}\log^l x.$$

Now we know from (4.1) and (4.2) that

$$f_1(x)=f_0(x)+\Big(\sum_{l=0}^{l_1} c_{1l}x^{\alpha_1}\log^l x\Big)(\chi(x)-1)+\sum_{k=2}^{n}\sum_{l=0}^{l_k} c_{kl}x^{\alpha_k}(\log^l x)\chi(x)$$

$$=0(x^{\alpha})\quad \text{as}\quad x\to 0\quad \text{for every } \alpha<\alpha_2.$$

From the decay properties of $\hat{f}$ in the strip $\operatorname{Im}\lambda\in(\alpha_1,\alpha_2)$ and the definition of f_1^* we know that

$$f_1^*(x)=O(x^{\alpha})\quad \text{as}\quad x\to 0\quad \text{for every } \alpha\in(\alpha_1,\alpha_2).$$

Therefore $\sum_{l=0}^{l_1}(c_{1l}^*-c_{1l})x^{\alpha_1}\log^l x=O(x^{\alpha})$ as $x\to 0$ for some $\alpha>\alpha_1$. This gives $c_{1l}^*=c_{1l}$ for $0\leqslant l\leqslant l_1$ and thus $f_1=f_1^*$. It remains to show estimate (4.7).

From (4.5) follows that (4.7) for $r\leqslant 0$ is trivial. We use the integral representation (4.4) of $\gamma_k^{i\alpha_k}(\hat{f})$, the Parseval formula (2.4) and the Cauchy–Schwarz inequality to obtain

$$(4.11)\quad |\gamma_r^{i\alpha_k}(\hat{f})|\leqslant C(\|f_{k-1}\|_{\mathring{W}^{s_1}}+\|f_k\|_{\mathring{W}^{s_2}})\quad \text{for}\quad \begin{matrix} s_1-1/2\in(\alpha_{k-1},\alpha_k);\\ s_2-1/2\in(\alpha_k,\alpha_{k+1}).\end{matrix}$$

Now we define

$$p_j(x)=\sum_{l=0}^{l_j} c_{jl}x^{\alpha_j}\log^l x$$

and note that

$$(4.12)\quad \begin{matrix} \chi p_j\in \mathring{W}_0^s(\boldsymbol{R}_+)\cap \tilde{H}^s(\boldsymbol{R}_+) & \text{for} & s<\alpha_j+1/2;\\ (1-\chi)p_j\in \mathring{W}_0^s(\boldsymbol{R}_+) & \text{for} & s>\alpha_j+1/2.\end{matrix}$$

(By what we have shown above, $\widehat{(1-\chi)p_j}$ is the analytic continuation of $-\widehat{\chi p_j}$ to $\operatorname{Im}\lambda>\alpha_j$.) Now by definitions (4.2) and (4.6) $f_k=f_k^0-(1-\chi)\sum_{j=1}^{k}p_j$. Therefore

$$(4.13)\quad \|f_k\|_{\mathring{W}_0^s(\boldsymbol{R}_+)}\leqslant \|f_k^0\|_{\mathring{W}_0^s(\boldsymbol{R}_+)}+C\sum_{j=1}^{k}\sum_{l=0}^{l_j}|c_{jl}|$$

$$\leqslant C\Big\{\|f_k^0\|_{\tilde{H}^s(\boldsymbol{R}_+)}+\sum_{j=1}^{k}\sum_{l=0}^{l_j}|c_{jl}|\Big\}\quad \text{for}\quad s>\alpha_k+1/2.$$

Also by (4.12) and (4.6)

$$(4.14)\quad \|f_k^0\|_{\tilde{H}^s(\boldsymbol{R}_+)}\leqslant \|f_{k'}^0\|_{\tilde{H}^s(\boldsymbol{R}_+)}+C\sum_{j=k+1}^{k'}\sum_{l=0}^{l_j}|c_{jl}|$$

for $s<\alpha_{k+1}+1/2$ and $k'\geqslant k$. Inserting (4.13) and (4.14) into (4.11), we get estimate (4.7). ∎

Remark 4.2. We note two special cases of (4.5), namely:

$$(4.15)\quad \begin{matrix} \gamma_0^{i\alpha_k}(\hat{f})=-ic_{k0} & (l=0),\\ \gamma_{-1}^{\alpha_k}(\hat{f})=c_{k1} & (l=1).\end{matrix}$$

The first one is important, e.g., if $\alpha_1=0$ and $l_1=0$, i.e., f is continuous at 0, and then

$$(4.16)\quad \gamma_0^0(\hat{f})=-ic_{10}=-if(0),$$

or if $\alpha_1=0$, $\alpha_2=1$ ($l_1=0=l_2$), i.e., f is differentiable at 0, and then

$$(4.17)\quad \gamma_0^i(\hat{f})=-ic_{20}=-if'(0).$$

whereas the second one occurs if $f=c\cdot x\log x+f_0$ with smooth f_0, and then $\gamma_{-1}^i(\hat{f})=c$.

Later on we shall need a certain converse to Lemma 4.1, which also goes back to Kondratiev [31] and can be proved in a similar way to Lemma 4.1.

LEMMA 4.3. *Let* $\hat{f}$ *be meromorphic in a strip* $\operatorname{Im}\lambda\in(\alpha_0,\alpha_{n+1})$ *and have poles at* $\operatorname{Im}\lambda=i\alpha_k$ *of order* l_k+1 $(k=1,\ldots,n;\ \alpha_0<\alpha_1<\ldots<\alpha_{n+1})$. *Assume that, for* $\operatorname{Im}\lambda=$ const, $\hat{f}(\lambda)$ *is rapidly decreasing as* $|\operatorname{Re}\lambda|\to\infty$. *Define* $f_{(h)}$ *by* (4.8). *Then for* $h\in(\alpha_0,\alpha_1)$ *and* $h'\in(\alpha_n,\alpha_{n+1})$ *we have*

$$f_{(h)}=\mathring{W}_0^s(\boldsymbol{R}_+)\quad \textit{for}\quad s-1/2\in(\alpha_0,\alpha_1);$$

$$f_{(h')}\in\mathring{W}_0^{s'}(\boldsymbol{R}_+)\quad \textit{for}\quad s'-1/2\in(\alpha_n,\alpha_{n+1})\quad \textit{and}$$

$$f_{(h)}(x)=\sum_{k=1}^{n}\sum_{l=0}^{l_k}c_{kl}x^{\alpha_k}\log^l x+f_{(h')}(x).$$

Formula (4.5) *holds in this case also.*

4.2. Now we apply this to the operators K_ω, V_0, and V_ω on $\boldsymbol{R}_+$, the Mellin transforms of which were given in Lemma 2.13. We assume that f has an expansion (4.1). Then $\hat{K}_\omega(\lambda)\cdot\hat{f}(\lambda)$, $\hat{V}_0(\lambda)\hat{f}(\lambda-i)$, and $\hat{V}_\omega(\lambda)\hat{f}(\lambda-i)$ are meromorphic in the whole plane. We compute the residue (only for K_ω and V_ω, because $V_0=V_\omega|_{\omega=0}$):

1.) $\lambda_0=i\alpha_k$, $\alpha_k\notin\boldsymbol{Z}$. Then $\hat{K}_\omega$ and $\hat{V}_\omega$ are regular at λ_0, and we get

$$(4.18)\quad \gamma_r^{i\alpha_k}(\hat{K}_\omega\hat{f})=\sum_{m=0}^{l_k+r}\frac{1}{(l_k+r-m)!}\Big(\frac{d}{d\lambda}\Big)^{l_k+r-m}\hat{K}_\omega(\lambda)|_{\lambda=i\alpha_k}\gamma_{m-l_k}^{i\alpha_k}(\hat{f});$$

$$(4.19)\quad \gamma_r^{i(\alpha_k+1)}(V_\omega f)=\sum_{m=0}^{l_k+r}\frac{1}{(l_k+r-m)!}\Big(\frac{d}{d\lambda}\Big)^{l_k+r-m}\hat{V}_\omega(\lambda)|_{\lambda=i(\alpha_k+1)}\gamma_{m-l_k}^{i\alpha_k}(\hat{f}).$$

A special case of (4.18) is ($\hat{K}_\omega$ is regular at 0!)

(4.20)

$$\gamma_0^0(\hat{K}_\omega \hat{f}) = \hat{K}_\omega(0)\gamma_0^0(\hat{f}) = \frac{\omega-\pi}{\pi}\gamma_0^0(\hat{f}) \quad \text{if } \hat{f} \text{ has a pole of order 1 in 0.}$$

2.) $\gamma_0 = i\alpha$, $\alpha \in \mathbf{Z}$, $\hat{f}$ regular at λ_0.

$$(4.21) \quad \gamma_r^{i\alpha}(\hat{K}_\omega \hat{f}) = \sum_{j=0}^{r} \frac{1}{(r-j)!}\left(\frac{d}{d\lambda}\right)^{r-j}[\hat{K}_\omega(\lambda)(\lambda - i\alpha)]|_{\lambda=i\alpha} \cdot \gamma_{j+1}^{i\alpha}(\hat{f});$$

$$(4.22) \quad \gamma_r^{i(\alpha+1)}(\widehat{V_\omega f}) = \sum_{j=0}^{r} \frac{1}{(r-j)!}\left(\frac{d}{d\lambda}\right)^{r-j}[\hat{V}_\omega(\lambda)(\lambda - i\alpha - i)]|_{\lambda=i\alpha+1} \cdot \gamma_{j+1}^{i\alpha}(\hat{f}).$$

3.) $\lambda_0 = i\alpha_k$, $\alpha_k \in \mathbf{Z}$.

$$(4.23) \quad \gamma_r^{i\alpha_k}(\hat{K}_\omega \hat{f}) = \sum_{m=0}^{l_k+r+1} \frac{1}{(l_k+r+1-m)!}\left(\frac{d}{d\lambda}\right)^{l_k+r+1-m}[\hat{K}_\omega(\lambda)(\lambda - i\alpha_k)]|_{\lambda=i\alpha_k} \cdot \gamma_{m-l_k}^{i\alpha_k}(\hat{f});$$

$$(4.24) \quad \gamma_r^{i(\alpha_k+1)}(\widehat{V_\omega f}) = \sum_{m=0}^{l_k+r+1} \frac{1}{(l_k+r+1-m)!}\left(\frac{d}{d\lambda}\right)^{l_k+r+1-m}[\hat{V}_\omega(\lambda)(\lambda - i\alpha_k - i)]|_{\lambda=i\alpha_k+1} \cdot \gamma_{m-l_k}^{i\alpha_k}(\hat{f}).$$

Special cases of (4.23), (4.24) are

$$(4.25) \quad \gamma_0^i(\hat{K}_\omega \hat{f}) = \frac{d}{d\lambda}[\hat{K}_\omega(\lambda)(\lambda - i)]|_{\lambda=1} \cdot \gamma_0^i(\hat{f}) + [K_\omega'(\lambda)(\lambda - i\alpha)]|_{\lambda=i} \cdot \gamma_1^i(\hat{f})$$

$$= \frac{\omega-\pi}{\pi}\cos\omega\,\gamma_0^i(\hat{f}) + \frac{i}{\pi}\sin\omega\gamma_1^i(\hat{f})$$

if $\hat{f}$ has a pole of first order at i;

$$(4.26) \quad \gamma_{-1}^i(\hat{K}_\omega \hat{f}) = \frac{i}{\pi}\sin\omega\,\gamma_0^i(\hat{f}) \quad \text{if } \hat{f} \text{ has a pole of first order at } i;$$

$$(4.27) \quad \gamma_{-1}^{i}(\widehat{V_\omega f}) = -\frac{i}{\pi}\cos\omega\,\gamma_0^0(\hat{f}) \quad \text{if } \hat{f} \text{ has a pole of first order at 0.}$$

4.3. Now we can investigate the regularity of the solution U of the integral equations (1.20) $\mathscr{A}_0 U = \mathscr{B}_0 G$. By the reduction to one of the three local forms (D), (N), (M), we get equations

$$(4.28) \qquad \mathscr{A}_0^{(x)} U = \mathscr{B}_0^{(x)} G + H \quad (x \in \{D, N, M\})$$

on Γ^∞, which correspond to (2×2)-systems on $\mathbf{R}_+$. The function H in (4.28)

comes from the localization and is thus C^∞. We will proceed as follows:

We assume that $G \in C^\infty$, and thus $G \mathrel{\hat{=}} \begin{bmatrix} g_- \\ g_+ \end{bmatrix}$, where $g_\pm$ has an expansion (4.1) with $\alpha_k = k = 0, 1, \ldots$, and $l_k = 0$. Then $\widehat{\mathscr{B}_0^{(x)} G} + \hat{H}$ will be meromorphic. Therefore also $\widehat{\mathscr{A}_0^{(x)} U} = \widehat{\mathscr{B}_0^{(x)} G} + \hat{H}$ is meromorphic and consequently also $\hat{U}$. Thus U has an expansion (4.1), where the regularity of the smooth remainder term can be deduced from Lemma 4.3. Also from the Parseval formula (2.4) and the equation

$$\hat{U} = \hat{\mathscr{A}}_0^{(x)-1}[\widehat{\mathscr{B}_0^{(x)} G} + \hat{H}]$$

it will be clear that the expansion coefficients of U as well as the smooth remainder depend continuously on G in some appropriate $\mathring{W}_0^s$ norm. By the local equivalence of norms, this leads finally to an a priori estimate for U on Γ of the same type which was given in (1.11) for the solution of the variational problem. By continuity, this holds not only for $G \in C^\infty$ but for G in the corresponding Sobolev spaces.

In the use of Lemma 4.3, we always start with the regularity of the weak solution, i.e., in some strip $\operatorname{Im}\lambda \in (0, \varepsilon)$, and then shift the path of integration for the inverse Mellin transform up to $\operatorname{Im}\lambda < 2$, which corresponds to Sobolev indices $s \in (1/2, 5/2)$. We will thus investigate the poles of the Mellin transforms in the strip $\operatorname{Im}\lambda \in (0, 2)$.

The Mellin transforms of the operators were calculated in Lemma 2.13, but from this for $\phi \in C^\infty[0, \infty)$ we can only deduce

$$\widehat{K_\omega \phi}(\lambda) = \hat{K}_\omega(\lambda) \cdot \hat{\phi}(\lambda) \quad \text{for} \quad \operatorname{Im}\lambda \in (-1, 0).$$

For the strip $\operatorname{Im}\lambda \in (0, 1)$, where we want to start, we have to use Lemma 4.3 together with (4.20) and (4.16), which gives

$$(4.29) \quad \begin{aligned} &K_\omega \phi(x) = \frac{\omega-\pi}{\pi}\phi(0) + \mathring{K}_\omega \phi(x) \quad \text{with} \\ &\widehat{\mathring{K}_\omega \phi}(\lambda) = \hat{K}_\omega(\lambda) \cdot \hat{\phi}(\lambda) \quad \text{for} \quad \operatorname{Im}\lambda \in (0, 1) \ (\phi \in C^\infty[0, \infty)). \end{aligned}$$

Now we compute the poles of $\hat{U}$ in the strip $\operatorname{Im}\lambda \in (0, 2)$ for the three local cases. The dependence on the given data G will be investigated, and several special cases will have to be distinguished. From the computations it will become clear how this procedure can be generalized to regions of higher regularity, i.e., to $\operatorname{Im}\lambda \geqslant 2$.

(D) Here the local forms of the operators are

$$(4.30) \quad \mathscr{A}_0 \mathrel{\hat{=}} \begin{bmatrix} V_0 & V_\omega \\ V_\omega & V_0 \end{bmatrix} + \begin{bmatrix} l & l \\ l & l \end{bmatrix}, \quad \mathscr{B}_0 \mathrel{\hat{=}} \begin{bmatrix} 1 & K_\omega \\ K_\omega & 1 \end{bmatrix}.$$

We assume $G\in C_0^\infty(\Gamma^\omega)$, i.e.,

(4.31) $$G=\begin{bmatrix} g_- \\ g_+\end{bmatrix},\quad \text{with}\quad g_\pm\in C_0^\infty[0,\infty),\quad g_+(0)=g_-(0),$$
$$\text{and}\quad g_+'(0)=-g_-'(0)\quad\text{if}\quad \omega=\pi.$$

For $U=\begin{bmatrix}\psi_-\\ \psi_+\end{bmatrix}$ we can assume an expansion (4.1)

(4.32) $$\psi_\pm(x)=\Big(\sum_{k=1}^{n}\sum_{l=0}^{l_k} c_{kl}^\pm x^{\alpha_k-1}\log^l x\Big)\chi(x)+\psi_\pm^0(x)\qquad(0<\alpha_1<\alpha_2<\ldots<\alpha_n)$$

where $\psi_\pm^0\in\tilde H^s(\boldsymbol{R}_+)$ for all $s<3/2$. First we look at the poles of $\widehat{\mathscr{B}_0 G}$: By (4.29) we have

(4.33) $$\mathscr{B}_0 G=\begin{bmatrix} g_-(0)+\dfrac{\omega-\pi}{\pi}g_+(0)\\[2mm] g_+(0)+\dfrac{\omega-\pi}{\pi}g_-(0)\end{bmatrix}+\mathring{\mathscr{B}}_0 G=\begin{bmatrix}1\\1\end{bmatrix}\frac{\omega}{\pi}g_+(0)+\mathring{\mathscr{B}}_0 G\quad\text{with}$$

$$\widehat{\mathring{\mathscr{B}}_0 G}(\lambda)=\hat{\mathring{\mathscr{B}}}_0(\lambda)\cdot\hat G(\lambda)=\begin{bmatrix}1 & \dfrac{-\sinh(\omega-\pi)\lambda}{\sinh\pi\lambda}\\[2mm] \dfrac{\sinh(\omega-\pi)\lambda}{\sinh\pi\lambda} & 1\end{bmatrix}\times$$
$$\times\begin{bmatrix}\hat g_-(\lambda)\\ \hat g_+(\lambda)\end{bmatrix}\qquad(\operatorname{Im}\lambda\in(0,1)).$$

In $\operatorname{Im}\lambda\in(0,2)$ there is only one pole at $\lambda_0=i$ of order 2 with residues

(4.34) $$\gamma_{-1}^i(\hat{\mathscr{B}}_0\hat G)=\frac{\sin\omega}{\pi}\begin{bmatrix}g_+'(0)\\ g_-'(0)\end{bmatrix}\quad\text{for}\quad\omega\neq\pi.$$

This follows from (4.26) and (4.17). In equation (4.28) there appears a function

(4.35) $$H=\begin{bmatrix}h_-\\ h_+\end{bmatrix}\in C_0^\infty(\Gamma^\omega),\quad h_\pm\in C_0^\infty[0,\infty);\quad h_+(0)=h_-(0);$$
$$\text{if}\quad\omega=\pi:\ h_+'(0)=h_-'(0).$$

$\hat H$ thus has a first order pole at $\lambda_0=i$, so that $\gamma_0^i(\hat{\mathscr{B}}_0\hat G+\hat H)$ is not determined by G alone.

(4.36) If $\omega=\pi$, then $\mathscr{B}_0=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, so $\mathscr{B}_0 G+H\in C^\infty$.

For $\omega\neq\pi$, (4.34) means that

(4.37) $$\mathscr{B}_0 G(x)=\begin{bmatrix}g_+'(0)\\ g_-'(0)\end{bmatrix}\cdot\frac{\sin\omega}{\pi}\cdot x\log x\cdot\chi(x)+H_0(x)\quad\text{with}\quad H_0\in C^\infty(\Gamma^\omega).$$

Now we calculate the coefficients in (4.32) from the requirement that $\widehat{\mathscr{A}_0 U}$ also has only a pole at $\lambda=i$ with the residues given by (4.34) or (4.36). We have

(4.38) $$\widehat{\mathscr{A}_0 U}(\lambda)=\hat{\mathscr{A}}_0(\lambda)\,\hat U(\lambda-i)$$
$$=\frac{1}{\lambda\sinh\pi\lambda}\begin{bmatrix}\cosh\pi\lambda & \cosh(\pi-\omega)\lambda\\ \cosh(\pi-\omega)\lambda & \cosh\pi\lambda\end{bmatrix}\begin{bmatrix}\hat\psi_-(\lambda-i)\\ \hat\psi_+(\lambda-i)\end{bmatrix}.$$

We use the operator $D=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$ to diagonalize $\hat{\mathscr{A}}_0$ and define

(4.39) $$D\begin{bmatrix}\psi_-\\ \psi_+\end{bmatrix}:=\begin{bmatrix}\psi_e\\ \psi_o\end{bmatrix}\quad\text{and have}$$
$$D\hat{\mathscr{A}}_0(\lambda)D^{-1}=\frac{1}{\lambda\sinh\pi\lambda}\begin{bmatrix}\cosh\pi\lambda+\cosh(\pi-\omega)\lambda & 0\\ 0 & \cosh\pi\lambda-\cosh(\pi-\omega)\lambda\end{bmatrix};$$

cf. (2.22). From

(4.40) $$\det\hat{\mathscr{A}}_0(\lambda)=\frac{\sinh(2\pi-\omega)\lambda\,\sinh\omega\lambda}{\lambda^2\sinh^2\pi\lambda}\quad\text{and}$$
$$\hat{\mathscr{A}}_0^{-1}=\frac{\lambda\sinh\pi\lambda}{\sinh(2\pi-\omega)\lambda\,\sinh\omega\lambda}\begin{bmatrix}\cosh\pi\lambda & -\cosh(\pi-\omega)\lambda\\ -\cosh(\pi-\omega)\lambda & \cosh\pi\lambda\end{bmatrix}$$

we see that $\hat U$ can have poles only at (λ_0-i) with $\det\hat{\mathscr{A}}_0(\lambda_0)=0$ of order at most 2 and at $\lambda_0=0$ of order at most one. This gives ($\lambda_0=i\alpha$; $\operatorname{Im}\lambda_0\in(0,2)$)

(4.41) (α) $$\sinh((2\pi-\omega)\cdot i\alpha)=0\ \Leftrightarrow\ \alpha=\frac{l\pi}{2\pi-\omega},\qquad l=1,2,3;$$

(4.42) (β) $$\sinh(\omega\cdot i\alpha)=0\ \Leftrightarrow\ \alpha=\frac{k\pi}{\omega},\qquad k=1,2,3.$$

Of course, it is not for all ω that all the six cases can appear. (α) and (β) together can appear only in the following cases:

(4.43) $$\omega=\frac{2\pi}{3},\quad\alpha=\frac{3}{2},\quad l=2,\quad k=1\quad\text{and}$$
$$\omega=\frac{4\pi}{3},\quad\alpha=\frac{3}{2},\quad l=1,\quad k=2.$$

In these cases $\cos \pi\alpha = 0 = \cos(\pi-\omega)\alpha$, so that $\hat{\mathscr{A}}_0(i\alpha) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and

$$\frac{d}{d\lambda}\hat{\mathscr{A}}_0(\lambda)|_{\lambda=i\alpha} = \frac{\pi}{i\alpha}\begin{bmatrix} -1 & 1/3 \\ 1/3 & -1 \end{bmatrix}. \tag{4.44}$$

Therefore from the conditions

$$\gamma_0^{i\alpha}(\hat{\mathscr{A}}_0\hat{U}) = \gamma_{-1}^{i\alpha}(\hat{\mathscr{A}}_0\hat{U}) = 0$$

it follows by (4.22) and the regularity of $\widehat{\mathscr{A}_0 U}$ at $\lambda = i$ that

$$\gamma_{-1}^{i(\alpha-1)}(\hat{U}) = 0; \text{ whereas} \tag{4.45}$$
$$\gamma_0^{i(\alpha-1)}(\hat{U}) = \begin{bmatrix} c_- \\ c_+ \end{bmatrix} \text{ is an arbitrary 2-dimensional vector.}$$

This means that the contribution of this pole is of the form

$$\begin{bmatrix} c_- \\ c_+ \end{bmatrix}\cdot x^{\alpha-1}.$$

When only (α) is fulfilled, we have $\cos \pi\alpha \neq 0$ and

$$\cos(\pi-\omega)\alpha = (-1)^l \cos \pi\alpha. \tag{4.46}$$

Therefore

$$D\hat{\mathscr{A}}_0(i\alpha)D^{-1} = \frac{\cos \pi\alpha}{-\alpha \sin \pi\alpha}\begin{bmatrix} 1+(-1)^l & 0 \\ 0 & 1-(-1)^l \end{bmatrix}$$

and this implies $\gamma_0^{i(\alpha-1)}(\hat{\psi}_e) = 0$ for l even and $\gamma_0^{i(\alpha-1)}(\hat{\psi}_0) = 0$ for l odd, or

$$\gamma_0^{i(\alpha-1)}(\hat{U}) = \begin{bmatrix} 1 \\ (-1)^{l+1} \end{bmatrix}\cdot c_\alpha \quad \text{if} \quad \alpha = \frac{l\pi}{2\pi-\omega}. \tag{4.47}$$

Similarly in case (β) one obtains $\cos(\pi-\omega)\alpha = (-1)^k \cos \pi\alpha$, whence

$$\gamma_0^{i(\alpha-1)}(\hat{U}) = \begin{bmatrix} 1 \\ (-1)^{k+1} \end{bmatrix}\cdot c_\alpha \quad \text{if} \quad \alpha = \frac{k\pi}{\omega}. \tag{4.48}$$

In both cases this gives a contribution $c_\alpha\begin{bmatrix} 1 \\ \pm 1 \end{bmatrix}\cdot x^{\alpha-1}$, which runs through a 1-dimensional space if G and H vary.

At the point $\lambda_0 = i$ by (4.34) ($\omega \neq \pi$) $\widehat{\mathscr{A}_0 U}$ must have a pole of second order. From the form of $\hat{\mathscr{A}}_0^{-1}$ in (4.40) and $\sin(2\pi-\omega) \neq 0 \neq \sin \omega$ it follows that U has a first order pole at $\lambda = 0$ with residues (4.27).

(4.49)
$$\gamma_0^0(\hat{U}) = i\pi\begin{bmatrix} 1 & \cos \omega \\ \cos \omega & 1 \end{bmatrix}^{-1}\gamma_{-i}^i(\widehat{\mathscr{A}_0 U}) \doteq \frac{i}{\sin \omega}\begin{bmatrix} 1 & -\cos \omega \\ \cos \omega & 1 \end{bmatrix}\begin{bmatrix} g'_+(0) \\ g'_-(0) \end{bmatrix}.$$

This shows that the smoth parts of $\psi_\pm$ will in general have different limits at 0, i.e., will not be contained in $H^{s-1}(\Gamma^\omega)$ but rather in $H^{s-1}(\boldsymbol{R}_+)\times$ $\times H^{s-1}(\boldsymbol{R}_+)$.

For $\omega = \pi$ one has

$$\hat{\mathscr{A}}_0(\lambda) = \frac{1}{\lambda \sinh \pi\lambda}\begin{bmatrix} \cosh \pi\lambda & 1 \\ 1 & \cosh \pi\lambda \end{bmatrix}; \tag{4.50}$$

$$\hat{\mathscr{A}}_0^{-1}(\lambda) = \frac{\lambda}{\sinh \pi\lambda}\begin{bmatrix} \cosh \pi\lambda & -1 \\ -1 & \cosh \pi\lambda \end{bmatrix}.$$

By (4.36), one has to require that $\widehat{\mathscr{A}_0 U}$ should have a first order pole at $\lambda_0 = i$ with residue

$$\gamma_0^i(\widehat{\mathscr{A}_0 U}) = C\begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{4.51}$$

Thus $\widehat{\mathscr{A}_0 U}(\lambda) = \dfrac{c}{\lambda-i}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \hat{U}_0(\lambda)$, where $\hat{U}_0$ is regular at $\lambda = i$. This gives

$$\hat{U}(\lambda-i) = \frac{c\lambda}{\sinh \pi\lambda\cdot(\lambda-i)}\begin{bmatrix} -\cosh \pi\lambda+1 \\ -\cosh \pi\lambda-1 \end{bmatrix} + \hat{\mathscr{A}}_0^{-1}(\lambda)\hat{U}_0(\lambda). \tag{4.52}$$

Since $\cosh \pi\lambda + 1$ has a double zero at $\lambda = i$, the first term in (4.52) is regular at $\lambda = i$. The second one can be written as

$$\frac{i}{\sinh \pi\lambda}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\hat{U}_0(i) + \hat{U}_1(\lambda) \quad \text{with } \hat{U}_1 \text{ regular at } \lambda = i.$$

Therefore we obtain

$$\hat{U}(\lambda-i) = \frac{c}{\lambda-i}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \hat{U}_2(\lambda) \quad \text{with } \hat{U}_2 \text{ regular at } \lambda = i \text{ or}$$

$$\gamma_0^0(\hat{U}) = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.53}$$

This means that, in contrast to the case $\omega \neq \pi$, the smooth part of U is continuous at the "corner point". There are no poles of case (α) or (β) for $\omega = \pi$, and so U will be contained in $H^s(\Gamma^\omega)$ for all $s < 3/2$.

(N) Here the local forms of the operators are

$$\mathscr{A}_0 \mathrel{\hat{=}} \begin{bmatrix} 1 & K_\omega \\ K_\omega & 1 \end{bmatrix}; \quad \mathscr{B}_0 \mathrel{\hat{=}} \begin{bmatrix} V_0 & V_\omega \\ V_\omega & V_0 \end{bmatrix} + \begin{bmatrix} l & l \\ l & l \end{bmatrix}. \tag{4.54}$$

For G we now assume

$$G = \begin{bmatrix} g_- \\ g_+ \end{bmatrix} \quad \text{with} \quad g_\pm \in C_0^\infty[0, \infty), \text{ and } g_+(0) = g_-(0) \text{ if } \omega = \pi. \tag{4.55}$$

For H we again have $H \in C_0^\infty(\Gamma^\omega)$, i.e., (4.33). The pole of $\widehat{\mathcal{B}_0 G}$ at $\lambda = i$ has residues (by (4.27))

(4.56)
$$\gamma_{-1}^i(\widehat{\mathcal{B}_0 G}) = -\frac{i}{\pi}\begin{bmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{bmatrix}\gamma_0^0(\hat{G}) = -\frac{1}{\pi}\begin{bmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{bmatrix}\begin{bmatrix} g_-(0) \\ g_+(0) \end{bmatrix}.$$

For $\omega \neq \pi$, this means that $\mathcal{B}_0 G$ contains terms $x \log x$. For $\omega = \pi$, there is only a first order pole (note $g_-(0) = g_+(0)$) with residue

$$\gamma_0^i(\widehat{\mathcal{B}_0 G}) = \frac{1}{\pi}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}[\gamma_0^0(\hat{G}) - i\gamma_1^0(\hat{U})] = c\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{4.57}$$

which means that $\mathcal{B}_0 G$ (which is continuous at 0 by theorem 2.17) also has a continuous derivative at 0. For $U = \begin{bmatrix} v_- \\ v_+ \end{bmatrix}$ we can assume an expansion (4.1)

$$v_\pm(x) = \Big(\sum_{k=1}^{n}\sum_{l=0}^{l_k} c_{kl}^\pm x^{\alpha_k}\log^l x\Big)\chi(x) + v_\pm^0(x) \quad (0 < \alpha_1 < \ldots < \alpha_n) \tag{4.58}$$

where $v_\pm^0 \in \tilde{H}^s(\boldsymbol{R}_+)$ for all $s < 5/2$. From (4.29) follows

$$\mathcal{A}_0 U = \begin{bmatrix} 1 & \dfrac{\omega-\pi}{\pi} \\ \dfrac{\omega-\pi}{\pi} & 1 \end{bmatrix}\cdot\begin{bmatrix} v_-(0) \\ v_+(0) \end{bmatrix} + \mathring{\mathcal{A}}_0 U \quad \text{with} \tag{4.59}$$

$$\widehat{\mathring{\mathcal{A}}_0 U}(\lambda) = \hat{\mathcal{A}}_0(\lambda)\cdot\hat{U}(\lambda) \text{ for } \operatorname{Im}\lambda \in (0, \alpha_0) \ (\alpha_0 = \min\{\alpha_1, 1\}).$$

As $\mathcal{A}_0 U$ has to be continuous and $\mathring{\mathcal{A}}_0 U(0) = 0$, it follows that

$$v_+(0) = v_-(0), \quad \text{i.e., } U \text{ is continuous.} \tag{4.60}$$

In $\operatorname{Im}\lambda \in (0, 1) \cup (1, 2)$ poles of $\hat{U}$ can only appear where $\det \hat{\mathcal{A}}_0(\lambda) = 0$. We have

(4.61)
$$\det \hat{\mathcal{A}}_0(\lambda) = \frac{\sinh(2\pi-\omega)\lambda \sinh \pi\lambda}{\sinh^2 \pi\lambda},$$
$$\hat{\mathcal{A}}_0^{-1}(\lambda) = \frac{\sinh \pi\lambda}{\sinh(2\pi-\omega)\lambda \sinh \omega\lambda}\begin{bmatrix} \sinh \pi\lambda & \sinh(\pi-\omega)\lambda \\ \sinh(\pi-\omega)\lambda & \sinh \pi\lambda \end{bmatrix}.$$

Therefore poles occur in the same cases (α) and (β) as for (D) ((4.41), (4.42)). In case (4.43), i.e., if (α) and (β) are simultaneously satisfied we now obtain for $\hat{U}$ a second order pole, where the residues have to statisfy

$$\begin{bmatrix} 1 & (-1)^k \\ (-1)^k & 1 \end{bmatrix}\gamma_{-1}^{i\alpha}(\hat{U}) = 0 = \begin{bmatrix} 1 & (-1)^k \\ (-1)^k & 1 \end{bmatrix}\gamma_0^{i\alpha}(\hat{U}), \tag{4.62}$$

in order that $\widehat{\mathcal{A}_0 U}$ should be regular at $\lambda_0 = i\alpha$. This implies that in this case U contains a term

$$\begin{bmatrix} 1 \\ (-1)^{k+1} \end{bmatrix}(c_1 + c_2 \log x)\, x^{3/2}\cdot\chi(x) \tag{4.63}$$

if $\widehat{\mathcal{B}_0 G}(3i/2) + \hat{H}(3i/2)$ is arbitrary. As follows later on from the equivalence theorem, for the actual problem it is not the case that even all poles from case (α) are absent. This means especially that terms of the form (4.63) will not appear in the actual solution of system (1.20).

When only (α) is fulfilled, we have

$$\sin(\pi-\omega)\alpha = (-1)^{l+1}\sin \pi\alpha, \tag{4.64}$$

and thus $\hat{\mathcal{A}}_0(i\alpha) = \begin{bmatrix} -1 & (-1)^{l+1} \\ (-1)^{l+1} & -1 \end{bmatrix}$, which gives

$$\gamma_0^{i\alpha}(\hat{U}) = \begin{bmatrix} 1 \\ (-1)^l \end{bmatrix}\cdot c_\alpha, \tag{4.65}$$

whereas in case (β) $\sin(\pi-\omega)\alpha = (-1)^k \sin \pi\alpha$ and thus

$$\hat{\mathcal{A}}_0(i\alpha) = -\begin{bmatrix} -1 & (-1)^k \\ (-1)^k & -1 \end{bmatrix},$$

whence

$$\gamma_0^{i\alpha}(\hat{U}) = \begin{bmatrix} 1 \\ (-1)^k \end{bmatrix}\cdot c_\alpha. \tag{4.66}$$

For $\lambda_0 = i$, $\omega \neq \pi$, $\widehat{\mathcal{A}_0 U}$ must have a second order pole with residues (4.56). From (4.61) it follows that $\hat{U}$ has a first order pole there, and the residues satisfy (4.26)

$$-\frac{1}{\pi}\begin{bmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{bmatrix}\begin{bmatrix} g_-(0) \\ g_+(0) \end{bmatrix} = \gamma_{-1}^i(\hat{\mathcal{A}}_0\hat{U}) = \begin{bmatrix} \gamma_{-1}^i(\hat{K}_\omega \hat{v}_+) \\ \gamma_{-1}^i(\hat{K}_\omega \hat{v}_+) \end{bmatrix} = \frac{i}{\pi}\sin\omega\, \gamma_0^i(\hat{U}).$$

This means that

$$\gamma_0^i(\hat{U}) = \frac{i}{\sin\omega}\begin{bmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{bmatrix}\begin{bmatrix} g_-(0) \\ g_+(0) \end{bmatrix}. \tag{4.67}$$

This equality shows that it makes no sense to assume that G is continuous, i.e., $g_+(0) = g_-(0)$, because in this case the smooth part of $\hat{U}$ would have to satisfy a compatibility condition of the form

$$v_+'(0) = v_-'(0), \tag{4.68}$$

which makes no sense. For $\omega = \pi$ we have $\hat{\mathscr{A}}_0(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, whence $\hat{U} = \mathscr{A}_0 \hat{U}$ has to have a first order pole at $\lambda_0 = i$ with residue (4.57)

$$\gamma_0^i(\hat{U}) = c \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{4.69}$$

which means that the smooth part of U is continuously differentiable at 0.

Of course, for $\omega = \pi$, there appear no poles of the form (α) or (β), and thus U itself is contained in $H^s(\Gamma^\omega)$ for all $s < 5/2$.

Remark 4.4. In both case (α) and case (β) we have $|\alpha| > 1/2$ for the poles of $\hat{U}(\lambda)$ at $\lambda = i\alpha$. Therefore in the strip $|\text{Im}\ \lambda| \leq 1/2$ there is only the pole at $\lambda = 0$, which is completely described by (4.59). This means that we can start at $\text{Im}\ \lambda = -1/2$, which corresponds to L^2, and then shift the path of integration up to $\text{Im}\ \lambda < \alpha_0 > 0$, and we only obtain an extra "singular" term from $\lambda = 0$, which, however, is constant near 0 by (4.56), (4.57). By using the density of $C_0^\infty(\Gamma^\omega)$ in $H^s(\Gamma^\omega)$ and Parseval's equation (2.4), we obtain.

COROLLARY 4.5. *Let $v \in L^2(\Gamma)$ and $s \in [0, 1]$. If $(1+K)v = h$ with $h \in H^s(\Gamma)$, then $v \in H^s(\Gamma)$.*

Proof. For $s \neq 1/2$ the result follows from the previous discussion. Then the result for $s = 1/2$ follows by interpolation. ∎

(M) We describe only the case (M_1), which means that Γ_- corresponds to Γ_1 and Γ_+ corresponds to Γ_2. Thus we assume $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ with $g_{1,2} \in C_0^\infty[0, \infty)$. We write $\hat{G}(\lambda) = \begin{bmatrix} \hat{g}_1(\lambda) \\ \hat{g}_2(\lambda - i) \end{bmatrix}$.

For H we assume again (4.35)

The local forms of the operators are now

$$\mathscr{A}_0 \hat{=} \begin{bmatrix} I & 0 \\ -I & 0 \end{bmatrix} + \begin{bmatrix} V_0 & -K_\omega \\ -V_\omega & 1 \end{bmatrix};$$
$$\mathscr{B}_0 \hat{=} \begin{bmatrix} 0 & -I \\ 0 & I \end{bmatrix} + \begin{bmatrix} 1 & -V_\omega \\ -K_\omega & V_0 \end{bmatrix}. \tag{4.70}$$

For $U = \begin{bmatrix} \psi \\ v \end{bmatrix}$ we can assume expansions (4.32) for ψ and (4.58) for v. Then $\hat{U}(\lambda) = \begin{bmatrix} \hat{\psi}(\lambda - i) \\ \hat{v}(\lambda) \end{bmatrix}$. From

$$\hat{\mathscr{B}}_0(\lambda) = \frac{1}{\lambda \sinh \pi\lambda} \begin{bmatrix} \lambda \sinh \pi\lambda & -\cosh(\pi - \omega)\lambda \\ \lambda \sinh(\pi - \omega)\lambda & \cosh \pi\lambda \end{bmatrix}.$$

we see that $\widehat{\mathscr{B}_0 G}$ has a second order pole at $\lambda = i$ with residues (compare (4.26), (4.27))

$$\gamma_{-1}^i(\widehat{\mathscr{B}_0 G}) = \frac{i}{\pi} \begin{bmatrix} 0 & \cos\omega \\ -\sin\omega & -1 \end{bmatrix} \begin{bmatrix} \gamma_0^i(\hat{g}_1) \\ \gamma_0^0(\hat{g}_2) \end{bmatrix} = \frac{1}{\pi} \begin{bmatrix} 0 & \cos\omega \\ -\sin\omega & -1 \end{bmatrix} \begin{bmatrix} g_1'(0) \\ g_2(0) \end{bmatrix}. \tag{4.71}$$

This implies that we have to distinguish several cases:

$\omega \notin \{\pi/2, \pi, 3\pi/2\}$: $\mathscr{B}_0 G$ contains a term $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x \log x\chi(x)$, $c_{1,2}$ being arbitrary,

$\omega = \pi$: $\mathscr{B}_0 G$ contains a term $-\frac{1}{\pi}\begin{bmatrix} 1 \\ 1 \end{bmatrix} g_2(0)\, x \log x\chi(x)$, which is smooth only if

$$g_2(0) = 0; \tag{4.72}$$

$\omega = \frac{\pi}{2}$: $\mathscr{B}_0 G$ contains a term $\frac{-1}{\pi}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(g_1'(0) + g_2(0))$, which is smooth for

$$g_1'(0) = -g_2(0); \tag{4.73}$$

$\omega = \frac{3\pi}{2}$: $\mathscr{B}_0 G$ contains a term $\frac{1}{\pi}\begin{bmatrix} 0 \\ 1 \end{bmatrix}(g_1'(0) - g_2(0))$, which is smooth for

$$g_1'(0) = g_2(0). \tag{4.74}$$

We have

(4.75)

$$\det \hat{\mathscr{A}}_0(\lambda) = \frac{\sinh(2\pi - \omega)\lambda \cosh \omega\lambda}{\lambda \sinh^2 \pi\lambda},$$

$$\hat{\mathscr{A}}_0^{-1}(\lambda) = \frac{\sinh \pi\lambda}{\sinh(2\pi - \omega)\lambda \cosh \omega\lambda} \begin{bmatrix} \lambda \sinh \pi\lambda & -\lambda \sinh(\pi - \omega)\lambda \\ \cosh(\pi - \omega)\lambda & \cosh \pi\lambda \end{bmatrix}.$$

For $\lambda = i$, $\omega \notin \{\pi/2, \pi, 3\pi/2\}$, $\hat{\mathscr{A}}_0^{-1}(\lambda)$ vanishes at $\lambda = i$; therefore $\hat{U}$ has there a first order pole whose residues satisfy

$$\frac{1}{\pi} \begin{bmatrix} 0 & \cos\omega \\ -\sin\omega & -1 \end{bmatrix} \begin{bmatrix} g_1'(0) \\ g_2(0) \end{bmatrix} = \gamma_{-1}^i(\widehat{\mathscr{A}_0 U}) = \frac{1}{\pi} \begin{bmatrix} -1 & -\sin\omega \\ \cos\omega & 0 \end{bmatrix} \begin{bmatrix} \psi^s(0) \\ v^{s\prime}(0) \end{bmatrix},$$

where ψ^s and v^s denote the smooth parts of ψ and v, respectively: that is, ψ^s

and v^s do not contain, respectively, the $x^{\alpha_k-1}\log^l x$ and the $x^{\alpha_k}\log^l x$ -terms for $\alpha_k < 1$. Therefore

(4.76) $$\begin{bmatrix}\psi^s(0)\\ v^{s'}(0)\end{bmatrix} = \frac{1}{\cos\omega}\begin{bmatrix}-\sin\omega & -1\\ 1 & \sin\omega\end{bmatrix}\begin{bmatrix}g_1'(0)\\ g_2(0)\end{bmatrix}.$$

This is also valid for $\omega = \pi$, where it gives

(4.77) $$\psi^s(0) = g_2(0),$$

which is a natural continuity property. There is no such condition on v^s. Note that if G is arbitrary, (4.77) is no restriction on U.

For $\omega = \pi/2$, $\hat{\mathscr{A}}_0^{-1}$ is regular at $\lambda = i$, but it does not vanish. Therefore $\hat{U}$ has a second order pole with residues

(4.78) $$\gamma_{-1}^i(\hat{U}) = \hat{\mathscr{A}}_0^{-1}(i)\gamma_{-1}^i(\widehat{\mathscr{A}_0 U}) = \begin{bmatrix}0 & -2\\ 0 & 2\end{bmatrix}\frac{1}{\pi}\begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}g_1'(0)\\ g_2(0)\end{bmatrix}$$
$$= \frac{2}{\pi}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}g_1'(0)\\ g_2(0)\end{bmatrix} = \frac{2}{\pi}\begin{bmatrix}1\\ -1\end{bmatrix}\cdot(g_1'(0)+g_2(0)).$$

Therefore, U will contain a term $\frac{2}{\pi}\begin{bmatrix}1\\ x\end{bmatrix}(g_1'(0)+g_2(0))\log x\cdot\chi(x)$, which is absent if (4.73) is satisfied. In this case, $\hat{U}$ has a first order pole at $\lambda = i$, and the residues satisfy

$$0 = \frac{1}{\pi}\begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}g_1'(0)\\ g_2'(0)\end{bmatrix} = \gamma_{-1}^i(\widehat{\mathscr{B}_0 G}) = \gamma_{-1}^i(\widehat{\mathscr{A}_0 U}) = \frac{1}{\pi}\begin{bmatrix}-1 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi^s(0)\\ v^{sl}(0)\end{bmatrix}.$$

This gives the necessary condition

(4.79) $$v^{sl}(0) = -\psi^s(0).$$

Similarly, for $\omega = 3\pi/2$, we obtain a term $\frac{2}{\pi}\begin{bmatrix}1\\ x\end{bmatrix}(g_1'(0)-g_2(0))\log x\cdot\chi(x)$, which disappears if (4.74) is satisfied. In this case we get

(4.80) $$v^{sl}(0) = \psi^s(0).$$

Except at $\lambda_0 = i$, there are poles at λ_0 with $\det\hat{\mathscr{A}}_0(\lambda_0) = 0$. This gives first the case (α) of (4.41), and secondly

(4.81) (γ) $$\cosh\omega\cdot i\alpha = 0 \Leftrightarrow \alpha = \frac{2k-1}{2}\frac{\pi}{\omega}, \quad k = 1, 2, 3.$$

Cases (α) and (γ) can simultaneously occur in several instance, namely for

$$\omega = \frac{2\pi}{3}, \quad \alpha = \frac{3}{4} \quad (l = 1, k = 1),$$

$$\omega = \frac{2\pi}{5}, \quad \alpha = \frac{5}{4} \quad (l = 2, k = 1),$$
$$\omega = \frac{6\pi}{5}, \quad \alpha = \frac{5}{4} \quad (l = 1, k = 2),$$
$$\omega = \frac{2\pi}{7}, \quad \alpha = \frac{7}{4} \quad (l = 3, k = 1),$$
$$\omega = \frac{6\pi}{7}, \quad \alpha = \frac{7}{4} \quad (l = 2, k = 2),$$
$$\omega = \frac{10\pi}{7}, \quad \alpha = \frac{7}{4} \quad (l = 1, k = 3).$$

In all these cases, U contains terms of the form

(4.82) $$x^{\alpha-1}\cdot\begin{bmatrix}c_1+c_2\log x\\ c_3 x+c_4 x\log x\end{bmatrix}\cdot\chi(x),$$

which run through a 2-dimensional space determined by the residue conditions:

(4.83) $$0 = \gamma_{-1}^{i\alpha}(\widehat{\mathscr{A}_0 U}) = \hat{\mathscr{A}}_0(i\alpha)\cdot\gamma_{-1}^{i\alpha}(\hat{U});$$
$$0 = \gamma_0^{i\alpha}(\widehat{\mathscr{A}_0 U}) = \frac{d}{d\lambda}\hat{\mathscr{A}}_0(i\alpha)\gamma_{-1}^{i\alpha}(\hat{U}) - i\hat{\mathscr{A}}_0(i\alpha)\gamma_0^{i\alpha}(\hat{U}).$$

We will not calculate the details, because in the actual solution of the boundary value problem the poles of case (α) and thus the double poles are absent. If only (α) is satisfied, we have $\cos(\pi-\omega)\alpha = (-1)^l\cos\pi\alpha$; $\sin(\pi-\omega)\alpha = (-1)^{l+1}\sin\pi\alpha$; therefore

(4.84) $$\hat{\mathscr{A}}_0^{-1}(\lambda) = \frac{1}{\sinh(2\pi-\omega)\lambda}\hat{\mathscr{A}}_0^0(\lambda), \quad \text{where } \hat{\mathscr{A}}_0^0(\lambda) \text{ is regular at } \lambda = i\alpha$$
$$\hat{\mathscr{A}}_0^0(i\alpha) = \frac{i\sin\pi\alpha}{\cos\omega\alpha}\begin{bmatrix}-\alpha\sin\pi\alpha & (-1)^{l+1}\sin\pi\alpha\\ (-1)^l\cos\pi\alpha & \cos\pi\alpha\end{bmatrix}.$$

This gives

$$\gamma_0^{i\alpha}(\hat{U}) = c\begin{bmatrix}-\alpha\sin\pi\alpha\\ \cos(\pi-\omega)\alpha\end{bmatrix},$$

which produces the term

(4.85) $$c\begin{bmatrix}-\alpha\sin\pi\alpha\cdot x^{\alpha-1}\\ \cos(\pi-\omega)\alpha\cdot x^\alpha\end{bmatrix}\cdot\chi(x) \quad (c \text{ is some constant})$$

in the expansion of U. If only case (γ) is satisfied, we have

$$\cos(\pi-\omega)\alpha=(-1)^{k+1}\sin\pi\alpha;\quad \sin(\pi-\omega)a=(-1)^k\cos\pi\alpha,$$

whence

$$\hat{\mathscr{A}}_0^{-1}(\lambda)=\frac{1}{\cosh\omega\lambda}\cdot\hat{\mathscr{A}}_0^1(\lambda)\text{ with}\tag{4.86}$$
$$\hat{\mathscr{A}}_0^1(i\alpha)=\frac{\sin\pi\alpha}{\sin(2\pi-\omega)}\begin{bmatrix}-\alpha\sin\pi\alpha & (-1)^k\cos\pi\alpha\\ (-1)^{k+1}\sin\pi\alpha & \cos\pi\alpha\end{bmatrix}.$$

This gives

$$\gamma_0^{i\alpha}(\hat{U})=c\begin{bmatrix}-\alpha\\ (-1)^{k+1}\end{bmatrix}=c\begin{bmatrix}-\alpha\\ \sin\omega\alpha\end{bmatrix},$$

which produces the term

$$\begin{bmatrix}-\alpha x^{\alpha-1}\\ (-1)^{k+1}x^\alpha\end{bmatrix}\chi(x)=c\begin{bmatrix}-\alpha x^{\alpha-1}\\ \sin\omega\alpha\cdot x^\alpha\end{bmatrix}\cdot\chi(x)\tag{4.87}$$

(c some constant) in the expansion of U.

Remark 4.6. If we compare the local expansions of U at the various corners which we have just studied with the expansion which is known for the solution of the boundary value problem (as described in (1.12), (1.13)), then, taking into account that $v=u|_{\Gamma_2}$ and $\psi=(\partial u/\partial n)|_{\Gamma_1}$, we see that the poles of cases (β) and (γ) yield just the singular functions of the latter problem. The singular functions from poles of case (α) would correspond to solutions of an exterior Dirichlet problem. This is natural, because the operator $\mathscr{A}_0$ alone does not "know" that we want to solve a mixed interior problem. The disappearance of these singular functions for the actual solution results from the fact that the right-hand side of the integral equations is of the form $\mathscr{B}_0 G$ and there is no global smooth perturbation H of this term such as occurred in the case of the localized problems (4.28).

Thus if we want to describe the mapping properties of the operator $\mathscr{A}_0$, we have to use the singular functions of case (α) in order to obtain the bijectivity of $\mathscr{A}_0$, but if we only want to describe the regularity ot the solution U of $\mathscr{A}_0 U=\mathscr{B}_0 G$, we do not need them.

4.4. Let us now sum up our calculations: Define

$$\alpha_{jl}:=\begin{cases}\dfrac{l\pi}{\omega_j} & \text{for}\quad j\in D\cup N \text{ and } \omega_j\neq\pi;\\[2mm] \dfrac{2l-1}{2}\dfrac{\pi}{\omega_j} & \text{for}\quad j\in M \text{ and } \omega_j\notin\left\{\dfrac{\pi}{2},\dfrac{3\pi}{2}\right\};\end{cases}$$

$$\alpha_{jl}^0:=\frac{l\pi}{2\pi-\omega_j}\quad\text{for all } j=1,\ldots,J.$$

Then we define, in terms of local coordinates which are given by the natural identification

$$\Gamma^j\cup\{t_j\}\cup\Gamma^{j+1}\subset\Gamma^{\omega j}=[0,\infty)\cup[0,\infty),$$

for $j\in D$ and $\omega_j\neq\pi$:

$$u_{jk}=\begin{bmatrix}1\\ (-1)^{k+1}\end{bmatrix}\cdot x^{\alpha_{jk}-1},\quad u_{jk}^0=\begin{bmatrix}1\\ (-1)^{k+1}\end{bmatrix}\cdot x^{\alpha_{jk}^0-1};\tag{4.88}$$

for $j\in N$ and $\omega_j\neq\pi$:

$$u_{jk}=\begin{bmatrix}1\\ (-1)^k\end{bmatrix}\cdot x^{\alpha_{jk}},$$

$$u_{jk}^0=\begin{bmatrix}1\\ (-1)^{k+1}\end{bmatrix}\cdot x^{\alpha_{jk}^0}\text{ if there is no } l\in N \text{ such that } \alpha_{jk}^0=\alpha_{jl};\tag{4.89}$$

$$u_{jk}^0=\begin{bmatrix}1\\ (-1)^{k+1}\end{bmatrix}\cdot x^{\alpha_{jk}^0}\log x\text{ if } \alpha_{jk}^0=\alpha_{jl}\text{ for some } l;$$

for $j\in M$ and $\omega_j\notin\{\pi/2, 3\pi/2\}$:

(4.90) (i) $j\in M_1$, i.e., $\Gamma^j\subset\Gamma_1$ and $\Gamma^{j+1}\subset\Gamma_2$:

$$u_{jk}=\begin{bmatrix}\alpha_{jk}x^{\alpha_{jk}-1}\\ (-1)^k x^{\alpha_{jk}}\end{bmatrix}$$

$$u_{jk}^0=\begin{bmatrix}-\alpha_{jk}^0\sin\pi\alpha_{jk}^0 x^{\alpha_{jk}^0-1}\\ \cos(\pi-\omega)\alpha_{jk}^0 x^{\alpha_{jk}^0}\end{bmatrix}\text{ if } \alpha_{jk}^0\neq\alpha_{jl}\text{ for all } l;$$

$$u_{jk}^0=\begin{bmatrix}a_{jk}+b_{jk}\log x\\ c_{jk}x+d_{jk}x\log x\end{bmatrix}\cdot x^{\alpha_{jk}^0-1}\text{ if } \alpha_{jk}^0=\alpha_{jl}\text{ for some } l,$$

where $a_{jk},\ldots,d_{jk}$ are determined from (4.82), (4.83).

(4.91) (ii) $j\in M_2$, i.e., $\Gamma^j\subset\Gamma_2$ and $\Gamma^{j+1}\subset\Gamma_1$: as for $j\in M_1$ but the two components of each vector are interchanged.

For the angles which are excluded at each corner we define no α_{jk}, α_{jk}^0, u_{jk}, or u_{jk}^0.

Finally, for every $j=1,\ldots,J$ we define

$$h_j^-=\begin{bmatrix}x\log x\\ 0\end{bmatrix},\quad h_j^+=\begin{bmatrix}0\\ x\log x\end{bmatrix}.\tag{4.92}$$

By A we denote the set of all exceptional exponents, i.e.,

$$A=\{\alpha_{jk}|\ j\in\{1,\ldots,J\},\ k\in N\}\cap(0,2),\tag{4.93}$$

and similarly

(4.94) $$A^0 = \{\alpha_{jk}^0 |\ j\in\{1,\dots,J\},\ k\in N\}\cap(0,2).$$

Now we can define the spaces $\mathscr{X}^s$, $\mathscr{Y}^s$, and $\mathscr{Z}^s$, in which, respectively, the functions G, $\mathscr{B}_0 G$, and U will be contained.

Let $\{\chi_j |\ j=1,\dots,J\}$ be a collection of cut-off functions with properties (2.8). The *space* $\mathscr{X}^s$ is defined for all $s\in[1/2, 5/2)$:

(4.95) $g\in\mathscr{X}^s$: $\Leftrightarrow$ (i) $g|_{\Gamma_1}\in H^s(\Gamma_1)$;

(ii) $g|_{\Gamma_2}\in\tilde{H}^{s-1}(\Gamma_2)$ for $s\in[1/2, 3/2)$;

(iii) $g|_{\Gamma^j}\in H^{s-1}(\Gamma^j)$ for $\Gamma^j\subset\Gamma_2$, $s\in[3/2, 5/2]$;

(iv) $\chi_j g\in H^{s-1}(\Gamma^{\omega_j})$ for $j\in N$, $s\in[3/2, 5/2]$, $\omega_j=\pi$;

(v) $\chi_j g = \begin{bmatrix} g_- \\ g_+\end{bmatrix}$ with $\left.\begin{matrix} g'_- - g_+ \\ g'_- + g_+ \\ g_- + g'_+ \\ g_- - g'_+\end{matrix}\right\}\in\tilde{H}^{s-1}(R_+)$ for $\begin{cases} j\in M_1,\ \omega_j=3\pi/2, \\ j\in M_1,\ \omega_j=\pi/2, \\ j\in M_2,\ \omega_j=\pi/2, \\ j\in M_2,\ \omega_j=2\pi/2,\end{cases}$

and $s\in[3/2, 5/2]$.

The *space* $\mathscr{Y}^s$ is defined for all $s\in[1/2, 5/2)\setminus\{3/2\}$ (and for $s=3/2$ if all angles are of the type excluded in (4.88), (4.89), (4.90)):

(4.96) $h\in\mathscr{Y}^s$: $\Leftrightarrow$ $h = h_0 + \sum_{\varepsilon=\pm}\sum_{j=1}^{J} c_j^\varepsilon h_j^\varepsilon \chi_j$ with $c_j^\pm$ constants,

$h_j^\pm$ defined in (4.92),

(i) $h_0|_{\Gamma_1}\in H^s(\Gamma_1)$ and $h_0|_{\Gamma_2}\in H^s(\Gamma_2)$;

(ii) all $c_j^\pm = 0$ for $s<3/2$;

(iii) $c_j^\pm = 0$ for $j\in D\cup N$ and $\omega_j=\pi$ and for $j\in M$ and $\omega_j\in\{\pi/2, 3\pi/2\}$;

(iv) $c_j^+ = c_j^-$ for $j\in M$ and $\omega_j=\pi$.

The *space* $\mathring{\mathscr{Z}}^s$ is defined for all s with $s-1/2\in(0,2)\setminus(A\cup A^0)$:

(4.97) $$u\in\mathring{\mathscr{Z}}^s:\ \Leftrightarrow\ u = u_0 + \sum_{j=1}^{J}\Big\{\sum_{\alpha_{jk}<s-1/2} c_{jk}u_{jk} + \sum_{\alpha_{jk}^0<s-1/2} c_{jk}^0 u_{jk}^0\Big\}\chi_j,$$

(i) where u_0 satisfies exactly the same conditions as the elements of $\mathscr{X}^s$ if Γ_1 and Γ_2 are interchanged.

(ii) $c_{jk}=0=c_{jk}^0$ if u_{jk}, u_{jk}^0 are not defined.

The *space* $\mathscr{Z}^s$ is defined for all s with $s-1/2\in(0,2)\setminus A$:

(4.98) $$u\in Z^s:\ \Leftrightarrow\ u = u_0 + \Big(\sum_{j=1}^{J}\sum_{\alpha_{jk}<s-1/2} c_{jk}u_{jk}\Big)\chi_j,\ \text{where}$$

(i) u_0 is as in $\mathring{\mathscr{Z}}^s$,

(ii) $c_{jk}=0$ if u_{jk} is not defined.

The norms in the spaces $\mathscr{X}^s$, $\mathscr{Y}^s$, and $\mathscr{Z}^s$ are defined in the natural way, namely they are required to be equivalent to the smallest norms for which all the conditions that are to be satisfied in the individual case of s and Γ are defined by continuous mappings.

We give two examples:

(4.99) $$\|g\|_{\mathscr{X}^s}^2 = \|g|_{\Gamma_1}\|_{H^s}^2 + \sum_{\Gamma^j\subset\Gamma^2}\|g|_{\Gamma^j}\|_{H^{s-1}(\Gamma^j)}^2$$

if $s\in[3/2, 5/2)$ (even $s\in(1/2, 5/2)$) and no $\omega_j=\pi$ for $j\in N$

and no $\omega_j\in\{\pi/2, 3\pi/2\}$ for $j\in M$;

(4.100) $$\|u\|_{\mathscr{Z}^s}^2 = \|u_0|_{\Gamma_1}\|_{\tilde{H}^{s-1}(\Gamma_1)}^2 + \|u_0|_{\Gamma_2}\|_{H^s(\Gamma_2)}^2 + \sum_{j=1}^{J}\sum_{\alpha_{jk}<s-1/2}|c_{jk}|^2$$

if $s\in[1/2, 3/2)\setminus A$.

Remark 4.7. For $s_1\leqslant s_2$ the spaces $\mathscr{X}^{s_2}$, $\mathscr{Y}^{s_2}$, $\mathscr{Z}^{s_2}$ are continuously embedded in $\mathscr{X}^{s_1}$, $\mathscr{Y}^{s_1}$, $\mathscr{Z}^{s_1}$, respectively.

4.5. With these definitions we deduce from our calculations in § 4.3

THEOREM 4.8. (i) *For* $s\in[1/2, 5/2)\setminus\{3/2\}$ *the operator* $\mathscr{B}_0$: $\mathscr{X}^s\to\mathscr{Y}^s$ *is continuous.*

(ii) *For* $s\in[1/2, 5/2)\setminus(\{3/2\}\cup A\cup A^0)$ *the operator* $\mathscr{A}_0$: $\mathring{\mathscr{Z}}^s\to\mathscr{Y}^s$ *is continuous.*

(iii) *If* $U\in\mathscr{Z}^{1/2}=H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)$ *is a solution of* $\mathscr{A}_0 U = H$ *and* $H\in\mathscr{Y}^s$, *then* $U\in\mathring{\mathscr{Z}}^s$ *if* s *is such that* $\mathscr{Y}^s$ *and* $\mathring{\mathscr{Z}}^s$ *are defined.*

(i) *There is an a-priori-estimate for the solution of* $\mathscr{A}_0 U = \mathscr{B}_0 G$:

(4.101) $$\|U\|_{\mathring{\mathscr{Z}}^s}\leqslant C(\|G\|_{\mathscr{X}^s}+\|U\|_{\mathscr{Z}^{1/2}}).$$

COROLLARY 4.9. *If* $U=\begin{bmatrix} v\\ \psi\end{bmatrix}\in H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)$ *is a solution of the homogeneous equation*

(4.102) $$A_0 U = 0,$$

then $v\in\tilde{H}^{1/2+\varepsilon}(\Gamma_2)$ *and* $\psi\in H^{-1/2+\varepsilon}(\Gamma_1)\cap L^p(\Gamma_1)$ *for some* $\varepsilon>0$ *and* $p>1$.

Proof. If (4.102) is satisfied then $U \in \mathring{\mathscr{X}}^s$ for all $s < 5/2$. This means by the definition of $\mathring{\mathscr{X}}^s$ that

$$u = u_0 + \sum_{j=1}^{J} \{\sum_{\alpha_{jk}} c_{jk} u_{jk} + \sum_{\alpha_{jk}^0} c_{jk}^0 u_{jk}^0\} \chi_j$$

where the smooth part u_0 satisfies at least

$$v_0 := u_0|_{\Gamma_2} \in H^s(\Gamma_2) \quad \text{and} \quad \psi_0 := u_0|_{\Gamma_1} \in H^{s-1}(\Gamma_1) \quad \text{for} \quad s < 3/2.$$

The singular parts u_{jk}, u_{jk}^0 have the local form x^α or $x^\alpha \log x$ for v and $x^{\alpha-1}$ or $x^{\alpha-1} \log x$ for ψ with $\alpha > 0$. Therefore

$$u_{jk}^{(0)}|_{\Gamma_2} \in \tilde{H}^{1/2+\varepsilon}(\Gamma_2) \quad \text{and} \quad u_{jk}^{(0)}|_{\Gamma_1} \in \tilde{H}^{-1/2+\varepsilon}(\Gamma_1) \cap L^p(\Gamma_1)$$

with some $0 < \varepsilon < \alpha_{jk}^{(0)}$ and $1 < p < 1/(1-\alpha)$ (for $\alpha < 1$). It remains to show that $v(t_j) = 0$ for $j \in M$.

For this purpose we define

$$\tilde{v} := \begin{cases} v & \text{on} \quad \Gamma_2 \\ 0 & \text{on} \quad \Gamma_1 \end{cases} \in L^2(\Gamma).$$

Similarly

$$\tilde{\psi} := \begin{cases} \psi & \text{on} \quad \Gamma_1 \\ 0 & \text{on} \quad \Gamma_2 \end{cases} \in \tilde{H}^{-1/2+\varepsilon}(\Gamma).$$

The homogeneous integral equations (4.102) can then be written as $(1+K)\tilde{v} = V\tilde{\psi}$. By Theorem 2.17, $V\tilde{\psi} \in H^{1/2+\varepsilon}(\Gamma)$, and so we can apply Corollary 4.5 to obtain $\tilde{v} \in H^{1/2+\varepsilon}(\Gamma)$, which means $v \in \tilde{H}^{1/2+\varepsilon}(\Gamma_2)$. ∎

Remark 4.10. For the inhomogeneous equation (1.20) we obtain as in the preceding proof

$$\tilde{v} := \begin{cases} v & \text{on} \quad \Gamma_2 \\ g_1 & \text{on} \quad \Gamma_1 \end{cases} \in H^s(\Gamma) \tag{4.103}$$

if we assume that $G = \begin{bmatrix} g_1 \\ g_1 \end{bmatrix} \in \mathscr{X}^s$. Especially for $s > 1/2$ we find that the solution v is a continuous extension of the given data g_1 from Γ_1 to Γ.

4.6. Having proved Corollary 4.9, which was used in § 3, we may now use the result of that section, in particular the bijectivity of $\mathscr{A}_0\colon \mathscr{X}^{1/2} \to \mathscr{Y}^{1/2}$ (Theorem 3.8) and the equivalence of the integral equations (1.20) with the weak (variational) formulation (1.8) of the mixed boundary value problem (Theorem 3.9). With the help of this equivalence we are going to show that in the expansion (4.97) of the solution U of (1.20) the singular functions u_{jk}^0 which belong to solutions of the exterior Dirichlet problem do *not* appear:

Fix $s_0 \in [1/2, 5/2]$ such that $s_0 - 1/2 \notin A$. Choose the right-hand side $G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in \mathscr{X}^\infty$; this means that there is a function $\tilde{g} \in C^\infty(\mathbf{R}^2)$ such that $g_1 = \tilde{g}|_{\Gamma_1}$ and $g_2 = (\partial\tilde{g}/\partial n)|_{\Gamma_2}$. It turns out that then all the compatibility conditions of the definition of $\mathscr{X}^t$ are satisfied (for any t). It is clear that $\mathscr{X}^\infty$ is dense in $\mathscr{X}^t$ for all $t \in [1/2, 5/2]$. Therefore, if we show that in the expansion (4.97) of the solution U of $\mathscr{A}_0 U = \mathscr{B}_0 G$ the coefficients c_{jk}^0 are zero, then this will also be true for any right-hand side in $\mathscr{X}^{s_0}$, because by the *a*-priori estimate (4.101) we know that these coefficients depend continuously on G in the $\mathscr{X}^{s_0}$ norm.

Now from the equivalence theorem (Theorem 3.9) we know that $U = \begin{bmatrix} v \\ \psi \end{bmatrix}$ with $v = u|_{\Gamma_2}$ and $\psi = \left.\dfrac{\partial u}{\partial n}\right|_{\Gamma_1}$, where u is the weak solution of the boundary value problem with data G. For this u we have Grisvard's expansion (1.10) for any $s > 1/2$. Since a real proof of this expansion seems to exist only for $u \in H^k(\Omega)$, k integer, we use it for $k = 3$, which corresponds to $s = 5/2$. If we compare (1.10) with (4.97), we conclude that $(\sum_{\alpha_{jk}^0 < s_0 - 1/2} c_{jk}^0 u_{jk}^0)\chi_j$ has to result from the traces of the singular functions (1.13) (which do not appear in (4.97) due to (4.73), (4.74)) and of a function in $H^3(\Omega)$. Obviously this is only possible if all c_{jk}^0 with $\alpha_{jk}^0 < s_0 - 1/2$ are zero. Summing up, we have shown

THEOREM 4.11. *Let* $s \in [1/2, 5/2) \setminus (\{3/2\} \cup A)$. *Then*

(i) $\mathscr{A}_0\colon \mathring{\mathscr{X}}^s \to \mathscr{Y}^s$ *is bijective;*

(ii) $\mathscr{A}_0\colon \mathscr{X}^s \to \mathscr{B}_0 \mathscr{X}^s \subset \mathscr{Y}^s$ *is bijective;*

For the solution of $\mathscr{A}_0 U = \mathscr{B}_0 G$ *we have the a-priori estimate*

$$\|U\|_{\mathscr{X}^s} \leqslant C \|\mathscr{A}_0 U\|_{\mathscr{Y}^s} \leqslant C' \|G\|_{\mathscr{X}^s}. \tag{4.104}$$

Remark 4.12. The above regularity results are of course true not only for the solution of (1.20), $\mathscr{A}_0 U = \mathscr{B}_0 G$, but also for the solution of (2.26), $\mathscr{A}U = \mathscr{B}G$, because these two systems have the same solutions in $\mathscr{X}^{1/2}$.

§ 5. The Galerkin method for the integral equations

Now we use the results of the proceding sections in order to obtain asymptotic error estimates of optimal order (in the energy norm) for the Galerkin approximation of the solution of the integral equations (2.26). In virtue of the equivalence theorem therefore we approximate by the boundary element method the solution of the mixed boundary value problem (P). For the stability of the Galerkin operator we need Gårding's inequality in $L^2(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$; this corresponds to the $L^2(\Gamma)$-scalar product, which is used for the standard Galerkin equations. However, if we considered the original system of integral equations (1.20) we would be forced to formulate the Galerkin equations with the scalar product in $H^{1/2}(\Gamma_1)\times L^2(\Gamma_1)$.

For higher convergence estimates we perform the Fix method, which was successfully used for the boundary element method in the case of a smooth curve in [58]. Here obviously we have to augment the space of the test and trial function with special singular terms, corresponding not only to the change of boundary conditions but also to the corners. Thus the error analysis for the boundary element method is performed for a lot of problems which were originally studied by the 2–D finite element procedure ([5], [12]).

5.1. The one-dimensional finite element spaces used have the following convergence property (5.1) and inverse property (5.2) which are both well known for regular finite element functions ([2], [5], [44]):

For any $U\in\mathscr{X}^r$ there exist a $\tilde U\in S_h^{p,t,k}$ with $t\geqslant r$ and $p\geqslant r$ and a constant $c>0$, independent of h and U, such that for $q\leqslant\min\{k,r\}$

(5.1) $$\|U-\tilde U\|_{\mathscr{X}^q}\leqslant ch^{r-q}\|U\|_{\mathscr{X}^r}.$$

For $q\leqslant r$, $\varepsilon>0$, $k\geqslant r$ there exists a constant $M>0$, independent of h, such that for all $\tilde U\in S_h^{p,t,k}$

(5.2) $$\|\tilde U\|_{\mathscr{X}^r}\leqslant Mh^{k-r-\varepsilon}\|\tilde U\|_{\mathscr{X}^q}$$

with $\varepsilon=0$ if $A\cap[q-1/2,\,r'-1/2]=\emptyset$ where $r'=\max\{p,r\}$.

The augmented Sobolev spaces $\mathscr{X}^r$ are defined in (4.98). Note that $\mathscr{X}^r$ is only defined for $r-1/2\notin A$, where the set A of exceptional exponents is defined in (4.93). Therefore, if we write $\|\cdot\|_{\mathscr{X}^r}$, we always mean that the condition $r-1/2\notin A$ is included. The augmented finite elements spaces $S_h^{p,t,k}$ will be defined in Definition 5.6. For the moment it is enough to know that

(5.3) $$S_h^{p,t,k}\subset\mathscr{X}^k\subset\mathscr{X}^s\quad(s\leqslant k)\quad\text{and}\quad\mathscr{X}^{1/2}=H^{1/2}(\Gamma_2)\times\tilde H^{-1/2}(\Gamma_1).$$

This is sufficient to derive asymptotic error estimates in these norms for the Galerkin solution corresponding to the following problem: Find $U_h\in S_h^{p,t,k}\subset\mathscr{X}^{1/2}$ such that for all $\mathbf{V}\in S_h^{p,t,k}$

(5.4) $$\langle\mathscr{A}U_h,\mathbf{V}\rangle_{\mathscr{L}^2(\Gamma)}=\langle\mathscr{A}U,\mathbf{V}\rangle_{\mathscr{L}^2(\Gamma)}$$

where $U\in\mathscr{X}^s$. Later on U will be the solution of the system of integral equations (2.26), so that (5.4) can also be written as

(5.5) $$\langle\mathscr{A}U_h,\mathbf{V}\rangle_{\mathscr{L}^2(\Gamma)}=\langle\mathscr{B}G,\mathbf{V}\rangle_{\mathscr{L}^2(\Gamma)}$$

where G are the given boundary data of problem (P).

THEOREM 5.1. *Let $t\geqslant 1/2$, $0<h\leqslant h_0$. Then the Galerkin operator*

$$G_{\mathscr{A}}:\ U\to U_h:\ H^{1/2}(\Gamma_2)\times\tilde H^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)\times\tilde H^{-1/2}(\Gamma_1)$$

corresponding to (5.4) is uniformly bounded independent of h.

Proof. By a result of Hildebrandt and Wienholtz ([22]; Lemma 5.2) the convergence of the Galerkin equations follows if a Gårding inequality holds. But unfortunately we have a Gårding inequality only for the wrong norms, namely $L^2(\Gamma_2)\times\tilde H^{-1/2}(\Gamma_1)$. This difficulty will be overcome by using an idea of Nitsche (5.13) for one part of the (2×2)-matrix $G_{\mathscr{A}}$. In addition a Gårding inequality in $H^{1/2}(\Gamma_2)\times\tilde H^{-1/2}(\Gamma_1)$ will be used for another part of $G_{\mathscr{A}}$ (5.11). We start with the operator

(5.6) $$\mathscr{A}_1:=\begin{vmatrix}1+K_{22} & -V_{12}\\ 0 & C\end{vmatrix},$$

which in virtue of the proof of Theorem 2.19, differs from $\mathscr{A}$ only by a compact perturbation. We reduce $\mathscr{A}_1$ by a compact perturbation to an operator

(5.7) $$\mathscr{A}_2:=\begin{bmatrix}D_2 & -V_{12}\\ 0 & D_1\end{bmatrix}.$$

By Lemma 5.2 below, the result on the uniform boundedness of $G_{\mathscr{A}}$ follows from that on $G_{\mathscr{A}_2}$, which is defined analogously. The operators D_1, D_2 in (5.7) have the following meaning: By (2.33) for C a Gårding inequality holds. Therefore $C=D_1+T_1$, where T_1: $\tilde H^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_1)$ is compact and D_1 is positive definite, i.e., there exists a $\gamma>0$ such that for all $\psi\in\tilde H^{-1/2}(\Gamma_1)$

(5.8) $$\langle D_1\psi,\psi\rangle_{H^{1/2}(\Gamma_1)\times\tilde H^{-1/2}(\Gamma_1)}\geqslant\gamma\|\psi\|^2_{\tilde H^{-1/2}(\Gamma_1)}.$$

Similarly, from (2.32) we deduce

(5.9) $$I+K_{22}=D_2+T_2$$

where D_2 satisfies the estimate

(5.10) $$\langle D_2v,v\rangle_{L^2(\Gamma_2)\times L^2(\Gamma_2)}\geqslant\gamma\|v\|^2_{L^2(\Gamma_2)}$$

and T_2 is now compact not only as an operator in $L^2(\Gamma_2)$ but also in $H^{1/2}(\Gamma_2)$.

Our next step is to show that

(5.11) $$D_2:\ H^{1/2}(\Gamma_2)\to H^{1/2}(\Gamma_2)\quad\textit{is bijective}.$$

From the continuity properties of $1+K_{22}$ and T_2 in connection with (5.9) we know that D_2 is continuous in $H^{1/2}(\Gamma_2)$. From (5.10) it follows that D_2 is injective. Since by Theorem 2.24 the operator $1+K_{22}$ satisfies also a Gårding inequality in $H^{1/2}(\Gamma_2)$, it is a Fredholm operator of index 0 in $H^{1/2}(\Gamma_2)$. Therefore D_2 is also of index 0 and thus bijective in $H^{1/2}(\Gamma_2)$. Using a technique of Nitsche [45], we show that operator G_{D_2} is uniformly bounded not only in $L^2(\Gamma_2)$ but even in $H^{1/2}(\Gamma_2)$. Here $G_{D_2}|$ is defined by G_{D_2}: $v\mapsto\mathbf{v}_h\in S_h^{p,t,k}$ through the equations

(5.12) $$\langle D_2\mathbf{v}_h,\mathbf{w}\rangle_{L^2(\Gamma_2)}=\langle D_2v,\mathbf{w}\rangle_{L^2(\Gamma_2)}\quad\text{for all }\mathbf{w}\in S_h^{p,t,k}.$$

First it follows from (5.10) that G_{D_2} is uniformly bounded in $L^2(\Gamma_2)$. Hence together with the convergence property (5.1) and the inverse assumption (5.2) with $q=0$, $r=1/2$ we have for $\mathbf{v}=\tilde{U}|_{\Gamma_2}$

$$(5.13)\qquad \begin{aligned}\|G_{D_2}v-v\|_{H^{1/2}(\Gamma_2)} &\leqslant \|G_{D_2}v-\mathbf{v}\|_{H^{1/2}(\Gamma_2)}+\|v-\mathbf{v}\|_{H^{1/2}(\Gamma_2)}\\ &\leqslant Mh^{-1/2}\|G_{D_2}(v-\mathbf{v})\|_{L^2(\Gamma_2)}+c\|v\|_{H^{1/2}(\Gamma_2)}\\ &\leqslant Mh^{-1/2}\|G_{D_2}\|_{L^2}\|v-\mathbf{v}\|_{L^2(\Gamma_2)}+c\|v\|_{H^{1/2}(\Gamma_2)}\\ &\leqslant (cM\|G_{D_2}\|_{L^2}+c)\|v\|_{H^{1/2}(\Gamma_2)}.\end{aligned}$$

Hence

$$(5.14)\qquad \|G_{D_2}v\|_{H^{1/2}(\Gamma_2)}\leqslant \tilde{c}\|v\|_{H^{1/2}(\Gamma_2)}\quad \text{for any } v\in H^{1/2}(\Gamma_2),$$

where $\tilde{c}$ is independent of h and v.

Note that, for $q=0$, $r=1/2$, $\mathscr{Z}^q|_{\Gamma_2}=L^2(\Gamma_2)$, $\mathscr{Z}^r|_{\Gamma_2}=H^{1/2}(\Gamma_2)$. Now we return to system (5.7). The corresponding Galerkin equations are

$$(5.15)\qquad \left\langle\begin{bmatrix}D_2 & -V_{12}\\ 0 & D_1\end{bmatrix}\begin{bmatrix}v_h\\ \psi_h\end{bmatrix},\begin{bmatrix}\begin{pmatrix}\mathbf{v}\\ \boldsymbol{\phi}\end{pmatrix}\end{bmatrix}\right\rangle_{\mathscr{L}^2(\Gamma)}=\left\langle\begin{bmatrix}D_2 & -V_{12}\\ 0 & D_1\end{bmatrix}\begin{bmatrix}v\\ \psi\end{bmatrix},\begin{bmatrix}\mathbf{v}\\ \boldsymbol{\phi}\end{bmatrix}\right\rangle_{\mathscr{L}^2(\Gamma)}$$

for all $(\mathbf{v},\boldsymbol{\phi})\in S_h^{p,t,k}\subset H^{1/2}(\Gamma_2)\times\tilde{H}^{-1/2}(\Gamma_1)$ and define $G_{\mathscr{A}_2}\colon \begin{bmatrix}v\\ \psi\end{bmatrix}\mapsto\begin{bmatrix}v_h\\ \psi_h\end{bmatrix}$. For the choice $\psi=0$ we have $G_{\mathscr{A}_2}\colon \begin{bmatrix}v\\ 0\end{bmatrix}\mapsto\begin{bmatrix}w_h\\ \chi_h\end{bmatrix}$. Therefore (5.15) reduces to

$$(5.16)\qquad \langle D_2w_h-V_{12}\chi_h,\mathbf{v}\rangle_{L^2(\Gamma_2)}=\langle D_2v,\mathbf{v}\rangle_{L^2(\Gamma_2)};$$

$$(5.17)\qquad \langle D_1\chi_h,\boldsymbol{\phi}\rangle_{H^{1/2}(\Gamma_1)\times\tilde{H}^{-1/2}(\Gamma_1)}=0.$$

By (5.8) we know that the Galerkin operator G_{D_1} exists and is uniformly bounded in $\tilde{H}^{-1/2}(\Gamma_1)$. This implies that (5.17) yields $\chi_h=0$. Therefore (5.16) reduces to (5.12), which means that $w_h=G_{D_2}v$. For the choice $v=0$, (5.15) gives

$$(5.18)\qquad \langle D_2u_h-V_{12}\psi_h,\mathbf{v}\rangle_{L^2(\Gamma_2)}=\langle -V_{12}\psi,\mathbf{v}\rangle_{L^2(\Gamma_2)};$$

$$(5.19)\qquad \langle D_1\psi_h,\boldsymbol{\phi}\rangle_{H^{1/2}(\Gamma_1)\times\tilde{H}^{-1/2}(\Gamma_1)}=\langle D_1\psi,\boldsymbol{\phi}\rangle_{H^{1/2}(\Gamma_1)\times\tilde{H}^{-1/2}(\Gamma_1)}.$$

Now (5.19) means that $\psi_h=G_{D_1}\psi$, and (5.18) can be written as $\langle D_2u_h,\mathbf{v}\rangle_{L^2(\Gamma_2)}=\langle D_2(D_2^{-1}V_{12}(\psi_h-\psi)),\mathbf{v}\rangle_{L^2(\Gamma_2)}$, which means that

$$u_h=G_{D_2}D_2^{-1}V_{12}(\psi_h-\psi)=G_{D_2}D_2^{-1}V_{12}[(G_{D_1}-1)\psi].$$

Now, collecting the results, we write

$$(5.20)\qquad \begin{aligned}G_{\mathscr{A}_2}\begin{bmatrix}v\\ \psi\end{bmatrix}&=G_{\mathscr{A}_2}\begin{bmatrix}v\\ 0\end{bmatrix}+G_{\mathscr{A}_2}\begin{bmatrix}0\\ \psi\end{bmatrix}\\ &=\begin{bmatrix}G_{D_2} & G_{D_2}D_2^{-1}V_{12}(G_{D_1}-1)\\ 0 & G_{D_1}\end{bmatrix}\begin{bmatrix}v\\ \psi\end{bmatrix}.\end{aligned}$$

We have seen that G_{D_2} and G_{D_1} are uniformly bounded in $H^{1/2}(\Gamma_2)$ and $\tilde{H}^{-1/2}(\Gamma_1)$, respectively. This implies also that

$$G_{D_2}D_2^{-1}V_{12}(G_{D_1}-1)\colon \tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)$$

is uniformly bounded due to the continuity of $D_2^{-1}V_{12}\colon \tilde{H}^{-1/2}(\Gamma_1)\to H^{1/2}(\Gamma_2)$. ∎

LEMMA 5.2 (Hildebrand and Wienholtz [22]). *Let H be a Hilbert spaces with a dual H' (not necessarily indentified with H) and A, $D\colon H\to H'$ isomorphisms with $T=A-D\colon H\to H'$ compact. Let $\{S_h\}_{h>0}$ be a family of subspaces of H such that the equations*

$$(5.21)\qquad \langle Dw_h,\mathbf{v}\rangle=\langle Dw,\mathbf{v}\rangle\quad \textit{for all } \mathbf{v}\in S_h$$

define an operator $G_D^h\colon w\mapsto w_h\in S_h$ with the property

$$(5.22)\qquad \|G_D^hw-w\|\to 0\quad \textit{as}\quad h\to 0\quad \textit{for all } w\in H.$$

Then for small h the equations

$$(5.23)\qquad \langle Au_h,\mathbf{v}\rangle=\langle Au,\mathbf{v}\rangle\quad \textit{for all } \mathbf{v}\in S_h$$

define an operator $G_A^h\colon u\mapsto u_h\in S_h$ such that

$$\|G_A^h\|\leqslant C\quad \textit{with } C \textit{ independent of } h.$$

Proof. From (5.22) and the compactness of T follows $\|A^{-1}T(1-G_D^h)\|\to 0$ $(h\to 0)$. Therefore for small h $\tilde{G}_A^h:=G_D^h[1-A^{-1}T(1-G_D^h)]^{-1}$ exists and $\|\tilde{G}_A^h\|$ is uniformly bounded. From equations (5.21) and (5.23) it is easily verified that $\tilde{G}_A^h=G_A^h$. ∎

Remark 5.3. The assumptions on D and S_h are fulfilled if D is positive definite, i.e., there exists a $\gamma>0$ with $|\langle Dw,w\rangle|\geqslant\gamma\|w\|^2$ for all $w\in H$, and there exists a uniformly bounded family $\{P_h\}$ of operators $P_h\colon H\to S_h$ with $\|P_hu-u\|\to 0$ $(h\to 0)$ for all $u\in H$.

With the uniform boundedness ot the Galerkin operator $G_{\mathscr{A}}$ in $\mathscr{Z}^{1/2}$ (Theorem 5.1) and the regularity results (Theorem 4.11) on the exact solution of $\mathscr{A}U=\mathscr{B}G$ we are now able to derive higher convergence rates for the Galerkin approximation, corresponding to (5.4) for smoother right-hand sides. This is a standard technique (see [52]), which uses convergence (5.1) and stability properties (5.2) of the finite element spaces.

THEOREM 5.4. *Let $1/2\leqslant r\leqslant s<5/2$ be such that $r-1/2$, $s-1/2\notin A$, where A is the set of exceptional exponents defined in (4.93). Let $U\in\mathscr{Z}^{1/2}$ be the exact solution of the system of integral equations (2.26) with $G\in\mathscr{X}^s$ and let $U_h\in S_h^{p,t,k}\subset\mathscr{Z}^r$ be the Galerkin solution of (5.5). Then for $r\leqslant k$, $s\leqslant p$, $s\leqslant t$, $\varepsilon>0$, and $0<h\leqslant h_0$ we have*

$$(5.24)\qquad \|U-U_h\|_{\mathscr{Z}^r}\leqslant ch^{s-r-\varepsilon}\|U\|_{\mathscr{Z}^s}\leqslant c'h^{s-r-\varepsilon}\|G\|_{\mathscr{X}^s}$$

where the constants c, $c' > 0$ *are independent of* U, G *and* h, *and* $\varepsilon = 0$ *for* $[0, p-\frac{1}{2}] \cap A = \emptyset$ *or* $r = 1/2$.

Proof. Choosing $\tilde{U} \in S_h^{p,t,k}$ which satisfies the convergence property (5.1) and using the uniform boundedness of $G_{\mathscr{A}}$ in $\mathscr{Y}^{1/2}$, the projection property $G_{\mathscr{A}}\tilde{U} = \tilde{U}$ and the inverse property (5.2) of the finite element function $G_{\mathscr{A}}(\tilde{U} - U) \in S_h^{p,t,k} \subset \mathscr{Y}^r$, we can estimate the error:

$$
\begin{aligned}
(5.25)\quad \|U - U_h\|_{\mathscr{Y}^r} &= \|U - G_{\mathscr{A}}U + \tilde{U} - \tilde{U}\|_{\mathscr{Y}^r} \leqslant \|U - \tilde{U}\|_{\mathscr{Y}^r} + \|G_{\mathscr{A}}(\tilde{U} - U)\|_{\mathscr{Y}^r} \\
&\leqslant \tilde{c}h^{s-r}\|U\|_{\mathscr{Y}^s} + Mh^{1/2-r-\varepsilon}\|G_{\mathscr{A}}(\tilde{U} - U)\|_{\mathscr{Y}^{1/2}} \\
&\leqslant \tilde{c}h^{s-r}\|U\|_{\mathscr{Y}^s} + M\|G_{\mathscr{A}}\|_{1/2}\, h^{1/2-r-\varepsilon}\|\tilde{U} - U\|_{\mathscr{Y}^{1/2}} \\
&\leqslant ch^{s-r-\varepsilon}\|U\|_{\mathscr{Y}^s}.
\end{aligned}
$$

With the help of the a-priori estimate (4.104) we thus obtain (5.24). Note that for $r = 1/2$ we do not need the inverse property. ∎

Remark 5.5. The convergence rate in (5.24) is optimal for $r = \frac{1}{2}$, which means that the error is estimated in the energy norm. We obtain L^∞-estimates for the error of the trace v if $r > \frac{1}{2}$ and for the normal derivate ψ if $r > \frac{3}{2}$ in (5.24). Due to the definition of the norm in $\mathscr{Y}^r$ (compare (4.100)), the error estimates for the coefficients c_{jk} of the singular parts of the solution are obtained from (5.24) for $r > \alpha_{jk} + \frac{1}{2}$. Thus the error for these coefficients is of order $h^{s-\alpha_{jk}-1/2-\varepsilon}$, which is a loss of order $h^{\alpha_{jk}+\varepsilon}$ compared with the energy norm estimate. Estimate (5.24) is for $r = \frac{1}{2}$ of the same order as the corresponding energy norm estimate for the Galerkin approximations of the solution of the variational problem which are constructed by means of 2D finite element spaces augmented by the singular terms of expansion (1.10) (cf [5], p. 274).

Next we define the augmented finite element spaces $S_h^{p,t,k}$ used and show that they satisfy the assumed properties (5.1) and (5.2). The parameters in $S_h^{p,t,k}$ have (roughly) the following meaning

(i) $0 < h \leqslant h_0$ is the mesh size of the partition of Γ;

(ii) t describes the degree of the piecewise polynomials which constitute the regular part of $S_h^{p,t,k}$;

(iii) k describes conformity, i.e., $S_h^{p,t,k} \subset \mathscr{Y}^k$.

(iv) p gives the number of singular terms u_{jk} included in $S_h^{p,t,k}$.

More precisely, we have the following definition:

DEFINITION 5.6

$$
(5.26)\qquad \mathbf{u} \in S_h^{p,t,k} :\Leftrightarrow \mathbf{u} = \mathbf{u}_0 + \sum_{j=1}^{J} \sum_{\alpha_{jk} < p-1/2} \mathbf{c}_{jk} u_{jk} \chi_j
$$

where $\mathbf{c}_{jk} \in \mathbf{R}$ is arbitrary, and u_{jk}, χ_j are as in the definition of $\mathscr{Y}^p$ (4.98); an

$$
\mathbf{u}_0|_{\Gamma_2} = \mathbf{v}_0, \quad \mathbf{u}_0|_{\Gamma_1} = \psi_0
$$

with $\mathbf{v}_0$ belonging to a $S_h^{t,k}$-system on Γ_2 (Definition; see [5]; p. 83) and $\psi_0|_{\Gamma^j} \in S_h^{k-1,t^*}$ $(t^* = \max(1, t-1))$ for each j with $\Gamma^j \subset \Gamma_1$. Furthermore ψ_0 and $\mathbf{v}_0$ have to satisfy the following conformity conditions, which ensure that $\mathbf{u}_0 \in \mathscr{Y}^k$:

(i) no condition for $k < 3/2$;

(ii) for $k \geqslant 3/2$: ψ_0 continuous at t_j for $j \in N$ and $\omega_j = \pi$.

(iii) $\psi_0(t_j) = \pm \mathbf{v}_0'(t_j)$ with $+$ for $j \in M_2$, $\omega_j = \pi/2$ or $j \in M_1$, $\omega_j = 3\pi/2$ and with $-$ for $j \in M_1$, $\omega_j = \pi/2$ or $j \in M_2$, $\omega_j = 3\pi/2$.

Note that $\mathbf{v}_0 \in S_h^{t,k}$ implies that v_0 is a piecewise polynomial of degree $t-1$ and $\mathbf{v}_0 \in H^k(\Gamma_2)$; similarly $\psi_0|_{\Gamma^j} \in H^{k-1}(\Gamma^j)$.

The Galerkin procedure (5.5) reads more explicitly: Find $U_h := (v_h, \psi_h) \in S_h^{p,t,k}$ such that

$$
\begin{aligned}
(5.27)\quad & \langle (1+K_{22})v_h - V_{12}\psi_h, \tilde{v}\rangle_{L^2(\Gamma_2)\times L^2(\Gamma_2)} + \\
&\quad + \langle K_{21}K_{22}v_h - (V_{11} - K_{21}V_{12})\psi_h, \tilde{\psi}\rangle_{H^{1/2}(\Gamma_1)\times \tilde{H}^{-1/2}(\Gamma_1)} \\
&= \langle -K_{12}g_1 + V_{22}g_2, \tilde{v}\rangle_{L^2(\Gamma_2)\times L^2(\Gamma_2)} + \\
&\quad + \langle (1+K_{11} - K_{21}K_{12})g_1 - (V_{21} - K_{21}V_{22})g_2, \tilde{\psi}\rangle_{H^{1/2}(\Gamma_1)\times \tilde{H}^{-1/2}(\Gamma_1)} \\
&\qquad\qquad \text{for all } (\tilde{v}, \tilde{\psi}) \in S_h^{p,t,k}.
\end{aligned}
$$

With the trial functions

$$
(5.28)\qquad
\begin{aligned}
v^h &= v_0^h + \sum_{j \in N \cup M} c_{jk}^h u_{jk}\chi_k, \quad v_0^h = \sum_{l=0}^{N'} \gamma_l \mu_l \quad \text{on } \Gamma_2, \\
\psi^h &= \psi_0^h + \sum_{j \in D \cup M} c_{jk}^h u_{jk}\chi_k, \quad \psi_0^h = \sum_{l=N'+1}^{N''} \delta_l \mu_l \quad \text{on } \Gamma_1,
\end{aligned}
$$

(where μ_l are basis functions of regular finite element functions) the Galerkin method (5.27) means: Find real numbers γ_l, δ_l, c_{jk}^h such that (5.27) holds for the regular test functions

$$
(5.29)\qquad \tilde{v} \underset{\Gamma_2}{=} \mu_l \ (l = 0, \ldots, N'), \qquad \tilde{\psi} \underset{\Gamma_1}{=} \mu_l \ (l = N'+1, \ldots, N'')
$$

and for the singular test functions

$$
(5.30)\qquad \tilde{v} = u_{jk}|_{\Gamma_2} \ (j \in N \cup M), \qquad \tilde{\psi} = u_{jk}|_{\Gamma_1} \ (j \in D \cup M).
$$

Unless $j \in M$ and $(\tilde{v}, \tilde{\psi}) = u_{jk}$ the system (5.27) can be written as equations on Γ_1 and Γ_2, alone. By this choice of test functions (5.28) and trial functions (5.29), (5.30), the system turns out to be a quadratic system of linear algebraic equations which is uniquely solvable. The coefficients of this stiffness matrix can be computed with quadrature formulas (Gauss, Newton, Cotes, etc.) (see [25], [33]) since the scalar products in (5.27) are L^2-scalar products.

LEMMA 5.7. *The finite element spaces* $S_h^{p,t,k}$ *have properties* (5.1), (5.2).

Proof. For proving (5.1) we choose $\mathbf{c}_{jk} = c_{jk}$ for $\alpha_{jk} < r-\frac{1}{2}$ and $\mathbf{c}_{jk} = 0$ for $\alpha_{jk} \in (r-\frac{1}{2}, p-\frac{1}{2})$, where $\mathbf{c}_{jk}$ are the coefficients in (5.26) and c_{jk} those in (4.98). Thus it remains to estimate the smooth parts, i.e., $\|u_0 - \mathbf{u}_0\|_{\mathscr{X}^q}$, which reduces to an ordinary Sobolev norm where we can apply the convergence property of a $S_h^{t,k}$-system.

Thus we even get the estimate

$$\|U - \tilde{U}\|_{\mathscr{X}^q} = \|u_0 - \mathbf{u}_0\|_{\mathscr{X}^q} \leqslant ch^{r-q}\|u_0\|_{\mathscr{X}^r} \leqslant ch^{r-q}\|U\|_{\mathscr{X}^r}. \tag{5.31}$$

The inverse property (5.2) for $q \leqslant r \leqslant k$ and $A \cap [q-\frac{1}{2}, r'-\frac{1}{2}] = \emptyset$ ($r' = \max(r', p')$) follows immediately from the definition of norms and the inverse property of our $S_h^{t,k}$-systems (cf. [5], Theorem 4.1.3).

$$\|\tilde{U}\|^2_{\mathscr{X}^r} = \|\mathbf{u}_0\|^2_{\mathscr{X}^r} + \sum_{j=1}^{J} \sum_{\alpha_{jk} < r-1/2} |\mathbf{c}_{jk}|^2 \leqslant M^2 h^{2(q-r)} \|\mathbf{u}_0\|^2_{\mathscr{X}^q} + \sum_{j=1}^{J} \sum_{\alpha_{jk} < q-1/2} |\mathbf{c}_{jk}|^2 \leqslant M^2 h^{2(q-r)} \|\tilde{U}\|^2_{\mathscr{X}^q}. \tag{5.32}$$

Here we have used the fact that $\mathbf{u}_0 \in S_h^{0,t,k}$ is a piecewise polynomial.

In what follows we use two estimates fot the approximation of the singular functions u_{jl} by piecewise polynomials $\hat{u}_{jl} \in S_h^{0,t,k}$, namely.

(i) To any singular function $\mathbf{c}_{jl} u_{jl} \chi_j$ there exists a $\hat{u}_{jl} \in S_h^{0,t,k}$ and constants c_ε independent of h such that for all $\varepsilon > 0$

$$\|\mathbf{c}_{jl} u_{jl} \chi_j - \hat{u}_{jl}\|_{\mathscr{X}^q} \leqslant c_\varepsilon h^{\alpha_{jl}+1/2-q-\varepsilon} |\mathbf{c}_{jl}| \tag{5.33}$$

for all $q \leqslant \min\{k, \alpha_{jl}+1/2-\varepsilon\}$. This follows directly from the convergence property (5.1) applied to the regular finite elements $\hat{u}_{jl}$ and the function $\mathbf{c}_{jl} u_{jl} \chi_j \in \mathscr{X}^{\alpha_{jl}+1/2-\varepsilon}$ (for all $\varepsilon > 0$), considered as a "smooth" function.

(ii) For any set of singular functions $\{\mathbf{c}_{jl} u_{jl} \chi_j | \ l = 1, \ldots, L\}$ at the corner t_j and any $q < \alpha_{jl} + 1/2$ ($l = 1, \ldots, L$) there exists a constant γ independent of h such that

$$\Big\|\sum_{l=1}^{L} \mathbf{c}_{jl} u_{jl} \chi_j - \hat{w}_j\Big\|_{\mathscr{X}^q} \geqslant \gamma h^{\alpha_{jl_0}+1/2-q} |\mathbf{c}_{jl_0}| \text{ for all } l_0 = 1, \ldots, L \tag{5.34}$$

$$\text{and for all regular } \hat{w}_j \in S_h^{0,t,k} \quad (k \geqslant q).$$

This is analogous to Lemma 1.5 of [58]. The proof is given in Lemma 5.9.

For $A \cap [q-1/2, r'-1/2] \neq \emptyset$ we distinguish two cases, from which then the general case follows by induction.

1) $A \cap [q-1/2, r-1/2] = \emptyset$ (p arbitrary).

For $\mathbf{U} \in S_h^{p,t,k}$ we use expansion (5.26) to define for $q \leqslant \tau \leqslant r$

$$\mathbf{u}_\tau = \mathbf{u}_0 + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (\tau-1/2, p-1/2)} \mathbf{c}_{jl} u_{jl} \chi_j. \tag{5.35}$$

Now for every $\alpha_{jl} \in (r-1/2, p-1/2)$ we take $\hat{u}_{jl}$ as in (5.33). Then we have, with the inverse property for *regular* finite elements,

$$\begin{aligned} \|\mathbf{u}\|_{\mathscr{X}^r} &= \Big(\|\mathbf{u}_r\|^2_{\mathscr{X}^r} + \sum_{j=1}^{J} \sum_{\alpha_{jl} < r-1/2} |\mathbf{c}_{jl}|^2\Big)^{1/2} \\ &\leqslant \Big\|\mathbf{u}_0 + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (r-1/2, p-1/2)} \hat{u}_{jl}\Big\|_{\mathscr{X}^r} + \\ &\quad + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (r-1/2, p-1/2)} \|\mathbf{c}_{jl} u_{jl} \chi_j - \hat{u}_{jl}\|_{\mathscr{X}^r} + \sum_{j=1}^{J} \sum_{\alpha_{jl} < r-1/2} |\mathbf{c}_{jl}| \\ &= \Big\|\mathbf{u}_0 + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (q-1/2, p-1/2)} \hat{u}_{jl}\Big\|_{\mathscr{X}^r} + \\ &\quad + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (q-1/2, p-1/2)} \|\mathbf{c}_{jl} u_{jl} \chi_j - \hat{u}_{jl}\|_{\mathscr{X}^r} + \sum_{j=1}^{J} \sum_{\alpha_{jl} < q-1/2} |\mathbf{c}_{jl}| \\ &\leqslant Mh^{q-r} \Big\|\mathbf{u}_0 + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (q-1/2, p-1/2)} \hat{u}_{jl}\Big\|_{\mathscr{X}^q} + \\ &\quad + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (q-1/2, p-1/2)} c_\varepsilon h^{\alpha_{jl}+1/2-r-\varepsilon} |\mathbf{c}_{jl}| + \sum_{j=1}^{J} \sum_{\alpha_{jl} < q-1/2} |\mathbf{c}_{jl}|. \end{aligned} \tag{5.36}$$

For the first term on the right-hand side we estimate further

$$\begin{aligned} \Big\|u_0 + \sum_{j=1}^{J} \sum_{\alpha_{jl} \in (q-1/2, p-1/2)} \hat{u}_{jl}\Big\|_{\mathscr{X}^q} &\leqslant \|\mathbf{u}_q\|_{\mathscr{X}^q} + \sum_{j=1}^{J} \sum_{\alpha_{jl} > q-1/2} \|\mathbf{c}_{jl} u_{jl} \chi_j - \hat{u}_{jl}\|_{\mathscr{X}^q} \\ &\leqslant \|\mathbf{u}_q\|_{\mathscr{X}^q} + \sum_{j=1}^{J} \sum_{\alpha_{jl} > q-1/2} c_\varepsilon h^{\alpha_{jl}+1/2-q-\varepsilon} |\mathbf{c}_{jl}|. \end{aligned} \tag{5.37}$$

Inserting (5.37) in (5.36), we get

$$\|\mathbf{u}\|_{\mathscr{X}^r} \leqslant Mh^{q-r} \|\mathbf{u}_q\|_{\mathscr{X}^q} + 2c_\varepsilon \sum_{j=1}^{J} \sum_{\alpha_{jl} > q-1/2} h^{\alpha_{jl}+1/2-r-\varepsilon} |\mathbf{c}_{jl}| + \sum_{j=1}^{J} \sum_{\alpha_{jl} < q-1/2} |\mathbf{c}_{jl}|. \tag{5.38}$$

Now we use (5.34) for $\hat{w}_j = -\tilde{u}_0 \cdot \hat{\chi}_j$, where $\hat{\chi}_j$ is a cut-off function which is 1 on the support of χ_j. Thus $\sum_l \mathbf{c}_{jl} u_{jl} \chi_j + \mathbf{u}_0 \hat{\chi}_j = \hat{\chi}_j \mathbf{u}_q$ and therefore for $\alpha_{jl_0} > q-1/2$

$$|\mathbf{c}_{jl_0}| \leqslant \gamma' h^{-\alpha_{jl_0}-1/2+q} \|\hat{\chi}_j \mathbf{u}_q\|_{\mathscr{X}^q} \leqslant \gamma' c h^{-\alpha_{jl_0}-1/2+q} \|\mathbf{u}_q\|_{\mathscr{X}^q}. \tag{5.39}$$

We insert this into (5.38) and obtain

(5.40)
$$\|\mathbf{u}\|_{\mathscr{H}^r} \leqslant M\{h^{q-r}+2c_\varepsilon\gamma'\sum_{j=1}^{J}\sum_{\alpha_{jl}>q-1/2} h^{q-r-\varepsilon}\}\|\mathbf{u}_q\|_{\mathscr{H}^q}+\sum_{j=1}^{J}\sum_{\alpha_{jl}<q-1/2}|\mathbf{c}_{jl}|$$
$$\leqslant M_\varepsilon h^{q-r-\varepsilon}\|\mathbf{u}\|_{\mathscr{H}^q}.$$

2) There is one $\alpha\in A\cap[q-1/2, r-1]$ (p arbitrary). Then we have the following changes against with respect to the previous case:

(α) There is an additional term $\sum_{j=1}^{J}\sum_{\alpha_{jl}=\alpha}|\mathbf{c}_{jl}|$ which in view of (5.39) can be estimated by

$$Ch^{-\alpha-1/2+q}\|\mathbf{u}_q\|_{\mathscr{H}^q}\leqslant ch^{q-r}\|\mathbf{u}_q\|_{\mathscr{H}^q}.$$

(β) Instead of (5.37) we have

(5.41)
$$\|\mathbf{u}_0+\sum_{j=1}^{J}\sum_{\alpha_{jl}\in(r-1/2,p-1/2)}\hat{u}_{jl}\|_{\mathscr{H}^q}\leqslant\sum_{j=1}^{J}\sum_{\alpha_{jl}=\alpha}\|\mathbf{c}_{jl}u_{jl}\chi_j\|_{\mathscr{H}^q}+$$
$$+\|\mathbf{u}_q\|_{\mathscr{H}^q}+\sum_{j=1}^{J}\sum_{\alpha_{jl}>r-1/2}\|\mathbf{c}_{jl}u_{jl}\chi_j-\hat{u}_{jl}\|_{\mathscr{H}^q}.$$

Here the first term on the right-hand side is estimated, in view of (5.39), by

(5.42)
$$\|\mathbf{c}_{jl}u_{jl}\chi_j\|_{\mathscr{H}^q}=c_q|\mathbf{c}_{jl}|\leqslant c_q Ch^{-\alpha-1/2+q}\|\mathbf{u}_q\|_{\mathscr{H}^q}.$$

This gives, compared with (5.40), an extra term

$$MC_q h^{q-r}h^{-\alpha-1/2+q}\|\mathbf{u}_q\|_{\mathscr{H}^q}\leqslant M_q h^{2q-r-\alpha-1/2}\|\mathbf{u}_q\|_{\mathscr{H}^q}.$$

Here the exponent is smaller than in (5.36), and so this term dominates all the others, and we obtain finally

(5.43)
$$\|\mathbf{u}\|_{\mathscr{H}^r}\leqslant M_q h^{2q-r-\alpha-1/2}\|\mathbf{u}\|_{\mathscr{H}^q}.$$

This holds for all $q<\alpha+1/2$, and therefore we can improve it in the following way:

Choose $\tau\in(q,\alpha+1/2)\backslash A$ arbitrarily. Then apply case (1) to τ and q instead of r and q (note that $A\cap[q-1/2,\tau-1/2]=\emptyset$) and case (2) to r and τ instead of r and q. This gives

(5.44)
$$\|\mathbf{u}\|_{\mathscr{H}^r}\leqslant Mh^{2\tau-r-\alpha-1/2}\|\mathbf{u}\|_{\mathscr{H}^\tau}\leqslant M'h^{2\tau-r-\alpha-1/2}h^{q-\tau-\varepsilon}\|\mathbf{u}\|_{\mathscr{H}^q}$$
$$=M_{\tau,\varepsilon}h^{q-r-(\alpha+1/2-\tau)-\varepsilon}\|\mathbf{u}\|_{\mathscr{H}^q}.$$

Now τ is arbitrarily close to $\alpha+1/2$, whence (5.44) can be written as

(5.45)
$$\|\mathbf{u}\|_{\mathscr{H}^q}\leqslant M_\varepsilon h^{q-r-\varepsilon'}\|\mathbf{u}\|_{\mathscr{H}^q}.$$

The case where are several singular exponents between $q-1/2$ and $r-1/2$ is easily reduced to the previous case by induction.

Remark 5.8. This proof is a generalization of the proof of Lemma 3.1 in [58], where a corresponding result was proved for the case of a smooth curve and mixed boundary conditions yielding the singular exponent $\alpha=1/2$.

LEMMA 5.9. *For the regular finite element spaces $S_h^{0,t,k}$ and the singular functions u_{jl} we have the inverse estimate* (5.34)

$$\|\sum_{l=1}^{L}\mathbf{c}_{jl}u_{jl}\chi_j-\hat{w}_j\|_{\mathscr{H}^q}\geqslant\gamma h^{\alpha_{l_0}+1/2-q}|\mathbf{c}_{jl_0}|$$

for all $l_0=1,\ldots,L$, and all $\hat{w}_j\in S_h^{0,t,k}$ if $k\geqslant q$ and $\alpha_{jl}+1/2>q$ $(l=1,\ldots,L)$.

Proof. It is sufficient to show that for every $s\in R$, $\alpha_1,\ldots,\alpha_L>s-1/2$, $\alpha_j\neq\alpha_k$ for $j\neq k$, we have

(5.46)
$$\|\sum_{l=1}^{L}c_l x^{\alpha_l}\chi_j-p_j(x)\|_{H^s(\Gamma^j)}\geqslant\gamma h^{\alpha_{l_0}+1/2-s}|c_{l_0}|$$

for every function $p_j\in H^s(\Gamma^j)$ which is polynomial on $[0,h]$. If we have to estimate $\|\cdot\|_{\tilde{H}^s}$, then we use the fact that $\|\cdot\|_{\tilde{H}^s}\geqslant\|\cdot\|_{H^s}$. In order to show (5.46), we assume first that there is a positive definite bilinear form $(\cdot,\cdot)_{s,h}$ with the properties

(5.47)
$$\|\phi\|^2_{H^s(\Gamma^j)}\geqslant c\cdot(\phi|_{[0,h]},\phi|_{[0,h]})_{s,h}\quad\text{for all }\phi\in H^s(\Gamma^j)$$

and

(5.48)
$$(x^{\alpha_j},x^{\alpha_k})_{s,h}=h^{\alpha_j+\alpha_k-2s+1}\gamma_{\alpha_j,\alpha_k,s}\quad\text{for all }\alpha_{j,k}>s-1/2,$$

$0<h\leqslant h_0$, where $\gamma_{\alpha_j,\alpha_k,s}>0$ are constants not depending on h. We define

(5.49)
$$F(c_1,\ldots,c_N;h):=(\sum_{l=1}^{N}c_l x^{\alpha_l},\sum_{l=1}^{N}c_l x^{\alpha_l})_{s,h}.$$

For fixed c_l, the minimum $F^l_{\min}(c_l;h)$ of this quadratic form is attained for

$$\frac{\partial F}{\partial c_j}=2(x^{\alpha_j},\sum_{k=1}^{N}c_k x^{\alpha_k})_{s,h}=0\quad(j=1,\ldots,N).$$

This can be written as a $(N-1)\times(N-1)$ system of equations

$$\sum_{k\neq l}c_k(x^{\alpha_j},x^{\alpha_k})=-c_l(x^{\alpha_j},x^{\alpha_l})$$

or by (5.48) as

(5.50)
$$\sum_{k\neq l}c_k(h)\cdot h^{\alpha_k-\alpha_l}\gamma_{\alpha_j,\alpha_k,s}=-c_l\gamma_{\alpha_j,\alpha_l,s}.$$

Thus the unique solution of this system is of the form

(5.51) $\quad c_k(h) = h^{\alpha_j - \alpha_k} c_l \gamma_{kl}$ (for all $k = 1, \ldots, N$ if we set $\gamma_{ll} = 1$),

where γ_{kl} do not depend on h and c_l. Inserting (5.51) in (5.49), we find

$$F^l_{\min}(c_j; h) = F(c_1(h), \ldots, c_l, \ldots, c_N(h); h) = h^{2\alpha_l - 2s + 1} c_l^2 F^l_{\min}(1; 1)$$

and therefore

(5.52) $\quad F(c_1, \ldots, c_N; h) \geqslant \gamma h^{2\alpha_l - 2s+1} |c_l|^2 \quad$ with $\quad \gamma = F^l_{\min}(1; 1) > 0.$

By (5.47) we can deduce (5.46) from (5.52) if we put

$$\sum_{l=L+1}^{N} c_l x^{\alpha_l} = -p_j(x) \quad \textit{for} \quad x \in [0, h].$$

It remains to define $(\cdot, \cdot)_{s,h}$ and to verify (5.47), (5.48). For $s \in \mathbf{N}_0$ we define

$$(\phi, \phi)_{s,h} := \int_0^h \left| \left(\frac{d}{dx}\right)^s \phi(x) \right|^2 ds,$$

so that

(5.53) $$\|\phi\|^2_{H^s(\Gamma^j)} \geqslant \|\phi\|^2_{H^s[0,h]} \geqslant (\phi, \phi)_{s,h}.$$

Similarly, for $s = m + \sigma$, $m \in \mathbf{N}_0$, $\sigma \in (0, 1)$, we define with the Besov norm instead of the L^2-norm:

$$(\phi, \phi)_{s,h} := \int_0^h \int_0^h \left| \left(\frac{d}{dx}\right)^m \phi(x) - \left(\frac{d}{dy}\right)^m \phi(y) \right|^2 |x-y|^{-(1+2\sigma)} dx dy,$$

and (5.53) again holds because of the continuity of the restriction $H^s(\Gamma^j) \to H^s[0, h]$.

In both cases, (5.48) is easily derived by a change of integration variables.

For $s < 1/2$ we can define

$$(\phi, \phi)_{s,h} := \|\phi|_{[0,h]}\|^2_{\overset{0}{W}{}^s(\mathbf{R}_+)}.$$

Then (5.47) follows from the local equivalence of norms (Lemma 2.6). By Parseval's equation (Lemma 2.2) we obtain by the Mellin transform

$$\int_0^h x^{i\lambda - 1} x^\alpha dx = -i \frac{h^{\alpha + i\lambda}}{\lambda - i} \quad (\operatorname{Im} \lambda < \alpha):$$

$$(x^{\alpha_j}, x^{\alpha_k})_{s,h} = C \cdot \int_{\operatorname{Im}\lambda = s - 1/2} (1 + |\lambda|^2)^s \frac{h^{\alpha_j + i\lambda} h^{\alpha_k - i\bar{\lambda}}}{(\lambda - i\alpha_j)(\bar{\lambda} + i\alpha_k)} d\lambda$$

$$= h^{\alpha_j + \alpha_k - 2s + 1} C \cdot \int_{\operatorname{Im}\lambda = s - 1/2} \frac{(1 + |\lambda|^2)^s d\lambda}{(\lambda - i\alpha_j)(\bar{\lambda} + i\alpha_k)}.$$

For $s < 1/2$ this is finite, which proves (5.48) for this case. ∎

As a final result we want to present an error estimate for a Galerkin procedure with a different scalar product, namely that of $H^{1/2}(\Gamma_2) \times L^2(\Gamma_1)$ instead of $L^2(\Gamma)$ as in (5.5). This result is easily obtained by the methods used above but based on the Gårding inequality of Theorem 2.24 instead of Theorem 2.19.

THEOREM 5.10: *Let $G^h_{\mathscr{A}_0}$ be the Galerkin operator defined by $G^h_{\mathscr{A}_0}$: $U \mapsto U^*_h$ where $U^*_h \in S^{p,t,k}_h$ is the solution of*

$$\langle \mathscr{A}_0 U^*_h, V \rangle_{\mathscr{H}^{1/2}} = \langle \mathscr{A}_0 U, V \rangle_{\mathscr{H}^{1/2}} \quad \textit{for all } V \in S^{p,t,k}_h.$$

Here $\langle \cdot, \cdot \rangle_{\mathscr{H}^{1/2}}$ is defined in (2.43). Then

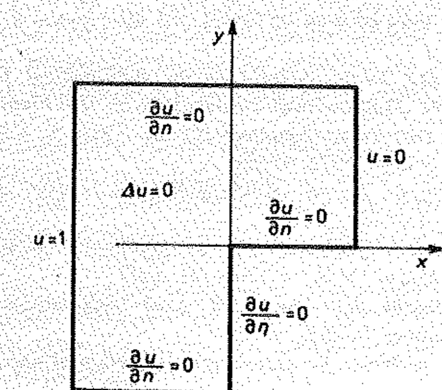
$$G^h_{\mathscr{A}_0}: H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1) \to H^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$$

is uniformly bounded independent of $0 < h \leqslant h_0$. Furthermore, if U is the exact solution of $\mathscr{A}_0 U = \mathscr{B}_0 G$, there is an estimate like (5.24)

$$\|U - U^*_h\|_{\mathscr{X}^r} \leqslant c h^{s-r-\varepsilon} \|G\|_{\mathscr{X}^s}.$$

Finally we demonstrate our error estimate (5.24) by an example of a mixed boundary value problem on a L-shaped region which was numerically treated by different methods in [27]. Here the data satisfy the compatibillity conditions (4.95) (v). Therefore $G \in \mathscr{X}^s$ for all $s < 5/2$ and thus the solution U of our system of integral equations (1.20) belongs to any $\mathscr{X}^s$, $s < 5/2$. Hence, by (4.98) and the definition of the singular functions, U contains two of them, namely at (0, 0):

$$u_{12} = \begin{bmatrix} x^{4/3} \\ |y|^{4/3} \end{bmatrix}; \quad u_{11} = \begin{bmatrix} x^{2/3} \\ -|y|^{2/3} \end{bmatrix}.$$



Now if we take as trial functions μ_l (cf. Definition 5.6) for v piecewise quadratics, and trial functions v_l for ψ as piecewise linear polynomials, and furthermore u_{11}, u_{12} above, we have the error estimate (5.24), which holds for any $r < 5/2$ with $r \neq \{\frac{3}{2}, \frac{2}{3} + \frac{1}{2}, \frac{4}{3} + \frac{1}{2}\}$. For $r > \frac{3}{2}$ the trial functions have to satisfy additional compatibility conditions, too (see definition 5.6) (iii)). Thus by (5.24) we have

$$\|U-U^h\|_{\mathscr{H}^r} \leqslant ch^{5/2-r-\varepsilon}\|G\|_{\mathscr{H}^{5/2}} \tag{5.54}$$

and for the coefficients of the singular functions u_{11}, u_{12}

$$|c_{11}-c_{11}^h| = O(h^{4/3-\varepsilon}), \quad |c_{12}-c_{12}^h| = O(h^{2/3-\varepsilon}).$$

We define the solution u in Ω as in Theorem 3.9 and the approximate solution u^h correspondingly. By Lemma 3.3 and the Sobolev embedding theorem we have the pointwise estimate (choosing $r = 1/2+\varepsilon$ in (5.54))

$$\|u-u^h\|_{L^\infty(\bar{\Omega})} \leqslant c\|U-U^h\|_{\mathscr{H}^{1/2+\varepsilon}} = O(h^{2-\varepsilon}).$$

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