The Curious Importance of Function Spaces for the Maxwell Equations

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From the WEB. Google search for "function spaces are important"

Function spaces are important and natural examples of abstract Banach lattices http://eom.springer.de

However. . .

Introducing Function Space Yielding

Are you already optimizing your revenues and profits on your function space? The workshop will show you practical techniques how to calculate the price you should quote, not leaving any money on the table.

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Somewhat more seriously…

Ω: bounded polygon or polyhedron

(Dir) $$\Delta u = f \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

One can consider (Dir) in

Sobolev spaces	$H^s(\Omega), W_p^m(\Omega)$
Besov spaces	$B^s_{p,q}(\Omega)$
Hölder spaces	$C^\alpha(\Omega)$
with or without weight	$W_p^{m,\vec{\beta},\vec{\delta}}(\Omega)$
etc.	

In some of these spaces

- there is no existence result (Example: $H^2(\Omega)$, nonconvex Ω)
- there is no uniqueness result (Example: $L^2(\Omega)$, nonconvex Ω)

There is, however, a sanity result...

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Observation

Let X_1 and X_2 be two “reasonable” function spaces on Ω from the afore-mentioned list.

Let $f \in \Delta X_1 \cap \Delta X_2$ and let $u_1 \in X_1$, $u_2 \in X_2$ both be solutions of (Dir).

If X_1 and X_2 are such that in both spaces the homogeneous Dirichlet problem has only the trivial solution,

then $\quad u_1 = u_2$.

$\Longrightarrow$ Function spaces are useful: description of singularities, stability, error estimates, ..., but not really important.

The Dirichlet problem

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Outline

1. Are function spaces important?
   - What google has to say
   - The Dirichlet problem
2. Regularized Variational Formulation of Maxwell Equations
   - The Maxwell eigenvalue problem
   - Regularized variational formulations
   - Numerical observations
   - Explanation
   - Weighted Regularization
   - Numerical evidence for WRM
3. Regularized Boundary Integral Equations for Maxwell Equations
   - The Electrical Field Integral Equation
   - The Regularized EFIE
4. Lippmann-Schwinger Equation for the Magnetic Scattering Problem
   - Electromagnetic Transmission Problems
   - The Volume Integral Equation
   - The dielectric problem
   - The magnetic problem

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$$\boxed{\begin{aligned}\mathbf{curl}\,\boldsymbol{E} &= i\omega\mu\boldsymbol{H}\\ \mathbf{curl}\,\boldsymbol{H} &= -i\omega\varepsilon\boldsymbol{E}+\boldsymbol{J}\end{aligned}}$$

In this section: Domain $\Omega \subset \mathbb{R}^3$, $\varepsilon = \mu = 1$, $\boldsymbol{J} = 0$.
The condition $\operatorname{div} E = \operatorname{div} H = 0$ follows if $\omega \neq 0$.

$$\boxed{\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \& \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega}$$

Eigenfrequencies of a cavity with perfectly conducting walls.

Second order system for E: $\quad \mathbf{curl}\,\mathbf{curl}\,E - \omega^2 E = 0$

Simplest variational formulation

Find $\omega \neq 0$, $\boldsymbol{E} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \setminus \{0\}$ such that

$$\forall F \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \quad \int_\Omega \mathbf{curl}\,E \cdot \mathbf{curl}\,F = \omega^2 \int_\Omega E \cdot F$$

Energy space: $\boldsymbol{H}_0(\mathbf{curl}, \Omega) = \{u \in L^2(\Omega) \mid \mathbf{curl}\,u \in L^2(\Omega),\ u \times n = 0\}$
$= \text{closure in } \boldsymbol{H}(\mathbf{curl}, \Omega) \text{ of } C_0^\infty(\Omega)^3$

$$\begin{aligned} \mathbf{curl}\, \boldsymbol{E} &= i\omega\mu \boldsymbol{H} \\ \mathbf{curl}\, \boldsymbol{H} &= -i\omega\varepsilon \boldsymbol{E} + \boldsymbol{J} \end{aligned}$$

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Galerkin discretization:

Restriction to finite-dimensional subspace V_h, $h \to 0$.

Good: Eigenfrequencies are non-negative, discrete.

Regularized formulation

Simple variational formulation

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Big Problem: $\omega = 0$ has infinite multiplicity

Kernel: Electrostatic fields: **gradients** of all $\phi \in H_0^1(\Omega)$ (+ harmonic forms).

Idea: $\operatorname{div}\boldsymbol{E} = 0$, so we can add a multiple of $0 = \int_\Omega \operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{F}$

Regularized formulation:

$$(\mathrm{Reg}_X) \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{F} + s\int_\Omega \operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{F} = \omega^2\int_\Omega \boldsymbol{E}\cdot\boldsymbol{F}$$

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Second order system: $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - s\nabla\operatorname{div}\boldsymbol{E} = \omega^2\boldsymbol{E}$: Strongly elliptic. OK

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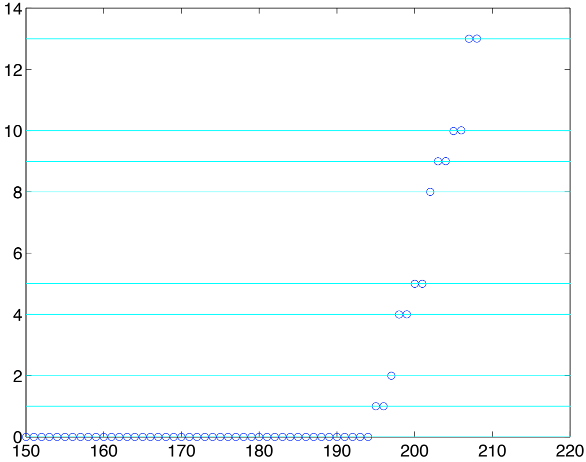
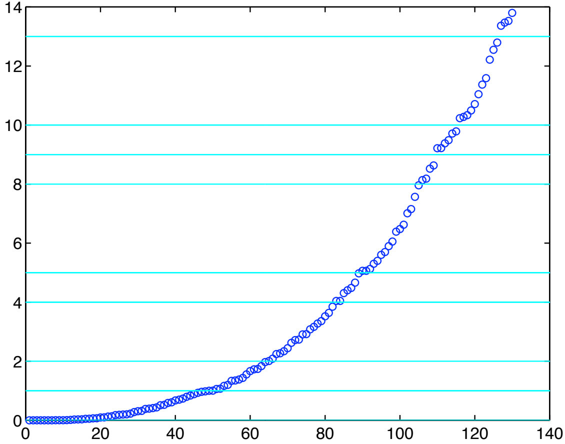
Good approximation: Triangular edge elements (15 nodes per side, $\mathbb{P}_1$)



Eigenvalue ω_k^2 vs. rank k

Approximation on the square $\Omega = (0, \pi) \times (0, \pi)$, $s = 0$

Bad approximation: Nodal triangular elements (15 nodes per side, $\mathbb{P}_1$)



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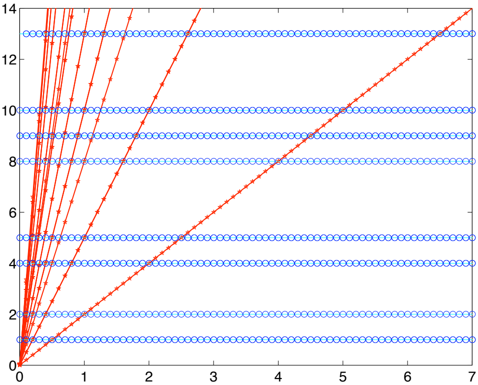
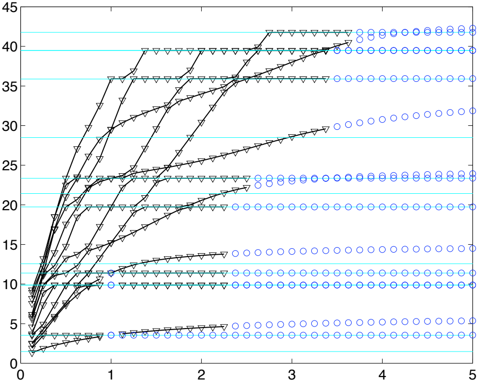


$\omega[s]^2$ vs. s

Blue circles: computed $\omega[s]^2$ with **curl**-dominant eigenfunctions.
Red stars: computed $\omega[s]^2$ with div-dominant eigenfunctions.
div $\boldsymbol{E}$ satisfies $\quad s\Delta \operatorname{div} \boldsymbol{E} = \omega^2 \operatorname{div} \boldsymbol{E}$
Extra eigenvalues: s times Dirichlet eigenvalues.

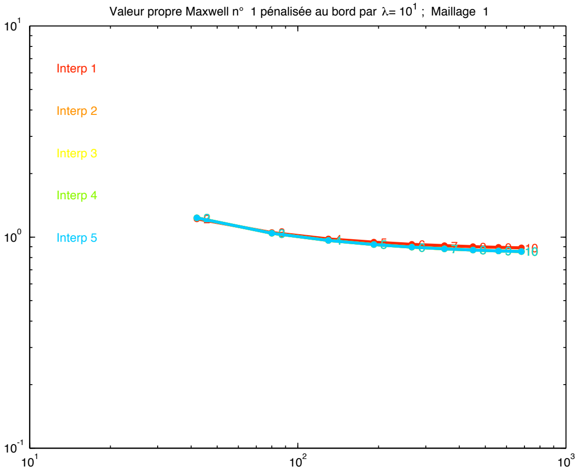
Regularized formulation in the "L"



$\omega[s]^2$ vs. s

Gray triangles: computed $\omega[s]^2$ with indifferent eigenfunctions.
Cyan-Lines: true Maxwell eigenvalues

Error of the first eigenvalue

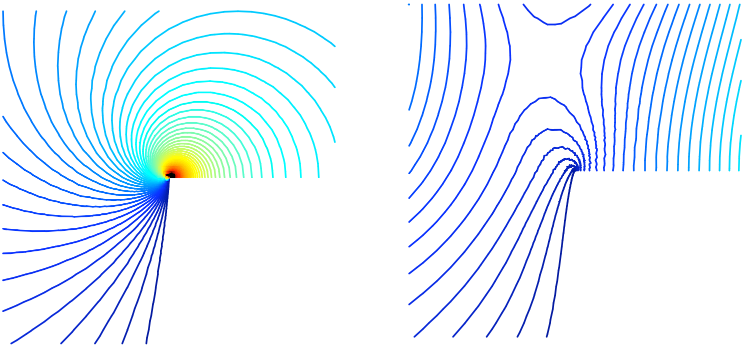


Error remains larger than 90%.

Error vs. number of d.o.f.

Exact solution
(2nd component $E_2 = r^{-\frac{1}{3}} \cos\frac{\theta}{3}$)).

Computation with $\mathbb{Q}_3$ elements.



$$\textbf{curl}\,\textbf{curl}\,\boldsymbol{E} - \nabla \operatorname{div} \boldsymbol{E} = 0 \quad \text{in } \Omega; \qquad \boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{E}_0 \quad \text{on } \partial\Omega$$

1991: M. COSTABEL
A coercive bilinear form for Maxwell's equations
J. Math. Anal. Appl. 157 (1991) 527–541.

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Numerical simulation of corner singularities: a paradox in Maxwell-like problems
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Weighted regularization of Maxwell equations in polyhedral domains. A rehabilitation of nodal finite elements
Numer. Math. 93 (2002) 239–277.

2009: A. BUFFA, P. CIARLET JR., E. JAMELOT
Solving electromagnetic eigenvalue problems in polyhedral domains with nodal finite elements
Numer. Math. 113 (2009) 497–518.

2011: A. BONITO, J.-L. GUERMOND
Approximation of the Eigenvalue Problem for Time Harmonic Maxwell System by Continuous Lagrange Finite Elements
Math. Comp. 80 (2011) 1887–1910.

An integration by parts formula (Co 1991)

Let $\Omega \subset \mathbb{R}^3$ be a polyhedron. Let $\boldsymbol{u} \in \boldsymbol{X}_N$. If $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, then

$$\int_\Omega |\nabla \boldsymbol{u}|^2 = \int_\Omega |\mathbf{curl}\,\boldsymbol{u}|^2 + \int_\Omega |\operatorname{div} \boldsymbol{u}|^2$$

Corollary

1. Define $\boldsymbol{H}_N = \boldsymbol{X}_N \cap \boldsymbol{H}^1(\Omega)$. Then $\boldsymbol{H}_N$ is a closed subspace of $\boldsymbol{X}_N$. If Ω is non-convex, then $\boldsymbol{H}_N \neq \boldsymbol{X}_N$.
2. For $s > 0$, the sesquilinear form

$$a_s(\boldsymbol{E}, \boldsymbol{F}) = \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{F} + s \int_\Omega \operatorname{div} \boldsymbol{E} \operatorname{div} \boldsymbol{F}$$

is $\boldsymbol{H}_N$-elliptic.

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Two Regularized Variational Formulations

Regularized formulation: $\boldsymbol{E} \in \boldsymbol{X}_N \ : \ \forall \boldsymbol{F} \in \boldsymbol{X}_N \ :$

$$(\mathrm{Reg}_{\boldsymbol{X}}) \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{F} + s \int_\Omega \operatorname{div} \boldsymbol{E} \operatorname{div} \boldsymbol{F} - \omega^2 \int_\Omega \boldsymbol{E} \cdot \boldsymbol{F} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{F}$$

Regularized formulation: $\boldsymbol{E} \in \boldsymbol{H}_N \ : \ \forall \boldsymbol{F} \in \boldsymbol{H}_N \ :$

$$(\mathrm{Reg}_{\boldsymbol{H}}) \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{E} \cdot \mathbf{curl}\,\boldsymbol{F} + s \int_\Omega \operatorname{div} \boldsymbol{E} \operatorname{div} \boldsymbol{F} - \omega^2 \int_\Omega \boldsymbol{E} \cdot \boldsymbol{F} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{F}$$

Consequence (For $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\Omega \subset \mathbb{R}^3$ polyhedron)

1. If ω^2 is not an eigenvalue, then both $(\mathrm{Reg}_{\boldsymbol{X}})$ and $(\mathrm{Reg}_{\boldsymbol{H}})$ have a unique solution.
2. Both are solutions of the boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} - s\nabla \operatorname{div} \boldsymbol{E} - \omega^2 \boldsymbol{E} = \boldsymbol{J} \quad \text{in } \Omega; \qquad \boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega$$

3. If Ω is non-convex, then the two solutions are different, in general. The eigenvalues of $(\mathrm{Reg}_{\boldsymbol{X}})$ and $(\mathrm{Reg}_{\boldsymbol{H}})$ are different, in general.

The role of the divergence

In the regularized variational formulation

$$(\text{Reg}_{\boldsymbol{H}}) \qquad \int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{F} + s\int_\Omega \operatorname{div}\boldsymbol{E}\operatorname{div}\boldsymbol{F} - \omega^2\int_\Omega \boldsymbol{E}\cdot\boldsymbol{F} = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{F}$$

one gets an equation for $\operatorname{div}\boldsymbol{E}$ by testing with gradients: $\boldsymbol{F} = \nabla\psi$.

$$s\int_\Omega \operatorname{div}\boldsymbol{E}\,\Delta\psi - \omega^2\int_\Omega \boldsymbol{E}\cdot\nabla\psi = \int_\Omega \boldsymbol{J}\cdot\nabla\psi$$

$$\iff \int_\Omega \operatorname{div}\boldsymbol{E}\,\big(s\Delta\psi + \omega^2\psi\big) = -\int_\Omega \operatorname{div}\boldsymbol{J}\,\psi$$

For $(\text{Reg}_{\boldsymbol{X}})$ one takes $\psi \in H^1_0(\Delta,\Omega)$.
For $(\text{Reg}_{\boldsymbol{H}})$ one takes $\psi \in H^1_0(\Omega)\cap H^2(\Omega)$.

Lemma

Let $\operatorname{div}\boldsymbol{J} = 0$ and $\frac{\omega^2}{s}$ not a Dirichlet eigenvalue.
For a solution $\boldsymbol{E}$ of $(\text{Reg}_{\boldsymbol{H}})$ there holds:
If $\operatorname{div}\boldsymbol{E} \in H^1(\Omega)$, then $\operatorname{div}\boldsymbol{E} = 0$,
and then $\boldsymbol{E}$ is a solution of $(\text{Reg}_{\boldsymbol{X}})$ and hence of the Maxwell problem.

Non-density of smooth functions

The space $\boldsymbol{H}_N$ is the completion of

$$\{\boldsymbol{u} \in \boldsymbol{C}^\infty(\overline{\Omega}) \mid \boldsymbol{u} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega\}$$

under the norm of $\boldsymbol{H}_0(\mathbf{curl}) \cap \boldsymbol{H}(\mathrm{div})$

$$\|\boldsymbol{u}\|^2_{\boldsymbol{X}} = \int_\Omega |\mathbf{curl}\,\boldsymbol{u}|^2 + \int_\Omega |\mathrm{div}\,\boldsymbol{u}|^2$$

$\boldsymbol{H}_N \neq \boldsymbol{X}_N$ means that smooth functions are not dense in $\boldsymbol{X}_N$. Finite element functions, which are piecewise polynomials on a triangulation of Ω, belong to $\boldsymbol{H}_N$ as soon as they belong to $\boldsymbol{X}_N$.

Consequence

Any Maxwell solution or eigenfunction that does not belong to $\boldsymbol{H}^1(\Omega)$ cannot be approximated by an $\boldsymbol{X}_N$ -conforming finite element method that uses the regularized variational formulation.

The Weighted Regularization Method (Co & Dauge 2002)

Idee : Replace the $\boldsymbol{L}^2$ norm in the regularizing term $s\int_\Omega \operatorname{div} \boldsymbol{E} \operatorname{div} \boldsymbol{F}$ by a weighted $\boldsymbol{L}^2$ norm.

Weighted Regularized formulation: $\boldsymbol{E} \in \boldsymbol{X}_N^w \ : \ \forall \boldsymbol{F} \in \boldsymbol{X}_N^w \ :$

$$(\mathrm{Reg}_{\boldsymbol{H}}^w) \qquad \int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \boldsymbol{F} + s \int_\Omega w \operatorname{div} \boldsymbol{E} \operatorname{div} \boldsymbol{F} - \omega^2 \int_\Omega \boldsymbol{E} \cdot \boldsymbol{F} = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{F}$$

Definition : $\boldsymbol{X}_N^w = \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \mid \int_\Omega w |\operatorname{div} \boldsymbol{u}|^2 < \infty\}$

We choose

$$w(x) = \big(\mathrm{dist}(x, S)\big)^\alpha$$

where S is the set of singular points (edges, corners) on the boundary.

Lemma

There exists $\alpha_0(\Omega) < 2$ such that
For $\alpha_0(\Omega) < \alpha < 2$: Smooth functions are dense in $\boldsymbol{X}_N^w$

Theorem

For $\alpha_0(\Omega) < \alpha < 2$:

If $\dfrac{\omega^2}{s}$ is not an eigenvalue of the Dirichlet problem for the

weighted Laplacian $\quad \operatorname{div} w\nabla$

then the solutions $\boldsymbol{E}$ of the weighted regularized problem $(\mathrm{Reg}_{\boldsymbol{H}}^{w})$ satisfy $\operatorname{div}\boldsymbol{E} = 0$ and are solutions of the Maxwell problem.

Numerical evidence follows...

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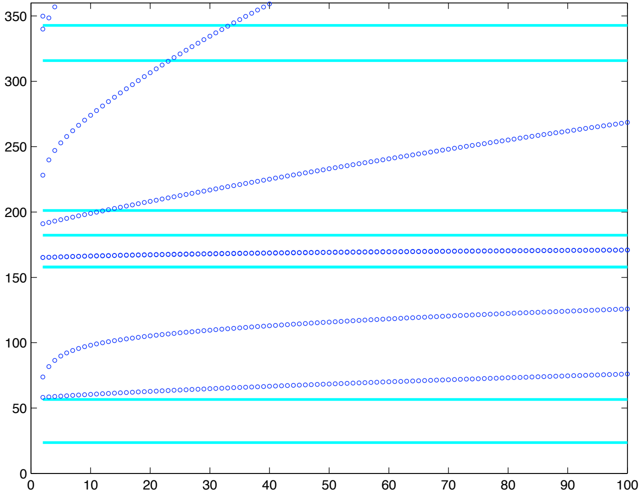
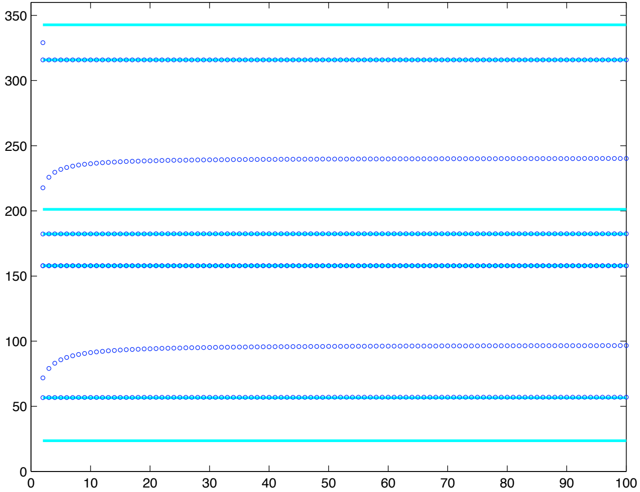
$\alpha = 0$

Degree $q = 1$



$\alpha = 0$

Degree $q = 4$



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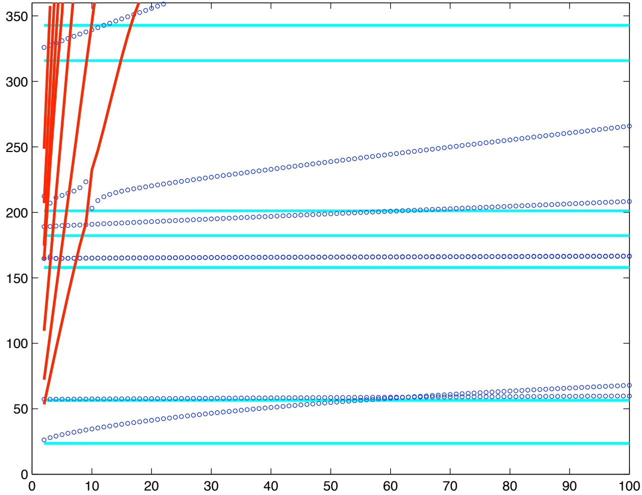

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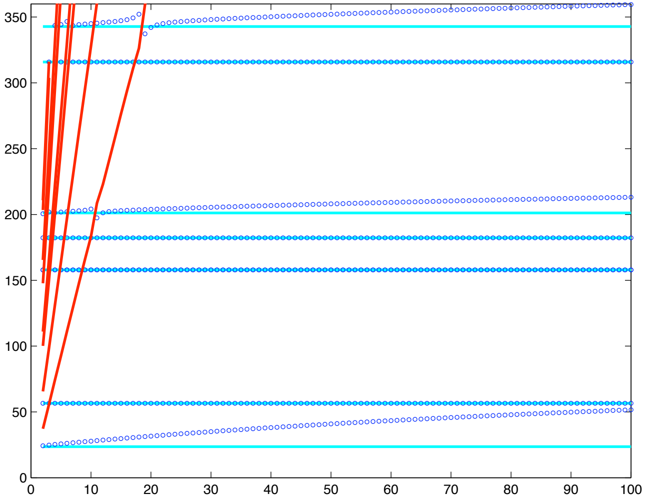
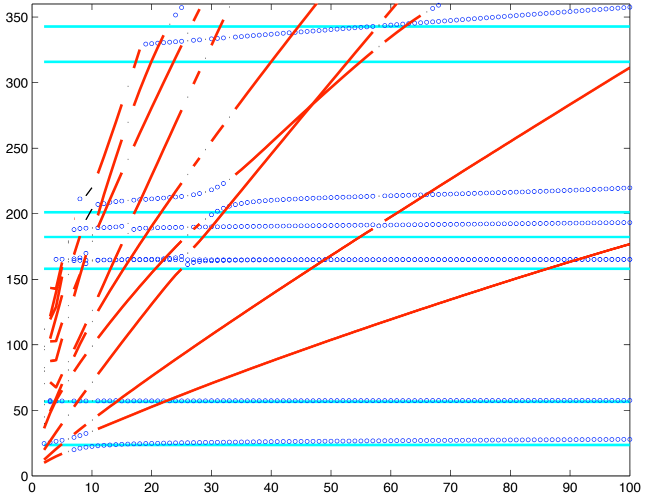


$\alpha = 2$

Degree $q = 1$



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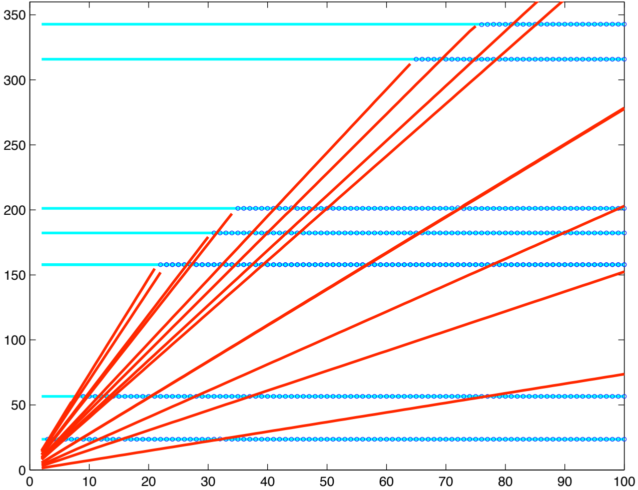
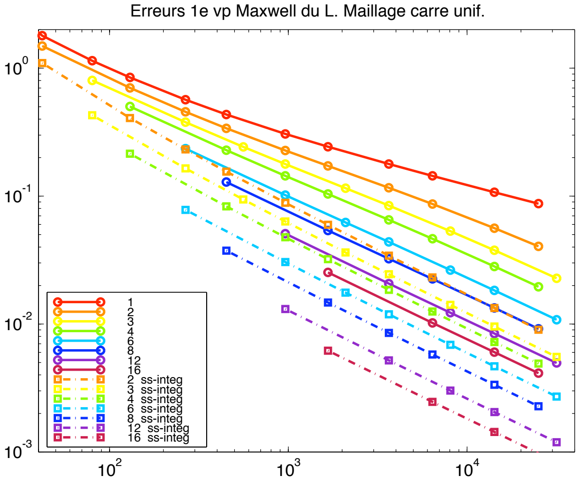
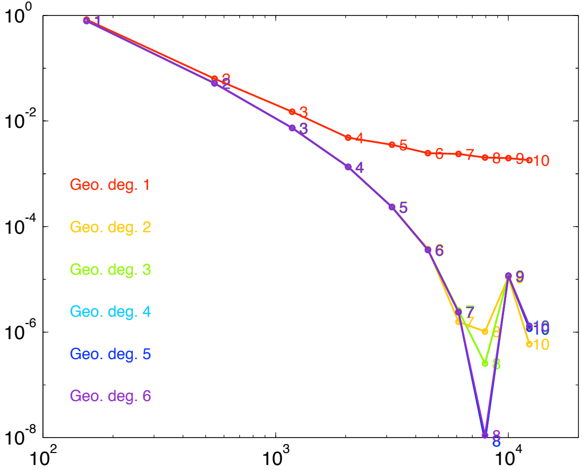
Degree $q = 4$





Uniform grids

Error vs. number of d.o.f.



Error vs. number of d.o.f.

hp version
Curved L

[Buffa-Jamelot-Ciarlet 2009] $\boldsymbol{E} \in \boldsymbol{X}_N^w,\ p \in L^{2,w}\ :\ \forall \boldsymbol{F} \in \boldsymbol{X}_N^w,\ q \in L^{2,w}\ :$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{F} + s\int_\Omega w\,\operatorname{div}\boldsymbol{E}\,\operatorname{div}\boldsymbol{F} - \omega^2\int_\Omega \boldsymbol{E}\cdot\boldsymbol{F} + \int_\Omega w\,p\operatorname{div}\boldsymbol{F} = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{F}$$

$$\int_\Omega w\,q\operatorname{div}\boldsymbol{E} = 0$$

Recent developments: Mixed formulations

[Buffa-Jamelot-Ciarlet 2009] $\boldsymbol{E} \in \boldsymbol{X}_N^w,\ p \in L^{2,w}\ :\ \forall \boldsymbol{F} \in \boldsymbol{X}_N^w,\ q \in L^{2,w}\ :$

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$$\int_\Omega w\,q\,\mathrm{div}\,\boldsymbol{E} = 0$$

[Bonito-Guermont 2011] $\boldsymbol{E} \in \boldsymbol{X}^{-\alpha},\ p \in H_0^1\ :\ \forall \boldsymbol{F} \in \boldsymbol{X}^{-\alpha},\ q \in H_0^1\ :$

$$\int_\Omega \mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{F} + \big(\mathrm{div}\,\boldsymbol{E}, \mathrm{div}\,\boldsymbol{F}\big)_{-\alpha} - \omega^2\int_\Omega \boldsymbol{E}\cdot\boldsymbol{F} + \int_\Omega \nabla p\cdot\boldsymbol{F} = \int_\Omega \boldsymbol{J}\cdot\boldsymbol{F}$$

$$\int_\Omega \nabla q\cdot\boldsymbol{E} + \big(p, q\big)_{\alpha-1} = 0$$

Here $\frac{1}{2} < \alpha < 1$.
Discretization of $H^{-\alpha}(\Omega)$ scalar product:

$$\big(p_h, q_h\big)_{-\alpha} \sim h^{2\alpha}\int_\Omega p_h\,q_h$$

The Electrical Field Integral Equation

Time-harmonic Maxwell transmission problem:

$$
\begin{array}{rclrcll}
\mathbf{curl}\,\boldsymbol{E} & = & i\omega \boldsymbol{H}\,; & \mathbf{curl}\,\boldsymbol{H} & = & -i\omega\boldsymbol{E} & \text{in } \mathbb{R}^3 \setminus \Gamma \\
[\boldsymbol{E}\times\boldsymbol{n}]_\Gamma & = & 0\,; & [\boldsymbol{n}\cdot\boldsymbol{H}]_\Gamma & = & 0 & \text{on } \Gamma \\
i\omega[\boldsymbol{H}\times\boldsymbol{n}]_\Gamma & = & \boldsymbol{j}\,; & [\boldsymbol{n}\cdot\boldsymbol{E}]_\Gamma & = & m & \text{on } \Gamma
\end{array}
$$

$+$ radiation condition

Representation formula, with $G_\omega(x) = \dfrac{e^{i\omega|x|}}{4\pi|x|}$:

$$\boldsymbol{E}(x) = \int_\Gamma G_\omega(x-y)\boldsymbol{j}(y)\,ds(y) + \nabla \int_\Gamma G_\omega(x-y)m(y)\,ds(y)$$

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Continuity condition $\operatorname{div}_\Gamma \boldsymbol{j} - \omega^2 m = 0$ & tangential trace on Γ:

(EFIE) $$\boxed{\left(V_\omega \boldsymbol{j}\right)_T + \frac{1}{\omega^2}\nabla_T V_\omega \operatorname{div}_\Gamma \boldsymbol{j} = \boldsymbol{j}_0}$$

Single layer potential $V_\omega m(x) = \int_\Gamma G_\omega(x-y)m(y)\,ds(y)$

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The EFIE: Principal properties

The EFIE on Γ

(EFIE) $$\big(V_\omega \boldsymbol{j}\big)_\top + \frac{1}{\omega^2}\nabla_\top V_\omega \operatorname{div}_\Gamma \boldsymbol{j} = \boldsymbol{j}_0$$

Lemma (Nedelec, second half of 20th century)

If ω^2 is not a Dirichlet eigenvalue in Ω, then

$$C_\omega : \boldsymbol{j} \mapsto \big(V_\omega \boldsymbol{j}\big)_\top + \frac{1}{\omega^2}\nabla_\top V_\omega \operatorname{div}_\Gamma \boldsymbol{j}$$

is an isomorphism between $\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ and its dual space. One has

$$\big(\boldsymbol{n} \times C_\omega\big)^2 = \frac{1}{4}\mathbb{I} - M_\omega^2$$

where M_ω is a compact operator in $\boldsymbol{H}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ if Γ is smooth.

Good: (EFIE) is given by a non-degenerate quadratic form

Big problem : The principal part of C_ω is indefinite.
No strong ellipticity in the sense of pseudodifferential operators, no convergence of arbitrary Galerkin methods.

Two ways out, as before:

1. Construct special finite elements, and prove a generalized strong ellipticity property, or
2. Regularize

We describe the regularization method introduced by MacCamy–Stephan.

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Write the EFIE as a system

$$\big(V_\omega \boldsymbol{j}\big)_\top + \nabla_\top V_\omega m = \boldsymbol{j}_0$$
$$V_\omega \operatorname{div}_\Gamma \boldsymbol{j} - \omega^2 V_\omega m = 0$$

This is still essentially indefinite.

The regularized EFIE (MacCamy & Stephan 1982)

Write the EFIE as a system

$$\big(V_\omega \boldsymbol{j}\big)_\top + \nabla_\top V_\omega m = \boldsymbol{j}_0$$
$$V_\omega \operatorname{div}_\Gamma \boldsymbol{j} - \omega^2 V_\omega m = 0$$

This is still essentially indefinite.
Now multiply the first equation by $\operatorname{div}_\Gamma$ and subtract:

$$(\mathrm{EFIE}_{\mathrm{reg}}) \qquad \begin{aligned} \big(V_\omega \boldsymbol{j}\big)_\top + \nabla_\top V_\omega m &= \boldsymbol{j}_0 \\ K_\omega \boldsymbol{j} - (\Delta_\Gamma + \omega^2) V_\omega m &= -\operatorname{div}_\Gamma \boldsymbol{j}_0 \end{aligned}$$

Here Δ_Γ is the Laplace-Beltrami operator, and
$K_\omega = V_\omega \operatorname{div}_\Gamma - \operatorname{div}_\Gamma V_\omega$ is an operator of order -1 if Γ is smooth.

Theorem (MacCamy–Stephan)

The system ($\mathrm{EFIE}_{\mathrm{reg}}$) is a strongly elliptic system of pseudodifferential operators. It defines a Fredholm operator of index 0

$$\boldsymbol{H}_\top^{s-\frac{1}{2}}(\Gamma) \times H^{s+\frac{1}{2}}(\Gamma) \to \boldsymbol{H}_\top^{s+\frac{1}{2}}(\Gamma) \times H^{s-\frac{1}{2}}(\Gamma) \qquad \forall s \in \mathbb{R}$$

Any Galerkin scheme for its approximation is stable in $\boldsymbol{H}_\top^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$.

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Another Tale of Two Function Spaces

We consider scattering by an open surface Γ (Screen problem)

1. Given $\boldsymbol{j}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\top, \Gamma)$, then for any $\omega > 0$ the Maxwell scattering problem and (EFIE) each have a unique solution

$$\boldsymbol{E} \in \boldsymbol{H}_{\rm loc}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma) + \text{ radiation condition}$$

$$\boldsymbol{j} \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

2. If $\boldsymbol{j}_0 \in \boldsymbol{H}_\top^{\frac{1}{2}}(\Gamma)$, then also $(\mathrm{EFIE}_{\rm reg})$ has a unique solution

$$(\boldsymbol{j}, m) \in \widetilde{\boldsymbol{H}}^{-\frac{1}{2}}(\Gamma) \times \widetilde{H}^{\frac{1}{2}}(\Gamma)$$

3. The solution of $(\mathrm{EFIE}_{\rm reg})$ can be approximated by any conforming finite element method [Stephan 1984, Heuer 1996].

Trap : The solutions $\boldsymbol{j}$ of (EFIE) and of $(\mathrm{EFIE}_{\rm reg})$ are different, in general, and the Maxwell solution does not satisfy $\mathrm{div}_\Gamma \boldsymbol{j} = \omega^2 m \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$.

Edge singularity: $m \sim r^{-\frac{1}{2}} \notin L^2(\Gamma)$

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  On the essential spectrum of the volume integral operator in electromagnetic scattering
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Electromagnetic scattering by a penetrable homogeneous object

$\mu = \mu_r$ in Ω, $\mu = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\varepsilon = \varepsilon_r$ in Ω, $\varepsilon = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.
Maxwell equations

$$\mathbf{curl}\, \boldsymbol{E} = i\omega\mu \boldsymbol{H}\,; \qquad \mathbf{curl}\, \boldsymbol{H} = -i\omega\varepsilon \boldsymbol{E} + \boldsymbol{J}$$

hold in $\mathbb{R}^3$ in the distributional sense (+ radiation condition).
supp $\boldsymbol{J}$ compact in $\mathbb{R}^3 \setminus \overline{\Omega}$.
$\Longrightarrow$ Transmission conditions on $\Gamma = \partial\Omega$:

$$\begin{aligned} [\boldsymbol{E} \times \boldsymbol{n}]_\Gamma &= 0\,; & [\boldsymbol{n} \cdot \mu \boldsymbol{H}]_\Gamma &= 0 \\ [\boldsymbol{H} \times \boldsymbol{n}]_\Gamma &= 0\,; & [\boldsymbol{n} \cdot \varepsilon \boldsymbol{E}]_\Gamma &= 0 \end{aligned}$$

Lippmann-Schwinger equation: One considers the obstacle as a perturbation: $\mathbf{curl}\, \frac{1}{\mu}\, \mathbf{curl}\, \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = i\omega \boldsymbol{J} \Leftrightarrow$

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{E} - \omega^2 \boldsymbol{E} = i\omega \boldsymbol{J} - \omega^2 p \boldsymbol{E} + \mathbf{curl}\, q\, \mathbf{curl}\, \boldsymbol{E}$$

with $p = (1 - \varepsilon_r)\chi_\Omega$, $q = (1 - \frac{1}{\mu_r})\chi_\Omega$.
The right-hand side has compact support: Convolution with fundamental solution of $\mathbf{curl}\,\mathbf{curl} - \omega^2$:

$$g_\omega = \left(\frac{1}{\omega^2}\nabla \operatorname{div} + 1\right) G_\omega\,; \qquad G_\omega(x) = \frac{e^{i\omega|x|}}{4\pi|x|}$$

$\mu = \mu_r$ in Ω, $\mu = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$, $\varepsilon = \varepsilon_r$ in Ω, $\varepsilon = 1$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.
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$$g_\omega = \Big(\frac{1}{\omega^2} \nabla \operatorname{div} + 1\Big) G_\omega\,; \qquad G_\omega(x) = \frac{e^{i\omega|x|}}{4\pi|x|}$$

Representation of $\boldsymbol{E}$ in $\mathbb{R}^3$ by volume integrals over Ω

$$\boldsymbol{E} = \omega^2 g_\omega * (p\boldsymbol{E}) + g_\omega * (\mathbf{curl}\, q\, \mathbf{curl}\, \boldsymbol{E}) + \boldsymbol{E}^{\mathrm{inc}}$$

$\Longrightarrow$ Volume integral equation in Ω

$$\boxed{\boldsymbol{E} = pA_\omega \boldsymbol{E} + qB_\omega \boldsymbol{E} + \boldsymbol{E}^{\mathrm{inc}}}$$

with

$$A_\omega \boldsymbol{E}(x) = -\nabla \operatorname{div} \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy - \omega^2 \int_\Omega G_\omega(x-y)\boldsymbol{E}(y)\,dy$$

$$B_\omega \boldsymbol{E}(x) = \mathbf{curl} \int_\Omega G_\omega(x-y)\,\mathbf{curl}\,\boldsymbol{E}(y)\,dy$$

Volume integral equation: $\boldsymbol{E} - pA_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$

Results (Co & E. Darrigrand & E.H. Koné 2009)

1. The operator A_ω can be extended to $\boldsymbol{L}^2(\Omega)$ as a bounded operator.
2. It has $\boldsymbol{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$ as invariant subspaces.
3. For $\boldsymbol{E}^{\mathrm{inc}}$ in $\boldsymbol{H}(\mathbf{curl},\Omega) \cap \boldsymbol{H}(\mathrm{div},\Omega)$, the integral equation in $\boldsymbol{L}^2$ has the same solutions as in $\boldsymbol{H}(\mathbf{curl},\Omega)$ or in $\boldsymbol{H}(\mathrm{div},\Omega)$.
4. $A_\omega - A_0$ is compact in $\boldsymbol{L}^2(\Omega)$.
5. $\mathrm{Sp}(A_0) \subset [0,1]$
   0 and 1 are eigenvalues of infinite multiplicity of A_0
   $\frac{1}{2}$ is accumulation point of eigenvalues, and
   $\mathrm{Sp}_{\mathrm{ess}}(A_0) = \{0, \frac{1}{2}, 1\}$ if Γ is smooth

The dielectric scattering problem can be solved by solving the volume integral equation in $\boldsymbol{L}^2(\Omega)$.

Volume integral equation: $\boldsymbol{E} - qB_\omega \boldsymbol{E} = \boldsymbol{E}^{\rm inc}$

The integral operator

$$B_\omega \boldsymbol{E}(x) = \mathbf{curl} \int_\Omega G_\omega(x-y)\,\mathbf{curl}\,\boldsymbol{E}(y)\,dy$$

is bounded from $\boldsymbol{H}(\mathbf{curl},\Omega)$ to itself and to $\boldsymbol{H}(\mathrm{div}\,0,\Omega)$.

For $\boldsymbol{E} \in C_0^\infty(\Omega)$ one has

$$\begin{aligned} B_\omega \boldsymbol{E} &= \mathbf{curl}\, G_\omega * \mathbf{curl}\,\boldsymbol{E} = \mathbf{curl}\,\mathbf{curl}\, G_\omega * \boldsymbol{E} \\ &= \nabla\,\mathrm{div}\, G_\omega * \boldsymbol{E} - (\Delta + \omega^2) G_\omega * \boldsymbol{E} + \omega^2 G_\omega * \boldsymbol{E} \\ &= \boldsymbol{E} - A_\omega \boldsymbol{E} \end{aligned}$$

Proposition

The operator B_ω can be extended to $\boldsymbol{L}^2(\Omega)$ as a bounded operator.

Trap alert !

Volume integral equation: $\boldsymbol{E} - q B_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$

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Proposition (Kirsch & Lechleiter 2010)

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Proposition (Kirsch & Lechleiter 2010)

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Trap alert !

A Third Tale of Two Function Spaces

Theorem 1

Solving the volume integral equation

$$\boldsymbol{E} - q B_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is equivalent to the magnetic Maxwell scattering problem.

Theorem 2

Let $\widehat{B}_\omega : \boldsymbol{L}^2(\Omega) \to \boldsymbol{L}^2(\Omega)$ be the extended operator. Solving the volume integral equation

$$\boldsymbol{E} - q \widehat{B}_\omega \boldsymbol{E} = \boldsymbol{E}^{\mathrm{inc}}$$

in $\boldsymbol{L}^2(\Omega)$ gives the Maxwell equations in $\mathbb{R}^3 \setminus \Gamma$ with the transmission conditions

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A Third Tale of Two Function Spaces

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Solving the volume integral equation

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in $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is equivalent to the magnetic Maxwell scattering problem.

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Explanation : For $\boldsymbol{E} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$, one has

$$\widehat{B}_\omega \boldsymbol{E} = B_\omega \boldsymbol{E} + \mathbf{curl} \int_\Gamma G_\omega(x-y) \boldsymbol{E}(y) \times \boldsymbol{n}(y) \, ds(y)$$

The latter term does not have a continuous extension to $\boldsymbol{L}^2(\Omega)$.

Proposition 1

The operator B_ω cannot be extended from $\boldsymbol{H}(\mathbf{curl}, \Omega)$ to $\boldsymbol{L}^2(\Omega)$ as a bounded operator.

Proposition 2

Although on $C_0^\infty(\Omega)$ we have

$$B_\omega = \mathbb{I} - A_\omega,$$

the commutator of A_ω and B_ω on $\boldsymbol{H}(\mathbf{curl}, \Omega)$ is not compact.

$\Longrightarrow$ No joint essential spectrum of A_ω and B_ω in the Lippmann-Schwinger operator

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Thank you for your attention!