ENUMATH ISCHIA 26/07/2001

A Wavelet Approximation Method for Antenna Problems

Martin Costabel & Cyril Safa

Institut de Recherche MAthématique de Rennes

http://www.maths.univ-rennes1.fr/~costabel

Outline

A simple wavelet-based fast boundary element method

for time-harmonic Maxwell scattering from open surfaces in $\mathbb{R}^3$

- The Electrical Field Integral Equation:

— Quasidiagonalization via Hodge decomposition

— Reformulations suitable for nodal finite element discretizations

— The $(p, \phi, m, \lambda, \alpha)$ -formulation

- Nodal wavelet discretization:

— Construction of a nodal wavelet basis

— Stability, matrix compression, preconditioning and all that

— Some problems with open surfaces

An specimen from electronics

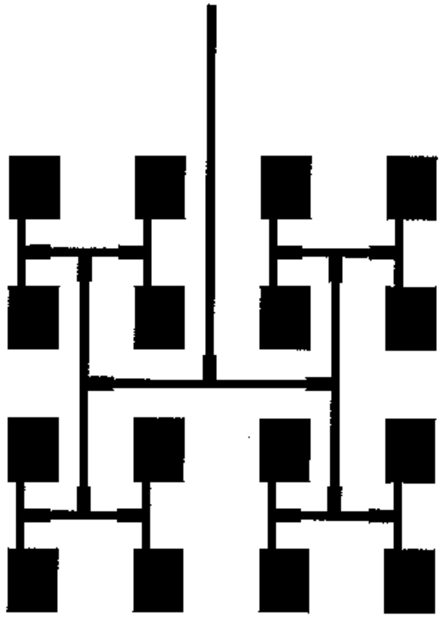


FIG. 2.28: Masque du réseau planaire à 16 éléments.

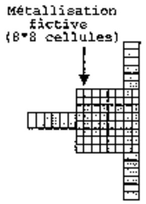


FIG. 2.29: Détail de description de la structure ; la ligne quart d'onde.

A small

antenna array

The geometry

Open surface with p/w smooth boundary $\Gamma_0 \subset \Gamma$ closed smooth surface $\subset \mathbb{R}^3$

Simple topologies

Sobolev spaces: $H^s(\Gamma_0) = H^s(\Gamma)\big|_{\Gamma_0}$

$$\widetilde{H}^s(\Gamma_0) = \{u \in H^s(\Gamma) \mid \operatorname{supp} u \subset \overline{\Gamma_0}\}$$

Surface differential operators:

$$\begin{aligned}
\mathbf{grad} &= \operatorname{grad}_T = -n \times (n \times \operatorname{grad}) \\
\mathbf{div} &= \operatorname{div}_\Gamma = -(\operatorname{grad}_T)^* \\
\mathbf{curl} &= \operatorname{curl}_\Gamma = -n \times \operatorname{grad} \\
\mathbf{rot} &= (\operatorname{curl}_\Gamma)^* \\
\boldsymbol{\Delta} &= \Delta_\Gamma = \operatorname{grad}_T \operatorname{div}_\Gamma = -\,\mathbf{rot}\,\mathbf{curl}
\end{aligned}$$

Some spaces on perfectly conducting boundaries:

$$\begin{aligned}
\widetilde{H}^{-1/2}_{\mathrm{div}} &= \{u \in \widetilde{H}^{-1/2}(\Gamma_0) \mid \mathbf{div}\, u \in \widetilde{H}^{-1/2}(\Gamma_0);\ u \cdot n = 0\} \\
H^{-1/2}_{\mathrm{curl}} &= \{u \in H^{-1/2}(\Gamma_0) \mid \mathbf{rot}\, u \in H^{-1/2}(\Gamma_0);\ u \cdot n = 0\} \\
H^1_{\boldsymbol{\Delta}} &= \{u \in H^1(\Gamma_0) \mid \boldsymbol{\Delta} u \in H^{-1/2}(\Gamma_0)\}
\end{aligned}$$

The Electrical Field Integral Equation

(Maxwell)
$$\begin{cases} \operatorname{curl} E - i\omega\mu H = 0 \quad , \quad \operatorname{curl} H + i\omega\varepsilon E = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Gamma_0}, \\ E \times n = -E^{\text{in}} \times n \quad \text{on} \quad \Gamma_0, \quad \text{Silver-Müller r.c.} \end{cases}$$

Single layer potential

$$Vu(x) = \int_{\Gamma_0} \frac{e^{ik|x-y|}}{4\pi|x-y|} u(y)\, ds(y) \;, \qquad k = \omega\sqrt{\varepsilon\mu}$$

Representation formula in $\mathbb{R}^3 \setminus \Gamma_0$

$$E = \frac{i}{\omega\varepsilon} \operatorname{grad} V \operatorname{div}_\Gamma u + i\omega\mu V u \;, \quad u = [H \times n]_{\Gamma_0} \in \widetilde{H}^{-1/2}_{\text{div}}, \text{ surface current}$$

Integral equation on Γ_0 (**EFIE**)

$$\boxed{\operatorname{grad} V \operatorname{div} u + k^2 V_T u = f} = i\omega\varepsilon E^{\text{in}}_T \;, \quad V_T = \pi_T V = -n \times (n \times V\cdot)$$

Hodge decomposition and quasi-diagonalization

Problem with EFIE: Wrong sign of k^2 !

V **: strongly elliptic (pos. def. + compact) pseudodifferential operator of order -1**

$$L = \operatorname{grad} V \operatorname{div} + k^2 V_T \Longrightarrow (Lu, v) = -(V \operatorname{div} u, \operatorname{div} v) + k^2 (V u, v)$$

Hodge decomposition in $\widetilde{H}^{-1/2}_{\mathrm{div}} = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma_0) = X^1 \oplus X^2$

$$\boxed{u = \operatorname{grad} p + \operatorname{curl} \phi}\ , \quad p \in H^1_\Delta\ ,\ \Delta^{\mathrm{Neu}} p = \operatorname{div} u\ , \quad \phi \in \widetilde{H}^{1/2}(\Gamma_0)$$

Hodge decomposition in the dual space $H^{-1/2}_{\mathrm{curl}} = H^{-1/2}(\mathrm{rot}_\Gamma, \Gamma_0)$

$$\boxed{u' = \operatorname{grad} p' + \operatorname{curl} \phi'}\ , \quad \phi' \in \widetilde{H}^1_\Delta\ ,\ \Delta^{\mathrm{Dir}} \phi' = \operatorname{rot} u'\ , \quad p' \in H^{1/2}(\Gamma_0)$$

(u^1, u^2) $\qquad L \sim \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$ **with L_{12}, L_{21} compact and L_{22} and $-L_{11}$ strongly elliptic.**

- **All is well for Galerkin methods: Stability, convergence etc...**

(p,ϕ) -formulation

Problem for finite element approximations: Discrete Hodge decomposition?

- **Exact H. d.:** (p,ϕ) **-formulation**
- **Approximate H. d.:** H_{div} **elements [Bendali ’82, Hiptmair-Schwab ’01]**
- **Approximate H. d.: Approximate or mixed**
  (p,ϕ) **-formulation [Safa, Buffa-Co.-Schwab ’01]**

(p,ϕ)

$$L \sim \begin{pmatrix} -\Delta V \Delta - k^2 \,\mathrm{div}\, V \,\mathrm{grad} & -k^2 \,\mathrm{div}\, V \,\mathrm{curl} \\ k^2 \,\mathrm{rot}\, V \,\mathrm{grad} & k^2 \,\mathrm{rot}\, V \,\mathrm{curl} \end{pmatrix}$$

Orders: $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **Problem: Energy space** $H^{3/2} \times \widetilde{H}^{1/2}$**:** C^1 **elements!**

Perturbed (m,ϕ) -formulation

Introduce the surface charge density $m = \operatorname{div} u \in \widetilde{H}^{-1/2}$, so that

(p,m) $$p = (\Delta^{\rm Neu})^{-1} m \in H^1_\Delta/\mathbb{C}$$

This would lead to a (m,ϕ) formulation containing the operator $(\Delta^{\rm Neu})^{-1}$.

Nothing gained so far.

Let $\Lambda_J \subset H^1(\Gamma_0)$ be a finite element space and

$p_J \in \Lambda_J$ be defined by the variational solution of (p-m):

(p_J,m) $$p_J = (\Delta_J^{\rm Neu})^{-1} m = \Pi_J (\Delta^{\rm Neu})^{-1} m \in \Lambda_J$$

We obtain the perturbed (m,ϕ) -formulation, a nonconforming (p,ϕ) -formulation:

p-(m,ϕ) $$L \sim \begin{pmatrix} \Delta V \Delta + k^2 (\Delta_J^{\rm Neu})^{-1} \operatorname{div} V \operatorname{grad} (\Delta_J^{\rm Neu})^{-1} & k^2 (\Delta_J^{\rm Neu})^{-1} \operatorname{div} V \operatorname{curl} \\ k^2 \operatorname{rot} V \operatorname{grad} (\Delta_J^{\rm Neu})^{-1} & k^2 \operatorname{rot} V \operatorname{curl} \end{pmatrix}$$

Analysis as a nonconforming method possible.

$(\phi, m, p, \lambda, \alpha)$ -formulation

The perturbed p- (m, ϕ) formulation allows an equivalent reformulation as a Galerkin method for a (4×4) system that can be considered as a mixed method where the equation (p, m) is considered as a constraint and a Lagrange parameter λ is introduced.

To take care of integrability conditions and free constants, one introduces $\alpha \in \mathbb{C}^3$.

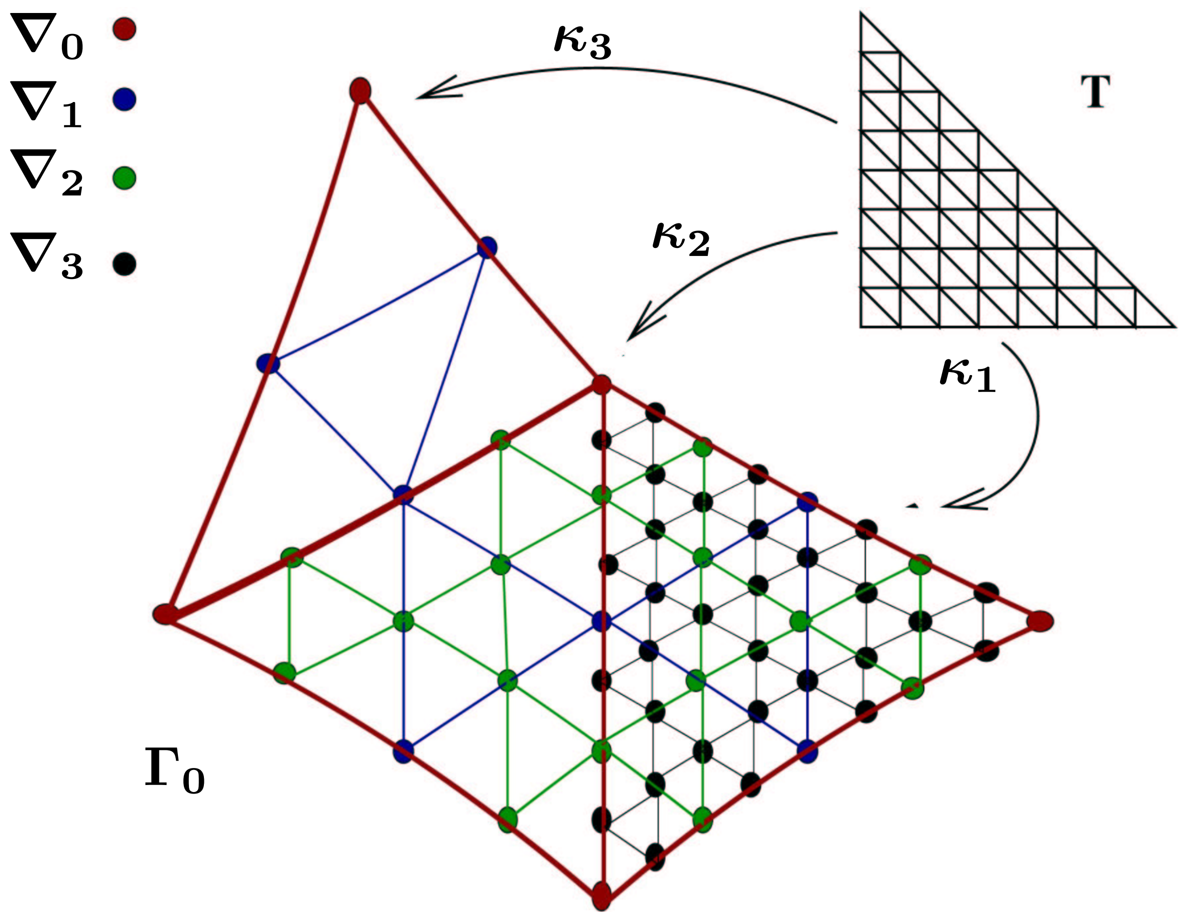
The final result is the system

$$(\phi, m, p, \lambda, \alpha) \qquad L \sim \begin{pmatrix} k^2 \operatorname{rot} V \operatorname{curl} & 0 & k^2 \operatorname{rot} V \operatorname{grad} & 0 & 0 \\ 0 & V & 0 & 1 & \\ 0 & 1 & -\Delta & 0 & I_3 \\ k^2 \operatorname{div} V \operatorname{curl} & 0 & k^2 \operatorname{div} V \operatorname{grad} & -\Delta & \\ 0 & & I_3 & & 0 \end{pmatrix}$$

Strongly elliptic system. SIOs of order ± 1.

Energy space: $\widetilde{H}^{1/2} \times \widetilde{H}^{-1/2} \times H^1 \times H^1 \times \mathbb{C}^3$. **Discretisation by** $\Lambda_J^4 \times \mathbb{C}^3$.

“Nodal” wavelets [Dahmen-Stevenson ’99, Rathsfeld]



Triangular patches on Γ_0

"Nodal" wavelets

- Reference triangle T, refinements T^j: 4^j triangles, meshsize $h_j = 2^{-j}$
- Mappings $\kappa_i : T \to \Gamma_0, i = 1, \dots, M,$ $+$ continuity conditions $\longrightarrow$ Coarse triangular patches on Γ_0, nodes Δ_0
- Refinement level j: $\bigcup_{i=1}^M \kappa_i(T^j)$, nodes Δ_j
- P_1 elements

$$\Lambda_j = \{u \in C^0(\Gamma_0) \mid u \circ \kappa_i\big|_{T_k^i} \in \mathbb{P}_1; i = 1, \dots, M; k = 1, \dots, 4^j\}$$

- Nodal basis $\varphi_\tau^j, \tau \in \Delta_j$
- Dual "basis" $\theta_\tau^{j+1} = \alpha_\tau \varphi_\tau^{j+1} - \beta_\tau \varphi_\tau^j \in \Lambda_{j+1}$: $(\varphi_\tau^j, \theta_\sigma^{j+1})_0 = c\,\delta_{\tau\sigma}$
- Wavelets: difference nodes $\tau \in \nabla_{j+1} = \Delta_{j+1} \setminus \Delta_j$

$$\psi_\tau = \varphi_\tau^{j+1} - \sum_{\operatorname{supp}\varphi_\sigma^j \cap \operatorname{supp}\varphi_\tau^{j+1} \neq \emptyset} \gamma_{\tau\sigma}^j \theta_\sigma^{j+1}$$

$$\psi_\tau = \varphi_\tau^0, \quad \tau \in \nabla_0 = \Delta_0$$

Basic properties of the wavelets, closed surface

- **Hierarchy of nodes:** $\Delta_J = \bigcup_{j=0}^{J} \nabla_j$
- **Basis of Λ_J :** $\{\psi_\tau \mid \tau \in \nabla_j, j = 0, \dots, J\}$
- **Support:** $\tau \in \nabla_j \Rightarrow \operatorname{diam} \operatorname{supp} \psi_\tau \sim 2^{-j}$
- **Stability:** $\forall s \in [-1, 1]$:

$$\| \sum_{j \geq 0} \sum_{\tau \in \nabla_j} c_\tau \psi_\tau \|^2_{H^s(\Gamma)} \sim \sum_{j \geq 0} 2^{2sj} \sum_{\tau \in \nabla_j} |c_\tau|^2$$

- **Approximation property (Jackson) and inverse estimate (Bernstein)**
- **Two vanishing moments:** For $d = 0, 1$:

$$|(v, \psi_\tau)_0| \leq C 2^{-(d+2)j} \sup\{|D^\alpha (v \circ \kappa_i)(x)| \mid x \in T; i \leq M; |\alpha| \leq d + 1\}$$

Here $(u, v)_0 = \sum_i \int_T u \circ \kappa_i \, \overline{v \circ \kappa_i} \, dx$ or alternatively, $(u, v)_0 = \int_\Gamma u \bar{v} ds$

Estimates for matrix elements

Let H be a singular integral operator on Γ with kernel $H(x,y)$ satisfying

$$\forall \alpha, \beta \; \forall x \neq y \in \Gamma : \; |D_x^\alpha D_y^\beta H(x,y)| \leq C_{\alpha,\beta} \, |x-y|^{-(2+r+\alpha+\beta)}$$

Then $\forall \tau \in \nabla_j, \tau' \in \nabla_{j'}, \operatorname{supp} \psi_\tau \cap \operatorname{supp} \psi_{\tau'} = \emptyset$:

$$|(H\psi_\tau, \psi_{\tau'})_0| \leq C \, 2^{-3(j+j')} d_{\tau\tau'}^{-(6+r)}$$

Here $r = \pm 1$ is the order of the operator and $d_{\tau\tau'} = \operatorname{dist}(\operatorname{supp} \psi_\tau, \operatorname{supp} \psi_{\tau'})$

The constant C is independent of the refinement level, but depends on the size of the bounded manifold Γ and on the wave number k.

This estimate is the basis for a matrix compression scheme based on the distance $d_{\tau\tau'}$. One defines a threshold matrix $(\epsilon_{jj'})_{j,j'=1,\dots,J}$. For a matrix (block) M corresponding to the operator H one defines the compressed matrix M^c by the rule

$$(\tau \in \nabla_j, \tau' \in \nabla_{j'}) : M^c_{\tau\tau'} = \begin{cases} M_{\tau\tau'} & : \; d_{\tau\tau'} \leq \epsilon_{jj'} \\ 0 & : \; d_{\tau\tau'} > \epsilon_{jj'} \end{cases}$$

Matrix compression, closed surface

Choice of the threshold matrix: Parameters $K, a, b > 0$,

$$\epsilon_{jj'} = K \max\{2^{-j}, 2^{-j'}, 2^{aJ-b(j+j')}\}$$

Theorem: Let $K > 1$, $0 < a < 1$, $b \geq (a+1)/2$. Then

$$\#\{M^c_{\tau\tau'} \neq 0\} = \mathcal{O}(J\, 2^{2J})$$

Define the diagonal order reduction matrix $D_s = \mathrm{diag}(2^{sj})$

Theorem: Let M be a block corresponding to an operator of order $r = \pm 1$. Define $d = r + 4$. Let $K > 1$, $0 < a < 1$, $b = (a+1)/2$. Then for all $s, s' < bd - 2$, there holds

$$\|D_{-s}(M - M^c)D_{-s'}\|_{\ell^2} \leq C\, K^{-d} 2^{(r-s-s')J}$$

In our case, we want to choose $s, s' \in \{0, \pm 1/2\}$ such that $s + s' \geq r + 1/2$. This requires

$$\frac{2}{3} < b = \frac{a+1}{2} < 1$$

Wavelets on open surfaces

- Two types of wavelets: with or without boundary condition.
- Without boundary condition: $\Lambda_J, \psi_\tau \dots$ as before
- With Dirichlet condition (Γ_0 has a polygonal boundary):

Nodes: $\widetilde{\Delta}_j = \Delta_j \setminus \partial\Gamma_0$, $\widetilde{\nabla}_j = \widetilde{\Delta}_j \setminus \widetilde{\Delta}_{j-1} = \nabla_j \setminus \partial\Gamma_0$

Shape functions: $\widetilde{\Lambda}_j = \Lambda_j \cap H_0^1(\Gamma_0) = \{\varphi_\tau^j \mid \tau \in \widetilde{\Delta}_j\}$

Wavelets: $\tau \in \widetilde{\nabla}_j : \widetilde{\psi}_\tau \in \widetilde{\Lambda}_j \ominus \widetilde{\Lambda}_{j-1}$

Attention: $(\Lambda_j \cap H_0^1(\Gamma_0)) \ominus (\Lambda_{j-1} \cap H_0^1(\Gamma_0)) \neq (\Lambda_j \ominus \Lambda_{j-1}) \cap H_0^1(\Gamma_0)$!

Interior wavelets: $\tau \in \widetilde{\nabla}_j^0 \Leftrightarrow \forall \sigma \in \partial\Gamma_0 \cap \Delta_j :\ \operatorname{supp} \widetilde{\psi}_\tau \cap \operatorname{supp} \varphi_\sigma^j = \emptyset$

Boundary layer: $\widetilde{\nabla}_j^\partial = \widetilde{\nabla}_j \setminus \widetilde{\nabla}_j^0$. Size: $\#\widetilde{\nabla}_j^\partial \sim 2^j$

Riesz-Stability: $\{\psi_\tau\}$ in H^s and $\widetilde{H}^{-s}$, $\{\widetilde{\psi}_\tau\}$ in $\widetilde{H}^s$ and H^{-s}, $0 \leq s \leq 1$

Estimates for wavelets on open surfaces

- **Vanishing moments:**

$$
\begin{aligned}
|(v,\psi_\tau)| &\leq C\,2^{-3j}\|v\|_{PW^{2,\infty}(\Gamma_0)} && \tau\in\nabla_j\\
|(v,\widetilde{\psi}_\tau)| &\leq C\,2^{-3j}\|v\|_{PW^{2,\infty}(\Gamma_0)} + C\,2^{-j}\|v\|_{L^\infty(\partial\Gamma_0)} && \tau\in\widetilde{\nabla}_j
\end{aligned}
$$

- **Matrix elements:** $\forall\tau\in\widetilde{\nabla}_j,\tau'\in\widetilde{\nabla}'_j$:

$$
\begin{aligned}
|(H\psi_\tau,\psi_{\tau'})_0| \lesssim\ & 2^{-3(j+j')}d_{\tau\tau'}^{-(6+r)}\\
&+2^{-(j+j')-2\min(j,j')}d_{\tau\tau'}^{-(4+r)}\\
&+2^{-(j+j')}d_{\tau\tau'}^{-(2+r)}
\end{aligned}
$$

Estimates cont'd

- Compression strategy:

— All matrix elements where both test and trial functions are close to the boundary are retained

— If both test and trial functions are interior wavelets, one uses the threshold matrix $\epsilon_{jj'}$ as before

— If one function is interior and the other one in the boundary layer, one uses a second threshold matrix

$$\epsilon^*_{jj'} = K \max\{2^{-j}, 2^{-j'}, 2^{aJ-bj-b^*j')}\}$$

with an additional parameter b^*.

- Choice of parameters:

$$K > 1, \ \frac{4}{5} < b = \frac{a+1}{2} < 1, \ b^* = b - 1/2$$

- Everything then works.

Conclusion

One obtains a linear system of $N \sim 2^{2J}$ equations with

- a simple diagonal preconditioner
- only $O(N \log N)$ non-zero matrix elements
- quasi-optimal error estimates

$$\begin{aligned} u &= u^1 + u^2, \quad \operatorname{rot} u^1 = 0 = \operatorname{div} u^2 \\ \|u^1 - u^1_J\|_{H^{-1/2}_{\mathrm{div}}} &\lesssim 2^{-J/2} \\ \|u^2 - u^2_J\|_0 &\lesssim 2^{-J} \\ \|\operatorname{div} u - m_J\|_{-1/2} &\lesssim 2^{-J/2} \end{aligned}$$

Future:

- Implementation
- Comparison with other methods . . .