On the Method of Volume Integral Equations

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Integral Methods in
Science and Engineering

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The scattering problem

$$\boxed{\operatorname{div} a(x)\nabla u + k(x)^2 u = f} \quad \text{in } \mathbb{R}^d, \quad \text{Sommerfeld rad. cond. at } \infty$$

$a(x) \equiv 1$, $k(x) \equiv k \in \mathbb{C}$ outside of a bounded domain Ω, $\operatorname{supp} f$ compact.

Tutorial: Acoustic scattering by a penetrable object

The scattering problem

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Rewritten as a perturbation problem

$$\boxed{(\Delta + k^2)u = f - (\operatorname{div} \alpha \nabla + \beta)u}, \quad \alpha(x) = a(x) - 1,\ \beta(x) = k(x)^2 - k^2$$

The Volume Integral Equation: Convolution with the fundamental solution

$$\boxed{u = -G_k * f + \operatorname{div} G_k * (\alpha \nabla u) + G_k * (\beta u)} \qquad G_k(x) = \frac{e^{ik|x|}}{4\pi|x|} \text{ for } d = 3$$

For any domain $\hat{\Omega}$ with $\Omega \subset \hat{\Omega} \subset \mathbb{R}^d$: $\quad u^{\mathrm{inc}}(x) := -\int G_k(x-y) f(y)\, dy$

$$\boxed{u(x) - \operatorname{div} \int_\Omega G_k(x-y)\alpha(y) \nabla u(y)\, dy - \int_\Omega G_k(x-y)\beta(y) u(y)\, dy = u^{\mathrm{inc}}(x)} \quad x \in \hat{\Omega}$$

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The VIE, classical: Lippmann-Schwinger equation

Situation: $a(x) \equiv 1 \Rightarrow \alpha = 0$

Fredholm alternative

Assumption: $\beta \in L^\infty$. Then the VIE

$$\boxed{u(x) - \int_\Omega G_k(x-y)\beta(y)u(y)\,dy = u^{\mathrm{inc}}(x)}$$

is a classical (weakly singular) second kind Fredholm integral equation. The Fredholm alternative holds in $L^2(\hat{\Omega})$ and, if β is smooth, in $H^s(\Omega)$ for any suitable s.

This is one of the standard methods for proving existence of a solution of the scattering problem.

Situation: $a \in C^1(\mathbb{R}^d) \Rightarrow \alpha = 0$ on $\partial\Omega$

The operator

$$u \mapsto \operatorname{div} G_k * (\alpha \nabla u)$$

maps $H^1(\Omega)$ boundedly to itself (and L^2 to L^2), but it is strongly singular and not compact.

One can obtain a weakly singular Fredholm integral equation by partial integration:

$$\begin{aligned}
\operatorname{div} G_k * (\alpha \nabla u)(x) &= -\operatorname{div} \int_\Omega \nabla_y \big(G_k(x-y)\alpha(y)\big)\, u(y)\, dy \\
&= \Delta \int_\Omega G_k(x-y)\alpha(y) u(y)\, dy \\
&\quad - \operatorname{div} \int_\Omega G_k(x-y)(\nabla \alpha)(y)\, u(y)\, dy \\
&= -\alpha(x) u(x) - k^2 \int_\Omega G_k(x-y)\alpha(y) u(y)\, dy \\
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The VIE, still classical

Situation: $a \in C^1(\mathbb{R}^d) \Rightarrow \alpha = 0$ on $\partial\Omega$

Fredholm alternative

Assumption: $\beta \in L^\infty$. Then the VIE

$$a(x)u(x) - \int_\Omega G_k(x-y)(\beta(y) - k^2\alpha)u(y)\,dy$$
$$+\,\mathrm{div}\int_\Omega G_k(x-y)(\nabla\alpha)(y)\,u(y)\,dy = u^{\mathrm{inc}}(x)$$

is a classical (weakly singular) second kind Fredholm integral equation
if and only if $a(x) \neq 0$ for all $x \in \Omega$.
The Fredholm alternative then holds in $L^2(\hat{\Omega})$ and, if α and β are smooth, in $H^s(\Omega)$ for any suitable s.

This is one of the standard methods for proving existence of a solution of the scattering problem [Colton-Kress 1983].

The VIE, no longer classical

Situation: $a \in C^1(\bar{\Omega})$, i.g. discontinuous across $\Gamma = \partial\Omega$.

Theorem: Fredholm alternative [??]

Assumption: $\beta \in L^\infty$. Then the VIE

$$\boxed{u(x) - \operatorname{div} \int_\Omega G_k(x-y)\alpha(y)\nabla u(y)\,dy - \int_\Omega G_k(x-y)\beta(y)u(y)\,dy = u^{\mathrm{inc}}(x)}$$

is Fredholm in $H^1(\Omega)$ if and only if

1. $a(x) \neq 0$ for all $x \in \Omega$
2. $a(x) \neq -1$ for all $x \in \Gamma$

if Γ is smooth.
If Γ is Lipschitz, then a sufficient condition is

1. and $a(x) \neq -\frac{\sigma}{1-\sigma}\ \ \forall \sigma \in \Sigma, x \in \Gamma$.

Here Σ is a compact subset of $(0,1)$, the essential spectrum of the double layer potential operator $\frac{1}{2}\mathbb{I} + K$ on $H^{1/2}(\Gamma)$.
$\Sigma = \{\frac{1}{2}\}$ if Γ is smooth.

Two motivations

1. Antennas for millimetre waves with dielectric lenses
   Collaboration with colleagues in electronics (Rennes)
2. Discrete Dipole Approximation
   A challenge...

Electromagnetic scattering by penetrable objects

Two motivations

1. Antennas for millimetre waves with dielectric lenses
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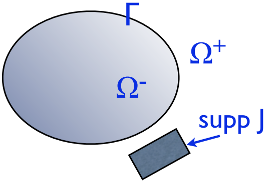
   A challenge...

For simplicity: Piecewise constant coefficients:



$\varepsilon = \varepsilon_r$ in $\Omega = \Omega^-$, $\quad \varepsilon = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega} = \Omega^+$,
$\eta = 1 - \varepsilon_r$
$\mu = \mu_r$ in Ω, $\mu = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\quad \nu = 1 - \frac{1}{\mu_r}$

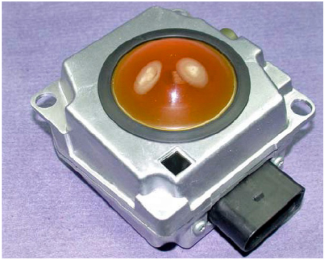
Time-harmonic Maxwell equations

$$\mathbf{curl}\, \boldsymbol{E} = i\omega\mu \boldsymbol{H}; \qquad \mathbf{curl}\, \boldsymbol{H} = -i\omega\varepsilon \boldsymbol{E} + \boldsymbol{J}$$

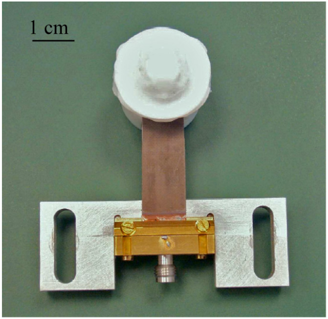
hold in $\mathbb{R}^3$ in the distributional sense ($+$ radiation condition).
supp $\boldsymbol{J}$ compact in $\mathbb{R}^3 \setminus \overline{\Omega}$.
$\Longrightarrow$ Transmission conditions on $\Gamma = \partial\Omega$:

$$\begin{aligned} [\boldsymbol{E} \times \boldsymbol{n}]_\Gamma &= 0; & [\boldsymbol{n} \cdot \mu \boldsymbol{H}]_\Gamma &= 0 \\ [\boldsymbol{H} \times \boldsymbol{n}]_\Gamma &= 0; & [\boldsymbol{n} \cdot \varepsilon \boldsymbol{E}]_\Gamma &= 0 \end{aligned}$$

The dielectric scattering problem: Dielectric lenses

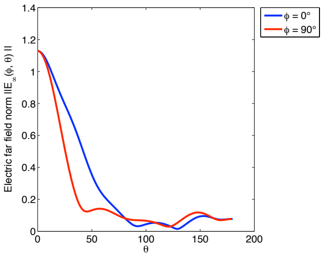


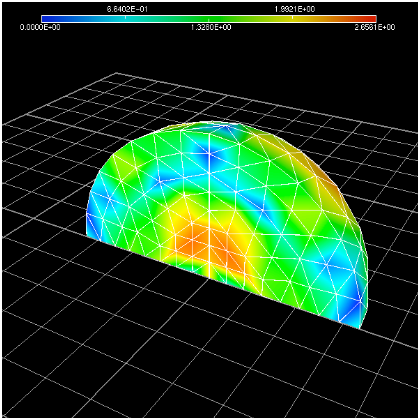
ILA pour communication indoor à 62GHz.
Source : IST



. 4 : ILA compacte pour communication par satellite à 49GHz [7]. Source : IETR

[E.H. Koné, PhD thesis Rennes 2010]





One considers the obstacle as a perturbation of the free-space situation:

$$\mathbf{curl}\,\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\varepsilon\boldsymbol{E} = i\omega\boldsymbol{J} \Leftrightarrow$$

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\boldsymbol{E} = i\omega\boldsymbol{J} - \omega^2\eta\,\chi_\Omega\boldsymbol{E} + \mathbf{curl}\,\nu\,\chi_\Omega\,\mathbf{curl}\,\boldsymbol{E}$$

with $\eta = 1 - \varepsilon_r, \;\; \nu = 1 - \frac{1}{\mu_r}6.$

The right-hand side has compact support:
Convolution with fundamental solution of $\mathbf{curl}\,\mathbf{curl} - \omega^2$:

$$g_\omega = \Big(\frac{1}{\omega^2}\nabla\,\mathrm{div} + 1\Big)G_\omega\,; \qquad G_\omega(x) = \frac{e^{i\omega|x|}}{4\pi|x|}$$

$$(\mathbf{curl}\,\mathbf{curl} - \omega^2)\Big(\frac{1}{\omega^2}\nabla\,\mathrm{div} + 1\Big) = -(\Delta + \omega^2)$$

The Volume Integral Equation

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Representation of $\boldsymbol{E}$ in $\mathbb{R}^3$ by volume integrals over Ω

$$\boldsymbol{E} = \omega^2 g_\omega * (\eta\,\chi_\Omega \boldsymbol{E}) + g_\omega * (\mathbf{curl}\,\nu\,\chi_\Omega\,\mathbf{curl}\,\boldsymbol{E}) + \boldsymbol{E}^{\mathrm{inc}}$$

$\Longrightarrow$ Volume integral equation in Ω (for piecewise continuous coefficients)

$$\boxed{\boldsymbol{E} = \eta A_\omega \boldsymbol{E} + \nu B_\omega \boldsymbol{E} + \boldsymbol{E}^{\mathrm{inc}}}$$

with

$$A_\omega \boldsymbol{E}(x) = -\nabla\,\mathrm{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy - \omega^2 \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy$$

$$B_\omega \boldsymbol{E}(x) = \mathbf{curl} \int_\Omega G_\omega(x-y)\,\mathbf{curl}\,\boldsymbol{E}(y)\,dy$$

A_ω is strongly singular:

$$-\nabla\,\mathrm{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy = \mathrm{p.v.} \int_\Omega \nabla_x \nabla_y G_\omega(x-y)\cdot\boldsymbol{E}(y)\,dy + \frac{1}{3}\boldsymbol{E}(x)$$

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The Problem

For which η and ν is the operator

$$\mathbb{I} - \eta A_\omega - \nu B_\omega$$

invertible [Fredholm] in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ [$\boldsymbol{L}^2(\Omega)$]?

Known: For "physically reasonable" (classical) material coefficients, for example $\eta < 1$ and $\nu < 1$, the transmission problem and therefore also the integral equation has a unique solution, $\forall \omega \in \mathbb{R}$.

The Spectral Problem

Determine the essential spectrum of the operator $\eta A_\omega + \nu B_\omega$, in particular, when is

$$1 \in \mathrm{Sp}_{\mathrm{ess}}(\eta A_\omega + \nu B_\omega) \qquad ?$$

Motivation:

Metamaterials, stability and convergence of numerical algorithms,

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The Discrete Dipole Approximation

What is the DDA?

Simplest delta-collocation

$$\int_\Omega k(x_i, y)\boldsymbol{E}(y)\,dy \sim \sum_{j\neq i} k(x_i, x_j)\boldsymbol{E}_j + \frac{1}{\alpha_i}\boldsymbol{E}_i$$

together with a recipe for the value of α_i.

From [Loke–Mengüç–Nieminen 2011]:

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From [Loke–Mengüç–Nieminen 2011]:

The original implementation of DDA Purcell and Pennypacker [1] used the Clausius–Mossotti polarizability, given by

$$\alpha_j^{CM} = \frac{3d^3}{4\pi}\left(\frac{m_j^2-1}{m_j^2+2}\right) = \frac{3d^3}{4\pi}\left(\frac{\varepsilon_j-1}{\varepsilon_j+2}\right). \quad (5)$$

A review of subsequent development and different formulations of the polarizability calculation is available from Yurkin and Hoekstra [5]. Currently, the most popular form is the lattice dispersion relation (LDR) [13]:

$$\alpha_j^{LDR} = \frac{\alpha_j^{CM}}{1+\frac{\alpha_j^{CM}}{d^3}[(b_1+m^2b_2+m^2b_3S)(kd)^2-\frac{2}{3}\mathrm{i}(kd)^3]}, \quad (6)$$

$$b_1 = -1.891531, \quad b_2 = 0.1648469,$$

$$b_3 = -1.7700004, \quad S \equiv \sum_{j=1}^{3}(\hat{a}_j\hat{e}_j)^2,$$

where $\hat{a}$ and $\hat{e}$ are unit vectors defining the direction and polarization of the incident light.

From [Yurkin-Hoekstra 2007]:

We present a review of the discrete dipole approximation (DDA), which is a general method to simulate light scattering by arbitrarily shaped particles. We put the method in historical context and discuss recent developments, taking the viewpoint of a general framework based on the integral equations for the electric field. We review both the theory of the DDA and its numerical aspects, the latter being of critical importance for any practical application of the method. Finally, the position of the DDA among other methods of light scattering simulation is shown and possible future developments are discussed.

The DDA is called the coupled dipole method or approximation by some researchers [12,13]. There are also other methods, such as the volume integral equation formulation [14] and the digitized Green's function (DGF) [7], which were developed completely independently from PP. However, later they were shown to be equivalent to DDA [8,15]. In

From [Yurkin-Hoekstra 2007]:

[59] Okamoto H, Macke A, Quante M, Raschke E. Modeling of backscattering by non-spherical ice particles for the interpretation of cloud radar signals at 94 GHz. An error analysis. Contrib Atmos Phys 1995;68:319-334.

[60] Liu CL, Illingworth AJ. Error analysis of backscatter from discrete dipole approximation for different ice particle shapes. Atmos Res 1997;44:231-241.

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[63] Yurkin MA, Maltsev VP, Hoekstra AG. Convergence of the discrete dipole approximation. II. An extrapolation technique to increase the accuracy. J Opt Soc Am A 2006;23:2592-2601.

Web search for 'discrete dipole approximation'

Google Scholar: 6180 hits

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Discrete Dipole Approximation: Theory?

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From [Yurkin-Hoekstra 2007]:

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[1] Purcell EM, Pennypacker CR. Scattering and adsorption of light by nonspherical dielectric grains. Astrophys J 1973;186:705-714.
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From [Okamoto-Xu 1998]:

The discrete-dipole approximation (DDA) was originally developed by Purcell and Pennypacker (1973). The DDA has been used in many fields, e.g., astronomy (e.g., Draine, 1988; Kozasa *et al*., 1992, 1993), planetary sciences (West, 1991; Lumme and Rahola, 1994; Okamoto *et al*., 1994) and atmospheric sciences (e.g., Flatau *et al*., 1990). The DDA seems

More recently:

Nano-science, "optical tweezers"

Discrete Dipole Approximation: An example

[Loke–Mengüç–Nieminen, JQSRT 2011]

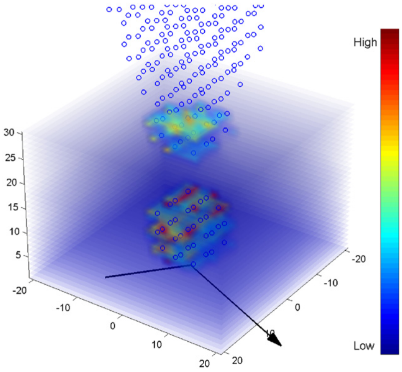

Fig. 14. Relative field intensity in the region comprising a simulated gold AFM probe tip in the vicinity of a 20 nm gold nanoparticle on a silicon surface, illuminated by an evanescent wave.

Back to Mathematics: Some References

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Volume integral equation in Ω, piecewise constant coefficients

$$\boxed{\boldsymbol{E} = \eta A_\omega \boldsymbol{E} + \nu B_\omega \boldsymbol{E} + \boldsymbol{E}^{\rm inc}}$$

with

$$A_\omega \boldsymbol{E}(x) = -\nabla \operatorname{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy - \omega^2 \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy$$

$$B_\omega \boldsymbol{E}(x) = \mathbf{curl} \int_\Omega G_\omega(x-y)\,\mathbf{curl}\,\boldsymbol{E}(y)\,dy$$

Two subproblems:

1. The dielectric problem ($\nu = 0$, operator A_ω)
2. The magnetic problem ($\eta = 0$, operator B_ω)

1. Partial integration

For simplicity, $a(x)$ piecewise constant, $k(x)$ constant:
$a(x) = a = \alpha + 1$ in Ω, $a(x) = 1$ in $\mathbb{R}^d \setminus \bar{\Omega}$, $\beta(x) \equiv 0$.
The VIE

$$\boxed{u(x) - \operatorname{div} \int_\Omega G_k(x-y)\alpha \nabla u(y)\, dy = u^{\text{inc}}(x)}$$

becomes after integration by parts

$$au(x) + \alpha k^2 \int_\Omega G_k(x-y) u(y)\, dy - \alpha \operatorname{div} \int_\Gamma G_k(x-y) \mathbf{n}(y)\, u(y)\, dy = u^{\text{inc}}(x)$$

or

$$\boxed{au + \alpha k^2 N_k u + \alpha D_k \gamma u = u^{\text{inc}} \quad \text{in } \Omega}$$

Here N_k is the Newton (volume) potential, D_k is the double layer potential, and γu is the trace of u on the boundary Γ.

How to deal with the operator $D_k \gamma$?

Trick: Treat γu as independent unknown. $\longrightarrow$ 2

1. Partial integration

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How to deal with the operator $D_k \gamma$?

Trick: Treat γu as independent unknown. $\longrightarrow$ 2

2. Extension

$$\boxed{au + \alpha k^2 N_k u + \alpha D_k \gamma u = u^{\mathrm{inc}} \quad \text{in } \Omega}$$

Taking the trace on Γ, we obtain a second equation with the double layer boundary integral operator K_k

$$\boxed{a\gamma u + \alpha k^2 \gamma N_k u + \alpha(\tfrac{1}{2}\mathbb{I} + K_k)\gamma u = \gamma u^{\mathrm{inc}} \quad \text{on } \Gamma}$$

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This gives the (2×2) system, triangular+compact,

Coupled boundary-domain integral equation system

$$\begin{pmatrix} a\mathbb{I} + \alpha k^2 N_k & \alpha D_k \\ \alpha k^2 \gamma N_k & \frac{a+1}{2}\mathbb{I} + (a-1)K_k \end{pmatrix} \begin{pmatrix} u \\ \gamma u \end{pmatrix} = \begin{pmatrix} u^{\rm inc} \\ \gamma u^{\rm inc} \end{pmatrix}$$

The system is Fredholm on $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$ if and only if

$$a \neq 0 \quad \text{and} \quad \frac{a}{a-1} \notin \Sigma = \mathrm{Sp}_{\rm ess}(\tfrac{1}{2}\mathbb{I} + K_k)$$

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The dielectric VIE

$$\boxed{\boldsymbol{E} - A_\omega \eta \, \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}}$$

with

$$A_\omega \boldsymbol{E}(x) = -\nabla \operatorname{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy - \omega^2 \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy$$

Theorem [Colton-Kress '98]

If $\varepsilon \in C^2(\mathbb{R}^3)$, then on the space of functions satisfying $\operatorname{div}(\varepsilon \boldsymbol{E}) = 0$, the operator $A_\omega \eta$ is equivalent to a weakly singular integral operator.

Proof: Integration by parts ($\eta = 1 - \varepsilon$)

$$\operatorname{div}(\varepsilon \boldsymbol{E}) = 0 \quad \Longrightarrow \quad \operatorname{div}(\eta \boldsymbol{E}) = \operatorname{div} \boldsymbol{E} = -\frac{1}{\varepsilon}(\nabla\varepsilon \cdot \boldsymbol{E})$$

$$\nabla \operatorname{div} G_\omega * (\eta \boldsymbol{E}) = \nabla G_\omega * \operatorname{div}(\eta \boldsymbol{E}) = -\nabla G_\omega * (\tfrac{\nabla\varepsilon}{\varepsilon} \cdot \boldsymbol{E}) .$$

Does not work for
discontinuous permittivity !

Next case: Dielectric VIE with Smooth permittivity

The dielectric VIE

$$\boxed{\boldsymbol{E} - A_\omega \eta \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}}$$

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$$A_\omega \boldsymbol{E}(x) = -\nabla \operatorname{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy - \omega^2 \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy$$

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Does not work for
discontinuous permittivity !

A numerical spectrum [from J. Rahola SIJSC 2000]

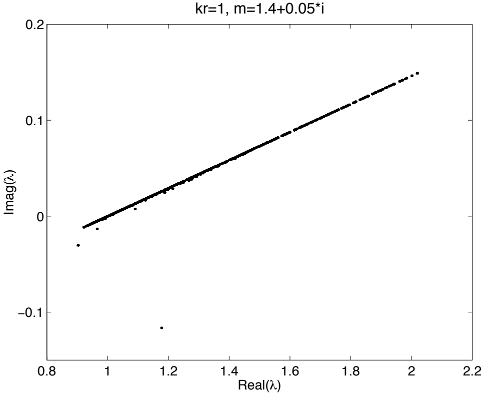

FIG. 3.1. *Eigenvalues of the coefficient matrix for a spherical scatterer of radius* $kr = 1$ *and refractive index* $m = 1.4 + 0.05i$. *The sphere is discretized with* 136 *computational cells (upper) and* 480 *computational cells (lower).*

$$\eta = 1 - m^2 = -0.9575 - 0.14\,i : \quad \text{line} \sim 1 - \eta.[0,1]$$

imaginary part Im[λ] are parameterized by three real numbers $\boldsymbol{x}$ ($\in \mathbb{R}^3$), but when ε is a constant, it degenerates to a curve connecting $\lambda = 1$ and this constant value as demonstrated by Rahola (2000) and Budko & Samokhin (2006a, 2006b). The

Translation:

Conjecture

$$\mathrm{Sp}_{\mathrm{ess}}(A_\omega) = [0, 1]$$

In reality:

Theorem (Ω regular) [illegible]

$$\mathrm{Sp}_{\mathrm{ess}}(A_\omega) = \left\{0, \tfrac{1}{2}, 1\right\}$$

A conjecture [quote from Fukumoto & Samokhin 2011]

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Conjecture

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In reality:

Theorem (Ω regular) [Pasciak 1984, Co 2008]

$$\mathrm{Sp}_{\mathrm{ess}}(A_\omega) = \left\{0, \tfrac{1}{2}, 1\right\}$$

The dielectric scattering problem: $\nu = 0$

Volume integral equation: $\boldsymbol{E} - \eta A_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$

Results (Co & E. Darrigrand & E.H. Koné 2009)

1. The operator A_ω can be extended to $\boldsymbol{L}^2(\Omega)$ as a bounded operator.
2. It has $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$ as invariant subspaces.
3. For $\boldsymbol{E}^{\mathrm{inc}}$ in $\boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, the integral equation in $\boldsymbol{L}^2$ has the same solutions as in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ or in $\boldsymbol{H}(\mathrm{div}, \Omega)$.
4. $A_\omega - A_0$ is compact in $\boldsymbol{L}^2(\Omega)$.
5. $\mathrm{Sp}(A_0) \subset [0, 1]$
   0 and 1 are eigenvalues of infinite multiplicity of A_0
   $\frac{1}{2}$ is accumulation point of eigenvalues, and
   $\mathrm{Sp}_{\mathrm{ess}}(A_0) = \{0, \frac{1}{2}, 1\}$ if Γ is smooth

The dielectric scattering problem can be solved by solving the volume integral equation in $\boldsymbol{L}^2(\Omega)$.

Orthogonal decompositions:

$$L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus V \; ; \qquad V = H(\operatorname{div} 0, \Omega)$$

$$= \nabla H^1(\Omega) \oplus V_0 \; ; \qquad V_0 = H_0(\operatorname{div} 0, \Omega)$$

Orthogonal decompositions:

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\begin{aligned}
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&= \nabla H^1(\Omega) \oplus V_0\,; &\qquad V_0 &= H_0(\operatorname{div} 0, \Omega)\\
&= \nabla H_0^1(\Omega) \oplus V_0 \oplus W\,; &\qquad W &= \nabla H^1(\Omega) \cap V \text{ : harmonic vector fields}
\end{aligned}
$$

Recall: $\quad A_0 u(x) = -\nabla \operatorname{div} \int_\Omega \frac{1}{4\pi|x-y|} u(y)\,dy$

Lemma

$$u \in \nabla H_0^1(\Omega) \Longrightarrow \quad A_0 u = u$$

Isomorphisms: $\quad W \ni u \leftrightarrow \gamma_n u = n \cdot u\big|_{\partial\Omega} \in H_*^{-1/2}(\partial\Omega)$

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Analysis of A_0 : Helmholtz decomposition

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S : single layer potential

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$$A_0\big|_W \leftrightarrow \partial_n S = \left(\tfrac{1}{2}\mathbb{I} + K'\right)\big|_{H_*^{-1/2}(\partial\Omega)}$$

The magnetic scattering problem: $\eta = 0$ (Co & E. Darrigrand & H. Sakly)

Volume integral equation: $\boxed{\boldsymbol{E} - \nu B_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}}$

Clear: The integral operator

$$B_\omega \boldsymbol{E}(x) = \mathbf{curl} \int_\Omega G_\omega(x-y)\,\mathbf{curl}\,\boldsymbol{E}(y)\,dy$$

is bounded from $\boldsymbol{H}(\mathbf{curl},\Omega)$ to itself and to $\boldsymbol{H}(\mathrm{div}\,0,\Omega)$.

Integration by parts: For $\boldsymbol{E} \in \boldsymbol{C}_0^\infty(\Omega)$ one has

$$\begin{aligned} B_\omega \boldsymbol{E} &= \mathbf{curl}\, G_\omega * \mathbf{curl}\,\boldsymbol{E} = \mathbf{curl}\,\mathbf{curl}\, G_\omega * \boldsymbol{E} \\ &= \nabla\,\mathrm{div}\, G_\omega * \boldsymbol{E} - (\Delta + \omega^2) G_\omega * \boldsymbol{E} + \omega^2 G_\omega * \boldsymbol{E} \\ &= \boldsymbol{E} - A_\omega \boldsymbol{E} \end{aligned}$$

Proposition

The operator B_ω can be extended from $\boldsymbol{C}_0^\infty(\Omega)$ to $\boldsymbol{L}^2(\Omega)$ as a bounded operator $\widehat{B}_\omega$.
There holds then

$$\widehat{B}_\omega = \mathbb{I} - A_\omega$$

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Proposition (Kirsch & Lechleiter 2010)

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The Function Space is Important Here

Theorem 1

Solving the volume integral equation

$$\boldsymbol{E} - \nu B_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is equivalent to the magnetic Maxwell scattering problem.

Theorem 2

Let $\widehat{B}_\omega : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ be the extended operator. Solving the volume integral equation

$$\boldsymbol{E} - \nu \widehat{B}_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{L}^2(\Omega)$ gives the Maxwell equations in $\mathbb{R}^3 \setminus \Gamma$ with the transmission conditions

$$\begin{aligned} [\tfrac{1}{\mu}\boldsymbol{E} \times \boldsymbol{n}]_\Gamma &= 0; & [\boldsymbol{n} \cdot \boldsymbol{H}]_\Gamma &= 0 \\ [\boldsymbol{H} \times \boldsymbol{n}]_\Gamma &= 0; & [\boldsymbol{n} \cdot \boldsymbol{E}]_\Gamma &= 0 \end{aligned}$$

Different problem !

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Theorem 1

Solving the volume integral equation

$$\boldsymbol{E} - \nu B_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is equivalent to the magnetic Maxwell scattering problem.

Theorem 2

Let $\widehat{B}_\omega : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ be the extended operator. Solving the volume integral equation

$$\boldsymbol{E} - \nu \widehat{B}_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{L}^2(\Omega)$ gives the Maxwell equations in $\mathbb{R}^3 \setminus \Gamma$ with the transmission conditions

$$\begin{aligned} [\tfrac{1}{\mu} \boldsymbol{E} \times \boldsymbol{n}]_\Gamma &= 0\,; & [\boldsymbol{n} \cdot \boldsymbol{H}]_\Gamma &= 0 \\ [\boldsymbol{H} \times \boldsymbol{n}]_\Gamma &= 0\,; & [\boldsymbol{n} \cdot \boldsymbol{E}]_\Gamma &= 0 \end{aligned}$$

Different problem !

Explanation : For $\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, one has

$$\widehat{B}_\omega \boldsymbol{E} = B_\omega \boldsymbol{E} + \mathbf{curl} \int_\Gamma G_\omega(x-y) \boldsymbol{E}(y) \times \boldsymbol{n}(y)\, ds(y)$$

The latter term does not have a continuous extension to $\boldsymbol{L}^2(\Omega)$.

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The operator B_ω cannot be extended from $\boldsymbol{H}(\mathbf{curl}, \Omega)$ to $\boldsymbol{L}^2(\Omega)$ as a bounded operator.

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Proposition 2

Although on $\boldsymbol{C}_0^\infty(\Omega)$ we have

$$B_\omega = \mathbb{I} - A_\omega,$$

the commutator of A_ω and B_ω on $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is not even compact.

$\Longrightarrow$ No joint essential spectrum of A_ω and B_ω in the Lippmann-Schwinger operator $\mathbb{I} - \eta A_\omega - \nu B_\omega$.

$$\mathrm{Sp}_{\mathrm{ess}}(\eta A_\omega + \nu B_\omega) \neq \mathrm{Sp}_{\mathrm{ess}}(\eta A_\omega + \nu \widehat{B}_\omega) = \{\eta, \nu, \frac{\eta+\nu}{2}\}$$

A coupled boundary-domain integral equation system

Definition: $\boldsymbol{X} = \boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\operatorname{div} 0, \Omega)$
For $\boldsymbol{E}^{\mathrm{inc}} \in \boldsymbol{X}$, the equation $B_0 \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$ is equivalent to the system

$$\begin{pmatrix} \mathbb{I} & -\nabla S_0 & \mathbf{curl}\, S_0 \\ 0 & \frac{1}{2}\mathbb{I} - K' & \gamma_n \mathbf{curl}\, S_0 \\ 0 & -\gamma_\times \nabla S_0 & \frac{1}{2}\mathbb{I} + M \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \gamma_n \boldsymbol{E} \\ \gamma_\times \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}^{\mathrm{inc}} \\ \gamma_n \boldsymbol{E}^{\mathrm{inc}} \\ \gamma_\times \boldsymbol{E}^{\mathrm{inc}} \end{pmatrix}$$

with the normal trace $\gamma_n \boldsymbol{u} = \boldsymbol{n} \cdot \boldsymbol{u}$, the tangential trace $\gamma_\times \boldsymbol{u} = \boldsymbol{n} \times \boldsymbol{u}$, the single layer potential S_0 and the magnetic field integral operator M

$$M\boldsymbol{u}(x) := \int_\Gamma \boldsymbol{n}(x) \times \mathbf{curl}_x\, g_0(x-y)\boldsymbol{u}(y)\, ds(y)$$

Attention: The boundary trace space is $H^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}, \Gamma)$
In order to be able to use symbols, one first has to use Helmholtz decomposition in $\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}, \Gamma)$. The boundary space becomes

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Principal symbols of boundary pseudodifferential operators [Co & Stephan 1990]
$\Rightarrow$ ellipticity $\Rightarrow$ essential spectrum.
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Result for the magnetic problem (Γ smooth)

Result (Co & E. Darrigrand & H. Sakly 2011)

If Γ is smooth:

1. $\mathrm{Sp}_{\mathrm{ess}}(B_\omega) = \{0, \frac{1}{2}, 1\}$
2. $\mathrm{Sp}_{\mathrm{ess}}(\eta A_\omega + \nu B_\omega) = \{0, \frac{\eta}{2}, \eta, \frac{\nu}{2}, \nu\}$
3. The Lippmann-Schwinger operator is Fredholm in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ if and only if

$$\varepsilon_r \notin \{0, -1\} \quad \text{and} \quad \mu_r \notin \{0, -1\}$$

What if Γ is not smooth?

Spectrum of the boundary integral operator system

$$\begin{pmatrix} \frac{1}{2}\mathbb{I} - K' & \gamma_n \,\mathbf{curl}\, S_0 \\ -\gamma_\times \nabla S_0 & \frac{1}{2}\mathbb{I} + M \end{pmatrix}$$

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Another coupled boundary-domain integral equation system

For $\boldsymbol{E}^{\mathrm{inc}} \in \boldsymbol{X} = \boldsymbol{H}(\mathbf{curl}, \Omega) \cap \boldsymbol{H}(\operatorname{div} 0, \Omega)$, the volume integral equation $(\mathbb{I} - \eta A_0 - \nu B_0)\boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$ is equivalent to the system

$$\begin{pmatrix} \mathbb{I} & -\eta \nabla S & -\nu\, \mathbf{curl}\, N & 0 \\ 0 & \mathbb{I} - \eta \partial_n S & -\nu \gamma_n \mathbf{curl}\, N & 0 \\ 0 & 0 & (1-\nu)\mathbb{I} & \nu \nabla S \\ 0 & 0 & 0 & (1-\nu)\mathbb{I} + \nu \partial_n S \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \gamma_n \boldsymbol{E} \\ \mathbf{curl}\, \boldsymbol{E} \\ \gamma_n \mathbf{curl}\, \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}^{\mathrm{inc}} \\ \gamma_n \boldsymbol{E}^{\mathrm{inc}} \\ \mathbf{curl}\, \boldsymbol{E}^{\mathrm{inc}} \\ \gamma_n \mathbf{curl}\, \boldsymbol{E}^{\mathrm{inc}} \end{pmatrix}$$

on the space $\boldsymbol{X} \times \boldsymbol{H}^{-\frac{1}{2}}(\Gamma) \times \boldsymbol{H}(\operatorname{div} 0, \Omega) \times \boldsymbol{H}^{-\frac{1}{2}}(\Gamma)$.

Here N is the Newton (volume) potential and $\partial_n S = \frac{1}{2}\mathbb{I} + K'$ is the normal derivative of the single layer potential (for the Laplace equation).

For Γ Lipschitz, it is known that $\mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2}\mathbb{I} + K') \subset (0,1)$.
[Poincaré 1896, Steinbach & Wendland 2001,]

Example: An edge of opening $\pi/2$ gives a contribution $[\frac{1}{4}, \frac{3}{4}]$.

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Final Result

Theorem [Co-Darrigrand-Sakly 2012] For Γ Lipschitz:

Let $\Sigma = \mathrm{Sp}_{\mathrm{ess}}(\frac{1}{2}\mathbb{I} + K') \subset (0,1)$.

1. $\mathrm{Sp}_{\mathrm{ess}}(A_\omega) = \{0,1\} \cup \Sigma$
2. $\mathrm{Sp}_{\mathrm{ess}}(B_\omega) = \{0,1\} \cup (1-\Sigma)$
3. $\mathrm{Sp}_{\mathrm{ess}}(\eta A_\omega + \nu B_\omega) = \{0,\eta,\nu\} \cup \eta\Sigma \cup \nu(1-\Sigma)$
4. The Lippmann-Schwinger operator $\mathbb{I} - \eta A_\omega - \nu B_\omega$ is Fredholm in $\boldsymbol{H}(\mathbf{curl},\Omega)$ if and only if

$$\varepsilon_r, \mu_r \neq 0 \quad \text{and} \quad \frac{1}{1-\varepsilon_r} \notin \Sigma\,, \quad \frac{1}{1-\mu_r} \notin \Sigma$$

Thank you for your attention!