The biharmonic double layer potential on Lipschitz domains

Martin Costabel

IRMAR, Université de Rennes 1, France

Workshop, Linz University 13 July 2006

Outline

Outline

1. What is the Biharmonic Double Layer Potential?
   - Smooth domains
   - Lipschitz domains
2. Properties of the Biharmonic Double Layer Potential
   - The Poincaré fundamental lemma
   - Positivity, Neumann series
3. References

Outline

1. What is the Biharmonic Double Layer Potential?
   - Smooth domains
   - Lipschitz domains
2. Properties of the Biharmonic Double Layer Potential
   - The Poincaré fundamental lemma
   - Positivity, Neumann series
3. References

Green formulas

$\Omega = \Omega^-$: Smooth bounded domain in $\mathbb{R}^2$; boundary Γ; exterior domain Ω^+

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \Delta u \Delta v + \int_\Gamma (\partial_n \Delta u\, v - \Delta u\, \partial_n v)\, ds$$

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma (\partial_n \Delta u\, v - \partial_\tau \partial_n u\, \partial_\tau v - \partial_n^2 u\, \partial_n v)\, ds$$

$$= \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma (\partial_n \Delta u\, v + \partial_s \partial_\tau \partial_n u\, v - \partial_n^2 u\, \partial_n v)\, ds$$

$0 \le \sigma \le 1$: $\int_\Omega \Delta^2 u\, v = a_\sigma(u,v) + \int_\Gamma (N_\sigma u\, v - M_\sigma u\, \partial_n v)\, ds$

$$a_\sigma(u,v) = \sigma \int_\Omega \Delta u \Delta v + (1-\sigma) \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v$$

$$M_\sigma = \sigma \Delta u + (1-\sigma)\partial_n^2 u \quad : \text{ bending moment}$$

$$N_\sigma = \partial_n \Delta u + (1-\sigma)\partial_s \partial_\tau \partial_n u \quad : \text{ twisting moment}$$

Green formulas

$\Omega = \Omega^-$: Smooth bounded domain in $\mathbb{R}^2$; boundary Γ; exterior domain Ω^+

$$\begin{aligned}
\int_\Omega \Delta^2 u\, v &= \int_\Omega \Delta u\, \Delta v + \int_\Gamma \big(\partial_n \Delta u\, v - \Delta u\, \partial_n v\big)\, ds \\
\int_\Omega \Delta^2 u\, v &= \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma \big(\partial_n \Delta u\, v - \partial_\tau \partial_n u\, \partial_\tau v - \partial_n^2 u\, \partial_n v\big)\, ds \\
&= \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma \big(\partial_n \Delta u\, v + \partial_s \partial_\tau \partial_n u\, v - \partial_n^2 u\, \partial_n v\big)\, ds
\end{aligned}$$

$0 \le \sigma \le 1$: $\int_\Omega \Delta^2 u\, v = a_\sigma(u,v) + \int_\Gamma \big(N_\sigma u\, v - M_\sigma u\, \partial_n v\big)\, ds$

$$\begin{aligned}
a_\sigma(u,v) &= \sigma \int_\Omega \Delta u\, \Delta v + (1-\sigma) \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v \\
M_\sigma &= \sigma \Delta u \quad + (1-\sigma)\partial_n^2 u \quad : \text{ bending moment} \\
N_\sigma &= \partial_n \Delta u + (1-\sigma)\partial_s \partial_\tau \partial_n u : \text{ twisting moment}
\end{aligned}$$

Green formulas

$\Omega = \Omega^-$: Smooth bounded domain in $\mathbb{R}^2$; boundary Γ; exterior domain Ω^+

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \Delta u \Delta v + \int_\Gamma (\partial_n \Delta u\, v - \Delta u\, \partial_n v)\, ds$$

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma (\partial_n \Delta u\, v - \partial_\tau \partial_n u\, \partial_\tau v - \partial_n^2 u\, \partial_n v)\, ds$$

$$= \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma (\partial_n \Delta u\, v + \partial_s \partial_\tau \partial_n u\, v - \partial_n^2 u\, \partial_n v)\, ds$$

$0 \le \sigma \le 1$: $\int_\Omega \Delta^2 u\, v = a_\sigma(u,v) + \int_\Gamma (N_\sigma u\, v - M_\sigma u\, \partial_n v)\, ds$

$$a_\sigma(u,v) = \sigma \int_\Omega \Delta u \Delta v + (1-\sigma) \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v$$

$$M_\sigma = \sigma \Delta u + (1-\sigma)\partial_n^2 u \quad : \text{ bending moment}$$

$$N_\sigma = \partial_n \Delta u + (1-\sigma)\partial_s \partial_\tau \partial_n u \quad : \text{ twisting moment}$$

Green formulas

$\Omega = \Omega^-$: Smooth bounded domain in $\mathbb{R}^2$; boundary Γ; exterior domain Ω^+

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \Delta u \Delta v + \int_\Gamma \big(\partial_n \Delta u\, v - \Delta u\, \partial_n v\big)\, ds$$

$$\int_\Omega \Delta^2 u\, v = \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma \big(\partial_n \Delta u\, v - \partial_\tau \partial_n u\, \partial_\tau v - \partial_n^2 u\, \partial_n v\big)\, ds$$

$$= \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v + \int_\Gamma \big(\partial_n \Delta u\, v + \partial_s \partial_\tau \partial_n u\, v - \partial_n^2 u\, \partial_n v\big)\, ds$$

$0 \le \sigma \le 1$: $\displaystyle\int_\Omega \Delta^2 u\, v = a_\sigma(u,v) + \int_\Gamma \big(N_\sigma u\, v - M_\sigma u\, \partial_n v\big)\, ds$

$$a_\sigma(u,v) = \sigma \int_\Omega \Delta u \Delta v + (1-\sigma) \int_\Omega \sum_{|\alpha|=2} \partial^\alpha u\, \partial^\alpha v$$

$M_\sigma = \sigma \Delta u + (1-\sigma)\partial_n^2 u$: bending moment

$N_\sigma = \partial_n \Delta u + (1-\sigma)\partial_s \partial_\tau \partial_n u$: twisting moment

Traces

$$\int_\Omega \Delta^2 u\, v = a(u,v) + \int_\Gamma (Nu\, v - Mu\, \partial_n v)\, ds$$

Cauchy data: $(\gamma_0 u, \gamma_1 u) := (u, \partial_n u, -Nu, Mu)$ on Γ

Traces: $(\gamma_0, \gamma_1) : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma) \times H^{s-\frac{3}{2}}(\Gamma) \times H^{s-\frac{7}{2}}(\Gamma) \times H^{s-\frac{5}{2}}(\Gamma)$

Energy norm ($s=2$): $X := H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) = \gamma_0 H^2(\Omega)$

$$(\gamma_0, \gamma_1) : H^2(\Delta^2; \Omega) \to X \times X'$$

First Green formula

$$a(u,v) = \int_\Omega \Delta^2 u\, v + \langle \gamma_1 u, \gamma_0 v \rangle$$

Second Green formula

$$\int_\Omega (\Delta^2 u\, v - u\, \Delta^2 v) = -\langle \gamma_1 u, \gamma_0 v \rangle + \langle \gamma_0 u, \gamma_1 v \rangle$$

Traces

$$\int_\Omega \Delta^2 u\, v = a(u,v) + \int_\Gamma (Nu\, v - Mu\, \partial_n v)\, ds$$

Cauchy data: $(\gamma_0 u, \gamma_1 u) := (u, \partial_n u, -Nu, Mu)$ on Γ

Traces: $(\gamma_0, \gamma_1) : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma) \times H^{s-\frac{3}{2}}(\Gamma) \times H^{s-\frac{7}{2}}(\Gamma) \times H^{s-\frac{5}{2}}(\Gamma)$

Energy norm ($s = 2$): $X := H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) = \gamma_0 H^2(\Omega)$

$$(\gamma_0, \gamma_1) : H^2(\Delta^2; \Omega) \to X \times X'$$

First Green formula

$$a(u,v) = \int_\Omega \Delta^2 u\, v + \langle \gamma_1 u, \gamma_0 v \rangle$$

Second Green formula

$$\int_\Omega (\Delta^2 u\, v - u\, \Delta^2 v) = -\langle \gamma_1 u, \gamma_0 v \rangle + \langle \gamma_0 u, \gamma_1 v \rangle$$

Traces

$$\int_\Omega \Delta^2 u\, v = a(u,v) + \int_\Gamma (Nu\, v - Mu\, \partial_n v)\, ds$$

Cauchy data: $(\gamma_0 u, \gamma_1 u) := (u, \partial_n u, -Nu, Mu)$ on Γ

Traces: $(\gamma_0, \gamma_1) : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma) \times H^{s-\frac{3}{2}}(\Gamma) \times H^{s-\frac{7}{2}}(\Gamma) \times H^{s-\frac{5}{2}}(\Gamma)$

Energy norm ($s = 2$): $X := H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) = \gamma_0 H^2(\Omega)$

$$(\gamma_0, \gamma_1) : H^2(\Delta^2; \Omega) \to X \times X'$$

First Green formula

$$a(u,v) = \int_\Omega \Delta^2 u\, v + \langle \gamma_1 u, \gamma_0 v \rangle$$

Second Green formula

$$\int_\Omega (\Delta^2 u\, v - u\, \Delta^2 v) = -\langle \gamma_1 u, \gamma_0 v \rangle + \langle \gamma_0 u, \gamma_1 v \rangle$$

Representation Formula, Single and Double Layer

Fundamental solution: $G(x) = \frac{1}{8\pi}|x|^2 \log|x|$ $(\Rightarrow \Delta G = \frac{1}{2\pi}(\log|x| + 1))$

Representation in Ω^-

$$\begin{aligned} u(x) &= \int_\Omega \Delta^2 u(y)\, G(x-y)\, dy \\ &\quad + \int_\Gamma \big(-Nu(y)G(x-y) + Mu(y)\partial_{n(y)} G(x-y)\big)\, ds(y) \\ &\quad - \int_\Gamma \big(\partial_n u(y) M(y) G(x-y) - u(y) N(y) G(x-y)\big)\, ds(y) \\ &= \mathscr{N} f(x) + \mathscr{S}\gamma_1 u(x) - \mathscr{D}\gamma_0 u(x) \end{aligned}$$

Distributional definitions

$$\begin{aligned} \mathscr{N} f &= G * f \\ \mathscr{S}\phi &= G * \gamma_0^* \phi \\ \mathscr{D} g &= G * \gamma_1^* g \end{aligned}$$

Representation Formula, Single and Double Layer

Fundamental solution: $G(x) = \frac{1}{8\pi}|x|^2 \log|x| \quad (\Rightarrow \Delta G = \frac{1}{2\pi}(\log|x| + 1))$

Representation in Ω^-

$$
\begin{aligned}
u(x) &= \int_\Omega \Delta^2 u(y)\, G(x-y)\, dy \\
&\quad + \int_\Gamma \big(-Nu(y) G(x-y) + Mu(y)\partial_{n(y)} G(x-y)\big)\, ds(y) \\
&\quad - \int_\Gamma \big(\partial_n u(y) M(y) G(x-y) - u(y) N(y) G(x-y)\big)\, ds(y) \\
&= \mathcal{N} f(x) + \mathcal{S}\gamma_1 u(x) - \mathcal{D}\gamma_0 u(x)
\end{aligned}
$$

Distributional definitions

$$
\begin{aligned}
\mathcal{N} f &= G * f \\
\mathcal{S}\phi &= G * \gamma_0^* \phi \\
\mathcal{D} g &= G * \gamma_1^* g
\end{aligned}
$$

Representation Formula, Single and Double Layer

Fundamental solution: $G(x) = \frac{1}{8\pi}|x|^2 \log|x| \quad (\Rightarrow \Delta G = \frac{1}{2\pi}(\log|x| + 1))$

Representation in Ω^-

$$
\begin{aligned}
u(x) &= \int_\Omega \Delta^2 u(y)\, G(x-y)\, dy \\
&\quad + \int_\Gamma \big(-Nu(y) G(x-y) + Mu(y) \partial_{n(y)} G(x-y)\big)\, ds(y) \\
&\quad - \int_\Gamma \big(\partial_n u(y) M(y) G(x-y) - u(y) N(y) G(x-y)\big)\, ds(y) \\
&= \mathcal{N} f(x) + \mathcal{S} \gamma_1 u(x) - \mathcal{D} \gamma_0 u(x)
\end{aligned}
$$

Distributional definitions

$$
\begin{aligned}
\mathcal{N} f &= G * f \\
\mathcal{S} \phi &= G * \gamma_0^* \phi \\
\mathcal{D} g &= G * \gamma_1^* g
\end{aligned}
$$

The double layer potential

$$\mathscr{D}\binom{g_0}{g_1}(x) = \int_\Gamma \big(-N(y)G(x-y)g_0(y) + M(y)G(x-y)g_1(y)\big)\,ds(y)$$

Jump relations: $[\gamma_0 \mathscr{D} g] = g\,;\quad [\gamma_1 \mathscr{D} g] = 0$

One-sided traces: $\gamma_0^{\pm} \mathscr{D} g = \pm\frac{1}{2} v + Kg\,;\quad \gamma_1^{\pm} \mathscr{D} g = -Wg$

$$Kg(x) = \int_\Gamma \begin{pmatrix} -N(y)G(x-y) & M(y)G(x-y) \\ -\partial_{n(x)} N(y)G(x-y) & \partial_{n(x)} M(y)G(x-y) \end{pmatrix} \binom{g_0}{g_1}(y)\,ds(y)$$

Integral equation for the interior Dirichlet problem

$$\left(\tfrac{1}{2}\mathrm{I} - K\right) g = f$$

Orders: $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$: Not a classical Fredholm second kind integral equation!

The double layer potential

$$\mathscr{D}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(x) = \int_\Gamma \big(-N(y)G(x-y)g_0(y) + M(y)G(x-y)g_1(y)\big)\,ds(y)$$

Jump relations: $[\gamma_0 \mathscr{D} g] = g\,;\quad [\gamma_1 \mathscr{D} g] = 0$

One-sided traces: $\gamma_0^{\pm} \mathscr{D} g = \pm\frac{1}{2} v + Kg\,;\quad \gamma_1^{\pm} \mathscr{D} g = -Wg$

$$Kg(x) = \int_\Gamma \begin{pmatrix} -N(y)G(x-y) & M(y)G(x-y) \\ -\partial_{n(x)}N(y)G(x-y) & \partial_{n(x)}M(y)G(x-y) \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(y)\,ds(y)$$

Integral equation for the interior Dirichlet problem

$$\left(\tfrac{1}{2}\mathrm{I} - K\right) g = f$$

Orders: $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$: Not a classical Fredholm second kind integral equation!

The double layer potential

$$\mathscr{D}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(x) = \int_\Gamma \big(-N(y)G(x-y)g_0(y) + M(y)G(x-y)g_1(y)\big)\,ds(y)$$

Jump relations: $[\gamma_0 \mathscr{D} g] = g\,; \quad [\gamma_1 \mathscr{D} g] = 0$

One-sided traces: $\gamma_0^{\pm} \mathscr{D} g = \pm\frac{1}{2}v + Kg\,; \quad \gamma_1^{\pm} \mathscr{D} g = -Wg$

$$Kg(x) = \int_\Gamma \begin{pmatrix} -N(y)G(x-y) & M(y)G(x-y) \\ -\partial_{n(x)}N(y)G(x-y) & \partial_{n(x)}M(y)G(x-y) \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(y)\,ds(y)$$

Integral equation for the interior Dirichlet problem

$$\left(\tfrac{1}{2}\mathrm{I} - K\right)g = f$$

Orders: $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$: Not a classical Fredholm second kind integral equation!

The double layer potential

$$\mathscr{D}\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(x) = \int_\Gamma \big(-N(y)G(x-y)g_0(y) + M(y)G(x-y)g_1(y)\big)\,ds(y)$$

Jump relations: $[\gamma_0 \mathscr{D} g] = g\,; \quad [\gamma_1 \mathscr{D} g] = 0$

One-sided traces: $\gamma_0^{\pm} \mathscr{D} g = \pm\frac{1}{2} v + Kg\,; \quad \gamma_1^{\pm} \mathscr{D} g = -Wg$

$$Kg(x) = \int_\Gamma \begin{pmatrix} -N(y)G(x-y) & M(y)G(x-y) \\ -\partial_{n(x)}N(y)G(x-y) & \partial_{n(x)}M(y)G(x-y) \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}(y)\,ds(y)$$

Integral equation for the interior Dirichlet problem

$$\left(\tfrac{1}{2}\mathrm{I} - K\right) g = f$$

Orders: $\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$: Not a classical Fredholm second kind integral equation!

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H_0^2(\Omega^-) = H^2(\Omega^+)/H_0^2(\Omega^+) = H^2(\mathbb{R}^2)/H_0^2(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2; \Omega) \to X' \subset H_\Gamma^{-2}(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v \rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v \rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v\rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v\rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Continuity

$$\mathscr{S} : X' \to H^2_{\mathrm{loc}}(\mathbb{R}^2) \quad ; \qquad \mathscr{D} : X \to H^2(\Delta^2;\Omega)$$

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v\rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Representation formula in Ω

$$\forall u \in H^2(\Delta^2;\Omega): \qquad u = G * \Delta^2 u + \mathscr{S}\gamma_1 u - \mathscr{D}\gamma_0 u$$

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v\rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Jump relations

$[\gamma_0 \mathscr{S}\phi] = 0$; $[\gamma_1 \mathscr{S}\phi] = -\phi$; $[\gamma_0 \mathscr{D}g] = g$; $[\gamma_1 \mathscr{D}] = 0$

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v \rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Definition of boundary integral operators

$$\begin{aligned} V\phi &= \{\gamma_0 \mathscr{S}\phi\} \quad , \quad K'\phi = \{\gamma_1 \mathscr{S}\phi\} \quad \text{on } X' \\ Kg &= \{\gamma_0 \mathscr{D}g\} \quad , \quad Wg = -\{\gamma_1 \mathscr{D}g\} \quad \text{on } X \end{aligned}$$

Lipschitz boundaries: Spaces, traces and potentials

Dirichlet trace:
$X = H^2(\Omega^-)/H^2_0(\Omega^-) = H^2(\Omega^+)/H^2_0(\Omega^+) = H^2(\mathbb{R}^2)/H^2_0(\mathbb{R}^2 \setminus \Gamma)$
$\gamma_0 : H^2(\Omega) \to X$: Canonical projection

Neumann trace: $\gamma_1 = \gamma_{1,\sigma} : H^2(\Delta^2;\Omega) \to X' \subset H^{-2}_\Gamma(\mathbb{R}^2)$
defined by the first Green formula: $\langle \gamma_1 u, \gamma_0 v\rangle := \int_\Omega \Delta^2 u\, v - a(u,v)$

Definition

Single layer potential: $\mathscr{S}\phi = G * \gamma_0^* \phi$ Double layer potential : $\mathscr{D}g = G * \gamma_1^* g$

With these definitions, many things work and look the same as for the Laplace operator or other second order operators:

Etc...

$K' = K^*\,,\ KV = VK'\,,\ K'W = WK\,,\ VW = \frac{1}{4}\mathrm{I} - K^2$...
Calderón projector, Poincaré-Steklov operator, Boundary integral equations...

Energy

Removal of zero-energy fields in Ω^-

$X_0' = \{g \in X' \mid \forall p \in \mathbb{P}_1 : \langle g, \gamma_0 p \rangle = 0\}$; $X_0 = X / \gamma_0 \mathbb{P}_1$; $X_0' = (X_0)'$

The Neumann problem $u \in H^2(\Omega) : \Delta^2 u = 0,\ \gamma_1 u = g$
is solvable $\Leftrightarrow g \in X_0'$

Finiteness of energy in Ω^+

For $u = \mathscr{S}\phi, \phi \in X' :\ a^+(u,u) < \infty \Leftrightarrow \phi \in X_0'$ and $a^+(u,u) = 0 \Rightarrow \phi = 0$
For $u = \mathscr{D}g, g \in X :\ a^+(u,u) < \infty$ and $a^+(u,u) = 0 \Leftrightarrow g \in \gamma_0 \mathbb{P}_1$

Lemma

The total energy $a^-(u,u) + a^+(u,u)$ defines positive quadratic forms
on X_0' via single layer potentials $u = \mathscr{S}\phi$
on X_0 via double layer potentials $u = \mathscr{D}g$

Energy

Removal of zero-energy fields in Ω^-

$X_0' = \{g \in X' \mid \forall p \in \mathbb{P}_1 : \langle g, \gamma_0 p\rangle = 0\}$; $X_0 = X/\gamma_0\mathbb{P}_1$; $X_0' = (X_0)'$

The Neumann problem $u \in H^2(\Omega) : \Delta^2 u = 0, \gamma_1 u = g$
is solvable $\Leftrightarrow g \in X_0'$

Finiteness of energy in Ω^+

For $u = \mathscr{S}\phi, \phi \in X' :\ a^+(u,u) < \infty \Leftrightarrow \phi \in X_0'$ and $a^+(u,u) = 0 \Rightarrow \phi = 0$
For $u = \mathscr{D}g, g \in X :\ a^+(u,u) < \infty$ and $a^+(u,u) = 0 \Leftrightarrow g \in \gamma_0\mathbb{P}_1$

Lemma

The total energy $a^-(u,u) + a^+(u,u)$ defines positive quadratic forms
on X_0' via single layer potentials $u = \mathscr{S}\phi$
on X_0 via double layer potentials $u = \mathscr{D}g$

Energy

Removal of zero-energy fields in Ω^-

$X_0' = \{g \in X' \mid \forall p \in \mathbb{P}_1 : \langle g, \gamma_0 p\rangle = 0\}$; $X_0 = X/\gamma_0\mathbb{P}_1$; $X_0' = (X_0)'$

The Neumann problem $u \in H^2(\Omega) : \Delta^2 u = 0,\ \gamma_1 u = g$
is solvable $\Leftrightarrow g \in X_0'$

Finiteness of energy in Ω^+

For $u = \mathscr{S}\phi, \phi \in X' :\ a^+(u,u) < \infty \Leftrightarrow \phi \in X_0'$ and $a^+(u,u) = 0 \Rightarrow \phi = 0$
For $u = \mathscr{D}g, g \in X :\ a^+(u,u) < \infty$ and $a^+(u,u) = 0 \Leftrightarrow g \in \gamma_0\mathbb{P}_1$

Lemma

The total energy $a^-(u,u) + a^+(u,u)$ defines positive quadratic forms
on X_0' via single layer potentials $u = \mathscr{S}\phi$
on X_0 via double layer potentials $u = \mathscr{D}g$

The Poincaré Fundamental Lemma

Lemma

There exists $\mu = \mu(\Gamma) \geq 1$ *such that*
if $u = \mathscr{S}\phi$, $\phi \in X_0'$, *or* $u = \mathscr{D}g$, $g \in X$ *then*

$$\frac{1}{\mu} a^-(u,u) \leq a^+(u,u) \leq \mu a^-(u,u)$$

Proof: Continuity of traces in one direction and estimates for the variational solution of the Dirichlet and Neumann problems in the other direction allow to compare both quadratic forms to the natural norms on the trace spaces X' and X. Proof of the Corollary:

$$\begin{aligned}
a(u,u) &\leq (\mu+1)a^-(u,u) \\
a(u,u) &\leq (1+\mu)a^+(u,u) \\
a^+(u,u) &= a(u,u) - a^-(u,u) \leq \frac{\mu}{\mu+1} a(u,u) \\
a^-(u,u) &= a(u,u) - a^+(u,u) \leq \frac{\mu}{\mu+1} a(u,u) \\
|a^+(u,u) - a^-(u,u)| &\leq \frac{\mu-1}{\mu+1} a(u,u)
\end{aligned}$$

The Poincaré Fundamental Lemma

Lemma

There exists $\mu = \mu(\Gamma) \geq 1$ *such that*
if $u = \mathscr{S}\phi\,,\ \phi \in X_0'$, *or* $u = \mathscr{D}g\,,\ g \in X$ *then*

$$\frac{1}{\mu} a^-(u,u) \leq a^+(u,u) \leq \mu\, a^-(u,u)$$

Corollary

On the Hilbert space X_0 *with the norm of the total energy*

$$\|g\|_a^2 = a(u,u) = a^-(u,u) + a^+(u,u)\ ; \quad (u = \mathscr{D}g)$$

the operators A^+ *and* A^- *defined by the bilinear forms* a^+ *and* a^- *are positive definite, selfadjoint bounded operators satisfying* $A^+ + A^- = \mathrm{I}$.
The 3 operators A^+, A^- *and* $A^+ - A^-$ *are contractions:*

$$\|A^+\|_a \leq \frac{\mu}{\mu+1}\ ; \quad \|A^-\|_a \leq \frac{\mu}{\mu+1}\ ; \quad \|A^+ - A^-\|_a \leq \frac{\mu-1}{\mu+1}$$

Proof of the Corollary:

The Poincaré Fundamental Lemma

Lemma

There exists $\mu = \mu(\Gamma) \geq 1$ *such that*
if $u = \mathscr{S}\phi\,,\ \phi \in X_0'$, *or* $u = \mathscr{D}g\,,\ g \in X$ *then*

$$\frac{1}{\mu} a^{-}(u,u) \leq a^{+}(u,u) \leq \mu\, a^{-}(u,u)$$

Proof of the Corollary:

$$\begin{aligned}
a(u,u) &\leq (\mu+1)a^{-}(u,u)\\
a(u,u) &\leq (1+\mu)a^{+}(u,u)\\
a^{+}(u,u) &= a(u,u) - a^{-}(u,u) \leq \frac{\mu}{\mu+1} a(u,u)\\
a^{-}(u,u) &= a(u,u) - a^{+}(u,u) \leq \frac{\mu}{\mu+1} a(u,u)\\
|a^{+}(u,u) - a^{-}(u,u)| &\leq \frac{\mu-1}{\mu+1} a(u,u)
\end{aligned}$$

The biharmonic double layer potential operator

From the jump relations, one has the expressions for the total and partial energies

$$\forall u = \mathscr{D}g, g \in X_0 \; : \qquad a^{\pm}(u,u) = \langle Wg, (\tfrac{1}{2}\mathrm{I} \pm K)g\rangle \; ; \quad a(u,u) = \langle Wg, g\rangle$$

Hence we can identify

$$\|g\|_a^2 = \langle Wg, g\rangle \; ; \quad A^{\pm} = \tfrac{1}{2}\mathrm{I} \pm K \; ; \quad A^+ - A^- = 2K$$

Theorem

The operators $\frac{1}{2}\mathrm{I} \pm K$ are positive definite selfadjoint operators on X_0 with the energy norm.
The operators $\frac{1}{2}\mathrm{I} \pm K$ and $2K$ are contractions.
The Dirichlet problem in Ω: $\Delta^2 u = 0, \gamma_0 u = f \in X$ can be solved by a double layer potential $u = \mathscr{D}g$, where g is given by the convergent Neumann series

$$g = (\tfrac{1}{2}\mathrm{I} - K)^{-1} f = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}\mathrm{I} + K)^{\ell} f$$

On the quotient space X_0 ... the following Neumann series is also

The biharmonic double layer potential operator

Theorem

The operators $\frac{1}{2}\mathrm{I} \pm K$ are positive definite selfadjoint operators on X_0 with the energy norm.
The operators $\frac{1}{2}\mathrm{I} \pm K$ and $2K$ are contractions.
The Dirichlet problem in Ω: $\Delta^2 u = 0, \gamma_0 u = f \in X$ can be solved by a double layer potential $u = \mathscr{D}g$, where g is given by the convergent Neumann series

$$g = (\tfrac{1}{2}\mathrm{I} - K)^{-1} f = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}\mathrm{I} + K)^{\ell} f$$

On the quotient space X_0, the following Neumann series is also convergent:

$$g = (\tfrac{1}{2}(\mathrm{I} - 2K))^{-1} f = 2 \sum_{\ell=0}^{\infty} (2K)^{\ell} f$$

The contraction constant (Poincaré estimate)

For the norm $\|\frac{1}{2}(\mathrm{I}-2K\|_W$ we have seen

$$\begin{aligned}
\|\tfrac{1}{2}\mathrm{I}-K\|_W &= \sup_{g\in X_0}\frac{\langle Wg,(\frac{1}{2}\mathrm{I}-K)g\rangle}{\langle Wg,g\rangle}\\
&= \sup\{\frac{a^-(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a double layer potential }\}\\
&= 1-\inf\{\frac{a^+(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a double layer potential }\}
\end{aligned}$$

In a similar way, we get, by representing single layer potentials by their Dirichlet data

$$\begin{aligned}
\|\tfrac{1}{2}\mathrm{I}+K\|_{V^{-1}} &= \sup_{g\in X}\frac{\langle V^{-1}g,(\frac{1}{2}\mathrm{I}+K)g\rangle}{\langle V^{-1}g,g\rangle}\\
&= \sup\{\frac{a^-(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a single layer potential }\}\\
&= 1-\inf\{\frac{a^+(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a single layer potential }\}
\end{aligned}$$

The contraction constant (Poincaré estimate)

For the norm $\|\frac{1}{2}(\mathrm{I}-2K\|_W$ we have seen

$$\begin{aligned} \|\tfrac{1}{2}\mathrm{I}-K\|_W &= \sup_{g\in X_0} \frac{\langle Wg, (\frac{1}{2}\mathrm{I}-K)g\rangle}{\langle Wg, g\rangle} \\ &= \sup\{\frac{a^-(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a double layer potential }\} \\ &= 1-\inf\{\frac{a^+(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a double layer potential }\} \end{aligned}$$

In a similar way, we get, by representing single layer potentials by their Dirichlet data

$$\begin{aligned} \|\tfrac{1}{2}\mathrm{I}+K\|_{V^{-1}} &= \sup_{g\in X} \frac{\langle V^{-1}g, (\frac{1}{2}\mathrm{I}+K)g\rangle}{\langle V^{-1}g, g\rangle} \\ &= \sup\{\frac{a^-(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a single layer potential }\} \\ &= 1-\inf\{\frac{a^+(u,u)}{a^+(u,u)+a^-(u,u)} \mid u \text{ is a single layer potential }\} \end{aligned}$$

The contraction constant (Steinbach-Wendland estimate)

Recall: The Poincaré-Steklov operator in Ω^-: $S : \gamma u \mapsto \gamma_1 u \, (Lu = 0)$

$$\begin{aligned} S &= (\tfrac{1}{2}I + K')V^{-1} && (\text{ S.L.: } u = \mathscr{S}\varphi;\, \gamma u = V\varphi;\, \gamma_1 u = (\tfrac{1}{2}I + K')\varphi \text{)} \\ &= W(\tfrac{1}{2}I - K)^{-1} && (\text{ D.L.: } u = \mathscr{D}v;\, \gamma u = (-\tfrac{1}{2}I + K)v;\, \gamma_1 u = -Wv \text{)} \\ &= W + S(\tfrac{1}{2}I + K) && (\, S(I - (\tfrac{1}{2}I + K)) = W \,) \\ &= W + (\tfrac{1}{2}I + K')V^{-1}(\tfrac{1}{2}I + K) && \text{symmetric form} \end{aligned}$$

The contraction constant (Steinbach-Wendland estimate)

Recall: The Poincaré-Steklov operator in Ω^-: $S : \gamma u \mapsto \gamma_1 u \, (Lu = 0)$

$$S = (\tfrac{1}{2} I + K')V^{-1} \quad (\text{ S.L.: } u = \mathscr{S}\varphi \,;\, \gamma u = V\varphi \,;\, \gamma_1 u = (\tfrac{1}{2} I + K')\varphi \text{ })$$

$$= W(\tfrac{1}{2} I - K)^{-1} \quad (\text{ D.L.: } u = \mathscr{D}v \,;\, \gamma u = (-\tfrac{1}{2} I + K)v \,;\, \gamma_1 u = -Wv \text{ })$$

$$= W + S(\tfrac{1}{2} I + K) \quad (\, S(I - (\tfrac{1}{2} I + K)) = W \,)$$

$$= W + (\tfrac{1}{2} I + K')V^{-1}(\tfrac{1}{2} I + K) \quad \text{symmetric form}$$

The contraction constant (Steinbach-Wendland estimate)

Recall: The Poincaré-Steklov operator in Ω^-: $S : \gamma u \mapsto \gamma_1 u \, (Lu = 0)$

$$S = (\tfrac{1}{2}I + K')V^{-1} \quad (\text{S.L.: } u = \mathscr{S}\varphi\,;\ \gamma u = V\varphi\,;\ \gamma_1 u = (\tfrac{1}{2}I + K')\varphi)$$

$$= W(\tfrac{1}{2}I - K)^{-1} \quad (\text{D.L.: } u = \mathscr{D}v\,;\ \gamma u = (-\tfrac{1}{2}I + K)v\,;\ \gamma_1 u = -Wv)$$

$$= W + S(\tfrac{1}{2}I + K) \quad (S(I - (\tfrac{1}{2}I + K)) = W)$$

$$= W + (\tfrac{1}{2}I + K')V^{-1}(\tfrac{1}{2}I + K) \quad \text{symmetric form}$$

The contraction constant (Steinbach-Wendland estimate)

Recall: The Poincaré-Steklov operator in Ω^-: $S : \gamma u \mapsto \gamma_1 u \, (Lu = 0)$

$$S = (\tfrac{1}{2} I + K') V^{-1} \quad (\text{S.L.: } u = \mathscr{S}\varphi \,;\, \gamma u = V\varphi \,;\, \gamma_1 u = (\tfrac{1}{2} I + K')\varphi)$$

$$= W(\tfrac{1}{2} I - K)^{-1} \quad (\text{D.L.: } u = \mathscr{D} v \,;\, \gamma u = (-\tfrac{1}{2} I + K) v \,;\, \gamma_1 u = -W v)$$

$$= W + S(\tfrac{1}{2} I + K) \quad (S(I - (\tfrac{1}{2} I + K)) = W)$$

$$= W + (\tfrac{1}{2} I + K') V^{-1} (\tfrac{1}{2} I + K) \quad \text{symmetric form}$$

The contraction constant (Steinbach-Wendland estimate)

Recall: The Poincaré-Steklov operator in Ω^-: $S : \gamma u \mapsto \gamma_1 u \, (Lu = 0)$

$$
\begin{aligned}
S &= (\tfrac{1}{2} I + K') V^{-1} && (\text{ S.L.: } u = \mathscr{S}\varphi \,;\, \gamma u = V\varphi \,;\, \gamma_1 u = (\tfrac{1}{2} I + K')\varphi \,) \\
&= W(\tfrac{1}{2} I - K)^{-1} && (\text{ D.L.: } u = \mathscr{D} v \,;\, \gamma u = (-\tfrac{1}{2} I + K) v \,;\, \gamma_1 u = -W v \,) \\
&= W + S(\tfrac{1}{2} I + K) && (\, S(I - (\tfrac{1}{2} I + K)) = W \,) \\
&= W + (\tfrac{1}{2} I + K') V^{-1} (\tfrac{1}{2} I + K) && \text{symmetric form}
\end{aligned}
$$

The contraction constant (Steinbach-Wendland estimate)

- If $a, b \in \mathbb{R}$ and $b \geq b^2 + a$ and $a > 0$, then

$$\frac{1}{2} - \sqrt{\frac{1}{4} - a} \leq b \leq \frac{1}{2} + \sqrt{\frac{1}{4} - a} < 1$$

- If A, B are bounded selfadjoint operators and $B = B^2 + A$ and $A \geq aI > 0$, then

$$\|B\| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - a} < 1$$

- Let $B = \frac{1}{2}\mathrm{I} + K$ in $X_{V^{-1}}$. The symmetric representation of S

$$V^{-1}(\tfrac{1}{2}\mathrm{I} + K) = S = (\tfrac{1}{2}\mathrm{I} + K')V^{-1}(\tfrac{1}{2}\mathrm{I} + K) + W$$

shows that $B = B^2 + A$, $A \geq c_0 I > 0$ with

$$c_0 = \inf_{v \in X_0} \frac{\langle v, Wv \rangle}{\langle v, V^{-1}v \rangle}$$

Hence $\|B\|_{V^{-1}} \leq \frac{1}{2} + \sqrt{\frac{1}{4} - c_0} < 1$

The contraction constant (Steinbach-Wendland estimate)

- If $a, b \in \mathbb{R}$ and $b \geq b^2 + a$ and $a > 0$, then

$$\frac{1}{2} - \sqrt{\frac{1}{4} - a} \leq b \leq \frac{1}{2} + \sqrt{\frac{1}{4} - a} < 1$$

- If A, B are bounded selfadjoint operators and $B = B^2 + A$ and $A \geq aI > 0$, then

$$\|B\| \leq \frac{1}{2} + \sqrt{\frac{1}{4} - a} < 1$$

- Let $B = \frac{1}{2}\mathrm{I} + K$ in $X_{V^{-1}}$. The symmetric representation of S

$$V^{-1}(\tfrac{1}{2}\mathrm{I} + K) = S = (\tfrac{1}{2}\mathrm{I} + K')V^{-1}(\tfrac{1}{2}\mathrm{I} + K) + W$$

shows that $B = B^2 + A$, $A \geq c_0 I > 0$ with

$$c_0 = \inf_{v \in X_0} \frac{\langle v, Wv \rangle}{\langle v, V^{-1}v \rangle}$$

Hence $\|B\|_{V^{-1}} \leq \frac{1}{2} + \sqrt{\frac{1}{4} - c_0} < 1$

The contraction constant (Steinbach-Wendland estimate)

- If $a, b \in \mathbb{R}$ and $b \geq b^2 + a$ and $a > 0$, then

$$\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - a} \leq b \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - a} < 1$$

- If A, B are bounded selfadjoint operators and $B = B^2 + A$ and $A \geq aI > 0$, then

$$\|B\| \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - a} < 1$$

- Let $B = \frac{1}{2}\mathrm{I} + K$ in $X_{V^{-1}}$. The symmetric representation of S

$$V^{-1}(\tfrac{1}{2}\mathrm{I} + K) = S = (\tfrac{1}{2}\mathrm{I} + K')V^{-1}(\tfrac{1}{2}\mathrm{I} + K) + W$$

shows that $B = B^2 + A$, $A \geq c_0 I > 0$ with

$$c_0 = \inf_{v \in X_0} \frac{\langle v, Wv \rangle}{\langle v, V^{-1} v \rangle}$$

Hence $\|B\|_{V^{-1}} \leq \frac{1}{2} + \sqrt{\frac{1}{4} - c_0} < 1$

H. POINCARÉ.
La méthode de Neumann et le problème de Dirichlet.
Acta Math. **20** (1896) 59–142.

W. STEKLOFF.
Les méthodes générales pour résoudre les problèmes fondamentaux.
Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (2) **2**(2) (1900) 207–272.

M. COSTABEL, M. DAUGE.
Invertibility of the biharmonic single layer potential operator.
Integral Equations Operator Theory **24**(1) (1996) 46–67.

G. SCHMIDT, B. N. KHOROMSKIJ.
Boundary integral equations for the biharmonic Dirichlet problem on nonsmooth domains.
J. Integral Equations Appl. **11**(2) (1999) 217–253.

O. STEINBACH, W. L. WENDLAND.
On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries.
J. Math. Anal. Appl. **262** (2001) 733–748.