Algebraic convergence for anisotropic edge elements in polyhedral domains

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Abstract

We study approximation errors for the h-version of Nédélec edge elements on anisotropically refined meshes in polyhedra. Both tetrahedral and hexahedral elements are considered, and the emphasis is on obtaining optimal convergence rates in the H(curl) norm for higher order elements. Two types of estimates are presented: First, *interpolation* error estimates for functions in anisotropic weighted Sobolev spaces. Here we consider not only the H(curl)-conforming Nédélec elements, but also the H(div)-conforming Raviart-Thomas elements which appear naturally in the discrete version of the de Rham complex. Our technique is to transport error estimates from the reference element to the physical element via highly anisotropic coordinate transformations. Second, *Galerkin* error estimates for the standard H(curl) approximation of time harmonic Maxwell equations. Here we use the anisotropic weighted Sobolev regularity of the solution on domains with three-dimensional edges and corners. We also prove the discrete compactness property needed for the convergence of the Maxwell eigenvalue problem. Our results generalize those of [40] to the case of polyhedral corners and higher order elements.

1 Introduction

Let Ω be a three-dimensional bounded domain filled with an isotropic, homogeneous material whose magnetic permeability and electric permittivity are given by positive constants μ and ε. In this work, we are interested in the approximation of solutions of the time-harmonic Maxwell equations in Ω by finite elements, in the situation where Ω is a Lipschitz polyhedron. Such a situation is very natural from the practical point of view, but is the cause of important difficulties, and even sometimes obstructions to a converging approximation. The main reason for this is the very poor regularity of solutions when Ω has non-convex edges (and corners).

Let λ be a fixed non-zero complex number. The time-harmonic Maxwell equations for the electric field can be written as:

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{u} - \lambda\varepsilon\mathbf{u} = \mathbf{f} \quad \Omega \tag{1}$$

$$\mathbf{u}\times\mathbf{n} = 0 \quad \partial\Omega. \tag{2}$$

Here $\mathbf{f}$ is the current density which we assume to belong to $\mathsf{L}^2(\Omega)$ and to satisfy $\operatorname{div}\mathbf{f} = 0$.

Note that when λ has a positive real part, (1) corresponds to the problem of electromagnetic wave propagation in a dielectric conducting (when $\lambda \in \mathbb{C}\setminus\mathbb{R}$) or non-conducting (when $\lambda\in\mathbb{R}$) medium. When the real part of λ is negative, the problem (1) can be seen as the stationary problem which has to be solved at each time step of the implicit time discretization of the electromagnetic time-domain wave problem in a dielectric medium (see, e.g., [20]) The corresponding variational space is $\mathbf{H}_0(\mathbf{curl},\Omega)$ defined as

$$\mathbf{H}_0(\mathbf{curl},\Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega)\ :\ \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega)\,,\ \mathbf{u}\times\mathbf{n}\big|_{\partial\Omega} = 0\},$$

endowed with the natural graph norm:

$$\|\mathbf{u}\|_X = \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}.$$

A classical variational formulation of problem (1)-(2) is:[1]

Find $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl},\Omega)$ *such that:*

$$\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} - \lambda\mathbf{u}\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in \mathbf{H}_0(\mathbf{curl},\Omega). \tag{3}$$

The following theorem is now well known ([37], [18]):

Theorem 1.1 *The problem* (3) *is well-posed for all* $\lambda \in \mathbb{C}$ except *for* λ *belonging to a non-negative increasing sequence of eigenvalues* $\{\lambda_j\}_{j\in\mathbb{N}}$, $\lambda_0 = 0$, $\lambda_j < \lambda_{j+1}$. *Moreover, all positive eigenvalues have finite multiplicity.*

Thus, for $\lambda \notin \{\lambda_j\}_{j\in\mathbb{N}}$ we are in the standard framework for well-posed variational problems. Such problems can in principle be approached by sequences of Galerkin approximations. But, when λ is positive, a generic difficulty is the infinite dimensional kernel (the eigenvalue λ_0), and this adds to the problem of the low regularity that we mentioned at the beginning: When Ω has non-convex edges (and corners), in general $\mathbf{u}$ is not even in $\mathbf{H}^1(\Omega)$, see [9, 21].

We know two main strategies to overcome these two categories of difficulties: *(i)* The use of special families of elements satisfying specific commuting diagram properties and the discrete compactness property, see [35], [10], [16], *(ii)* The regularization with weight, see [22, 25].

In this paper, we address strategy *(i)*, which is widely spread in practice, with the families of NÉDÉLEC edge elements, see [38], [20], [3]. Because of the poor regularity of the solution $\mathbf{u}$, the convergence rate is very low, and we loose all benefit of the use of higher degree elements.

In polygons, and for standard operators like the Laplacian, algebraic refinement towards the corners is a remedy and restores the optimal rate of convergence. Such a method makes use of shape-regular elements the size of which is adapted to the region were they are situated, without enlarging the total number of elements, and keeping the assumption that the mesh is regular. The pioneering work in this direction is [8] (see [43] for a list on references on the subject).

Now, in three dimensional domains, the refinement has to be performed towards edges and corners, which is much more difficult to do. The possibility of using anisotropic elements can make the design of the mesh easier, lower the number of elements and take advantage of the best regularity properties of the solution: In fact solutions of problems like the Dirichlet problem for the Laplace operator have more regularity in the direction of the edges than transversally to them.

The literature on anisotropic finite elements is nowadays rich and split basically into two categories: *(i)* the analysis of approximation properties of such elements for "regular" solutions, but under minimal requirements on the mesh (see [36], [2], [29], [1]); *(ii)* the analysis of approximation properties of "singular" solutions on suitably designed meshes, and with the aim of recovering algebraic convergence for finite elements of any order. This approach goes back to [8] and was developed in dimension three for low order Lagrangian finite elements in [4] (see also [7]), and

[1] We assume without restriction that $\varepsilon\mu = 1$.

partially extended to low order finite element approximation for vector problems in [30], [40] under restrictive assumptions on the geometry of the polyhedron Ω.

In this paper, we adopt the second point of view and combine the idea of anisotropic refined meshes with the use of edge elements to design algebraically optimal Galerkin methods for the electric Maxwell solution $\mathbf{u}$, in the sense of the following definition:

Definition 1.2 Let $\{X_h\}_{h\in\mathfrak{h}}$ be a family of finite element spaces, $X_h \subset \mathbf{H}_0(\mathbf{curl},\Omega)$, characterized by a family of triangulations $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ with $h \to 0$, a degree k for the spaces of polynomials and a set of degrees of freedom $\mathcal{D}$. We assume that 0 is the only accumulation point of $\mathfrak{h}$ and that the number of degrees of freedom $\dim X_h$ is $O(h^{-3})$.
We say that the Galerkin method based on the family $\{X_h\}_{h\in\mathfrak{h}}$ is *algebraically optimal of degree* k if the Galerkin projection $\mathbf{u}_h \in X_h$ of $\mathbf{u}$ satisfies for sufficiently regular right hand side:

$$\|\mathbf{u}-\mathbf{u}_h\|_X \leq C_k h^k$$

where C_k is a constant depending only on k. In terms of the number of degrees of freedom N, this means: $\|\mathbf{u}-\mathbf{u}_h\|_X \leq C'_k N^{-k/3}$. $\square$

In this paper, the finite element spaces X_h are realized by taking the tetrahedral or hexahedral edge elements of order k, $k = 1, 2$,...

The paper is structured in such a way that the different steps of a Galerkin error estimate are as clearly separated as possible. The plan is as follows: In Section 2, we consider only the reference element $\widehat{K}$. We recall the definitions of the reference spaces and degrees of freedom and prove interpolation error estimates, some of them new, in particular for tetrahedral elements, where we make use of the commuting diagram property.

In Section 3, we consider individual physical elements K. We define anisotropic elements by the introduction of admissible mappings, compatible with the Piola transformation, and we prove elementwise estimates for the projection operators. It is worth noting that, in contrast with anisotropic hexahedra where they are optimal, the local approximation properties of the edge-element interpolation operator are somewhat less satisfactory for anisotropic tetrahedra. This is due to the fact that, in the case of tetrahedra, the degrees of freedom have no tensor product structure and generate interpolation operators which do not commute with the projections on cartesian coordinates, which results in the appearance of aspect ratios in the estimates.

In Section 4 we state regularity results in anisotropic weighted spaces for our solution $\mathbf{u}$ in the form of a decomposition $\mathbf{u} = \mathbf{u}_0 + \nabla q$ with a more regular $\mathbf{u}_0$. In Section 5, we formulate the assumptions on the family of meshes $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ and we prove global interpolation error estimates. These imply the algebraic optimality under the condition that there holds a Céa-type estimate for our problem: i.e. that $\|\mathbf{u}-\mathbf{u}_h\|_X \lesssim \inf_{\mathbf{v}_h\in X_h} \|\mathbf{u}-\mathbf{v}_h\|_X$. In Section 6, we investigate the questions of over-refinement of the mesh and of piecewise heterogeneous materials. We prove generalization of the results of the previous section.

The Céa-type estimate holds when the variational formulation (3) is based on a *strongly coercive bilinear form on* X, which is the case as soon as λ is not a positive real number. When λ is positive, but different from the Maxwell eigenvalules λ_j, the obtention of the Céa-type estimate is subject to the validity of a *discrete compactness property*. The possibility of a correct approximation of eigenvalues is subject to the same condition. We investigate this discrete compactness property in Section 7. We prove it for hexahedral elements of any degree and for tetrahedral elements of lowest order 1. We rely on a splitting of the fields in $\mathbf{u}_0 + \nabla q$ and we use results of algebraic optimality for *scalar* interpolants [5, 7]. Finally, we draw conclusions in Section 8.

2 Edge elements

In this section we recall the definitions of the reference polynomial spaces on a tetrahedral or an hexahedral element, together with their associated degrees of freedom, in relation with the commuting diagram property. The reference hexahedral element $\widehat{K}$ is a cube and the spaces and degrees of freedom have a tensor product structure and, as a consequence, the projection operator is diagonal in the canonical basis. Such features do not hold for tetrahedra in general, but we prove in this section that, when restricted to subspaces of gradient fields or curl fields, the projection operators act componentwise. This fact will help for estimates on anisotropic elements, see §3.2 and 3.3.

2.1 Edge elements on the reference element: definitions

We denote by $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = \widehat{\mathbf{x}}$ the coordinates in $\mathbb{R}^3$ and by $\widehat{\mathbf{e}}_1$, $\widehat{\mathbf{e}}_2$ and $\widehat{\mathbf{e}}_3$ the canonical basis in $\mathbb{R}^3$, so that $\widehat{\mathbf{x}} = \widehat{x}_1\widehat{\mathbf{e}}_1 + \widehat{x}_2\widehat{\mathbf{e}}_2 + \widehat{x}_3\widehat{\mathbf{e}}_3$. The reference element $\widehat{K}$ is either the cube $]0,1[^3$ or the tetrahedron $\{(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in]0,1[^3 : \widehat{x}_1 + \widehat{x}_2 + \widehat{x}_3 < 1\}$.

In the sequel, $\mathbb{P}_k$ denotes the space of polynomials of degree $\leq k$ and $\overline{\mathbb{P}}_k$ stands for its subspace of homogeneous polynomials of degree k, whereas $\mathbb{Q}_{i,j,k}$ is the space of polynomials of *partial degrees* $\leq i$, j and k in the three variables $\widehat{x}_1$, $\widehat{x}_2$ and $\widehat{x}_3$ respectively.

The commuting diagram property is related to three families of finite elements, $\mathcal{P}_k$ for the approximation of potentials or divergences, $\mathcal{N}_k$ for the approximation of the electric fields and $\mathcal{R}_k$ for the approximation of their curls (magnetic fields):

1. Standard finite elements:

$$\mathcal{P}_k = \begin{cases} \mathbb{P}_k & \text{when } \widehat{K} \text{ is a tetrahedron} \\ \mathbb{Q}_{k,k,k} & \text{when } \widehat{K} \text{ is a cube.} \end{cases}$$

   On $\mathcal{P}_k$, we denote by $\widehat{\pi}$ the interpolation operator associated to the degrees of freedom:
   (i) values at the vertices,
   (ii) edge moments of order $\leq k-2$,
   (iii) face moments of order $\leq k-3$ (tetrahedron) or $\leq k-2$ (cube),
   (iv) volume moments of order $\leq k-4$ (tetrahedron) or $\leq k-2$ (cube).

   In addition to the interpolation projector $\widehat{\pi}$ we also consider the L^2 projection $\widehat{\pi}^*$ onto the same space $\mathcal{P}_k$.

2. First Nédélec family of edge elements (introduced in [39]):

$$\mathcal{N}_k = \begin{cases} \mathbb{P}^3_{k-1} \oplus \overline{\mathbb{P}}^3_{k-1} \times \widehat{\mathbf{x}} & \text{when } \widehat{K} \text{ is a tetrahedron} \\ \mathbb{Q}_{k-1,k,k} \times \mathbb{Q}_{k,k-1,k} \times \mathbb{Q}_{k,k,k-1} & \text{when } \widehat{K} \text{ is a cube.} \end{cases} \tag{4}$$

   The following degrees of freedom constitute unisolvent sets on $\mathcal{N}_k$:
   For the reference tetrahedron:

$$\begin{array}{lll} 1.\ \int_e (\mathbf{w}\cdot\tau)\, q \,\mathrm{d}e & \forall\, q \in \mathbb{P}_{k-1}(e) & \forall\, e \text{ edge of } \widehat{K} \\ 2.\ \int_f (\mathbf{w}\times\mathbf{n})\cdot\mathbf{q}\,\mathrm{d}f & \forall\, \mathbf{q} \in \mathbb{P}^2_{k-2}(f) & \forall\, f \text{ face of } \widehat{K} \\ 3.\ \int_{\widehat{K}} \mathbf{w}\cdot\mathbf{q}\,\mathrm{d}\widehat{\mathbf{x}} & \forall\, \mathbf{q} \in \mathbb{P}^3_{k-3}(\widehat{K}). & \end{array} \tag{5}$$

For the reference cube:

$$
\begin{array}{lll}
1.\ \int_e (\mathbf{w}\cdot\tau)\, q \,\mathrm{d}e & \forall\, q \in \mathbb{P}_{k-1}(e) \quad \forall\, e \text{ edge of } \widehat{K} \\
2.\ \int_f (\mathbf{w}\times\mathbf{n})\cdot\mathbf{q}\,\mathrm{d}f & \forall\, \mathbf{q} \in \big(\mathbb{Q}_{k-2,k-1}\times\mathbb{Q}_{k-1,k-2}\big)(f) \quad \forall\, f \text{ face of } \widehat{K} \\
3.\ \int_{\widehat{K}} \mathbf{w}\cdot\mathbf{q}\,\mathrm{d}\widehat{\mathbf{x}} & \forall\, \mathbf{q} \in \big(\mathbb{Q}_{k-1,k-2,k-2}\times\mathbb{Q}_{k-2,k-1,k-2}\times\mathbb{Q}_{k-2,k-2,k-1}\big)(\widehat{K}).
\end{array} \tag{6}
$$

We denote by $\widehat{\Pi}$ the associated interpolation operator.

3. Raviart-Thomas finite elements (first introduced in [41], [42], see also [31]):

$$
\mathcal{R}_k = \begin{cases} \mathbb{P}^3_{k-1} \oplus \overline{\mathbb{P}}_{k-1}\widehat{\mathbf{x}} & \text{for the reference tetrahedron,} \\ \mathbb{Q}_{k,k-1,k-1}\times\mathbb{Q}_{k-1,k,k-1}\times\mathbb{Q}_{k-1,k-1,k} & \text{for the reference cube.} \end{cases} \tag{7}
$$

The associated degrees of freedom for the reference tetrahedron are:

$$
\begin{array}{ll}
1.\ \int_f (\mathbf{w}\cdot\mathbf{n})\, q\,\mathrm{d}f & \forall\, q\in\mathbb{P}_{k-1}(f) \quad \forall\, f \text{ face of } \widehat{K} \\
2.\ \int_{\widehat{K}} \mathbf{w}\cdot\mathbf{q}\,\mathrm{d}\widehat{\mathbf{x}} & \forall\, \mathbf{q}\in\mathbb{P}^3_{k-2}(\widehat{K});
\end{array} \tag{8}
$$

For the reference cube:

$$
\begin{array}{ll}
1.\ \int_f (\mathbf{w}\cdot\mathbf{n})\, q\,\mathrm{d}e & \forall\, q\in\mathbb{Q}_{k-1,k-1}(f) \quad \forall\, f \text{ face of } \widehat{K} \\
2.\ \int_{\widehat{K}} \mathbf{w}\cdot\mathbf{q}\,\mathrm{d}\widehat{\mathbf{x}} & \forall\, \mathbf{q}\in\big(\mathbb{Q}_{k-2,k-1,k-1}\times\mathbb{Q}_{k-1,k-2,k-1}\times\mathbb{Q}_{k-1,k-1,k-2}\big)(\widehat{K}).
\end{array} \tag{9}
$$

We denote by $\widehat{R}$ the associated interpolation operator.

It is well known that these elements share the following important property (see e.g., [33], [34], and also [12]).

Theorem 2.1 *Let* $\mathbf{W}^{1,p}(\mathbf{curl},\widehat{K}) = \{\mathbf{w}\in\mathbf{W}^{1,p}(\widehat{K}) \ :\ \mathbf{curl}\,\mathbf{w}\in\mathbf{W}^{1,p}(\widehat{K})\}$ *and* $\mathbf{W}^{1,p}(\mathrm{div},\widehat{K}) = \{\mathbf{w}\in\mathbf{W}^{1,p}(\widehat{K}) \ :\ \mathrm{div}\,\mathbf{w}\in\mathrm{W}^{1,p}(\widehat{K})\}$.

Let $p\geq 2$. *The following diagram commutes (and all operators are continuous):*

$$
\begin{array}{ccccccc}
\mathrm{W}^{2,p}(\widehat{K}) & \xrightarrow{\nabla} & \mathbf{W}^{1,p}(\mathbf{curl},\widehat{K}) & \xrightarrow{\mathbf{curl}} & \mathbf{W}^{1,p}(\mathrm{div},\widehat{K}) & \xrightarrow{\mathrm{div}} & L^p(\widehat{K}) \\
\downarrow \widehat{\pi} & & \downarrow \widehat{\Pi} & & \downarrow \widehat{R} & & \downarrow \widehat{\pi}^\star \\
\mathcal{P}_k & \xrightarrow{\nabla} & \mathcal{N}_k & \xrightarrow{\mathbf{curl}} & \mathcal{R}_k & \xrightarrow{\mathrm{div}} & \mathcal{P}_{k-1}
\end{array}
$$

2.2 Properties of the projection operator for edge elements

This section and the following one are devoted to the properties of $\widehat{\Pi}$ and $\widehat{R}$ on the reference tetrahedron. We prove that there exist projection operators $\widehat{\Pi}_\pi$ and $\widehat{R}_\pi$ on the subspace $\mathbb{P}^3_{k-1}$ of $\mathcal{N}_k$ and $\mathcal{R}_k$, which *(i)* are diagonal in the canonical basis, *(ii)* coincide with $\widehat{\Pi}_\pi$ and $\widehat{R}_\pi$ on the gradients and the curls respectively. Moreover, these operators are used to provide non-standard Bramble-Hilbert estimates which will be useful later.

The tetrahedron $\widehat{K}$ has four faces and six edges. We agree to denote by f_i its face with outer normal $-\widehat{\mathbf{e}}_i$, and by e_i its edge along $\widehat{\mathbf{e}}_i$, for $i=1,2,3$.

Lemma 2.2 *Let* $\widehat{K}$ *be the reference tetrahedron. Let* $k \geq 1$.
(i) The degrees of freedom

$$\begin{array}{llll} 1. \int_{e_i} (\mathbf{v}\cdot\widehat{\mathbf{e}}_i)\, q \,\mathrm{d}e & \forall\, q \in \mathbb{P}_{k-1}(e_i) & i = 1,2,3 & \\ 2. \int_{f_j} (\mathbf{v}\cdot\widehat{\mathbf{e}}_i)\, q \,\mathrm{d}f & \forall\, q \in \mathbb{P}_{k-2}(f_j) & i = 1,2,3\,, & j \neq i \\ 3. \int_{\widehat{K}} (\mathbf{v}\cdot\widehat{\mathbf{e}}_i)\, q \,\mathrm{d}\widehat{\mathbf{x}} & \forall\, q \in \mathbb{P}_{k-3}(\widehat{K}) & i = 1,2,3; & \end{array} \tag{10}$$

are unisolvent on $\mathbb{P}^3_{k-1}(\widehat{K})$ *and define a projector* $\widehat{\Pi}_\pi$ *on* $\mathbf{W}^{1,p}(\widehat{K})$, $p > 2$.
(ii) The operator $\widehat{\Pi}_\pi$ *is diagonal in the basis* $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$, *i.e.*

$$\forall \mathbf{v} = \sum_{j=1}^{3} v_j \widehat{\mathbf{e}}_j, \qquad \widehat{\Pi}_\pi(v_i \widehat{\mathbf{e}}_i) = (\widehat{\Pi}_\pi \mathbf{v})_i\, \widehat{\mathbf{e}}_i, \quad i = 1,2,3.$$

Remark 2.3 The point *(ii)* can be formulated equivalently by asserting that the coordinate projections $\pi_i : \sum_{j=1}^3 v_j \widehat{\mathbf{e}}_j \mapsto v_i \widehat{\mathbf{e}}_i$ *commute* with $\widehat{\Pi}_\pi$, for $i = 1,2,3$. □

Proof: (i) The number of degrees of freedom in (10) is $3(k + k(k-1) + \frac{1}{6}k(k-1)(k-2))$ which is equal to $3(\frac{1}{6}k(k+1)(k+2))$ which is the dimension of $\mathbb{P}^3_{k-1}(\widehat{K})$. To prove that these degrees of freedom are unisolvent, it suffices to prove that if $\mathbf{v}$ belongs to $\mathbb{P}^3_{k-1}(\widehat{K})$ and has all its degrees (10) zero, then $\mathbf{v} = 0$. Let v_1 be the first component $\mathbf{v}\cdot\widehat{\mathbf{e}}_1$ of $\mathbf{v}$. We have

$$\begin{array}{ll} 1. \int_{e_1} v_1\, q \,\mathrm{d}e = 0 & \forall\, q \in \mathbb{P}_{k-1}(e_1) \\ 2. \int_{f_j} v_1\, q \,\mathrm{d}f = 0 & \forall\, q \in \mathbb{P}_{k-2}(f_j) \quad \forall\, j = 2,3 \\ 3. \int_{\widehat{K}} v_1\, q \,\mathrm{d}\widehat{\mathbf{x}} = 0 & \forall\, q \in \mathbb{P}_{k-3}(\widehat{K}). \end{array}$$

Condition 1. with $q = v_1|_{e_1}$, gives that $v_1|_{e_1} = 0$. Condition 2. with $q = \partial_1 v_1$ gives after integration by parts

$$\int_{\partial f_3} v_1^2\, (\mathbf{n}_{f_3}\cdot\widehat{\mathbf{e}}_1)\,\mathrm{d}f = 0, \qquad \int_{\partial f_2} v_1^2\, (\mathbf{n}_{f_2}\cdot\widehat{\mathbf{e}}_1)\,\mathrm{d}f = 0, \tag{11}$$

where $\mathbf{n}_{f_j}$ denotes the outer normal to the face f_j in the plane of f_j. Call e_{f_j} the only edge of $\widehat{K}$ belonging to ∂f_j which does not coincide with e_i for any $i = 1,2,3$. Then (11) implies that $v_1 = 0$ on e_{f_2} and e_{f_3}. Condition 2. with $q = \partial_2 v_1$ implies $\int_{\partial f_3} v_1^2(\mathbf{n}_{f_3}\cdot\widehat{\mathbf{e}}_2)\,\mathrm{d}f = 0$, which means $v_1 = 0$ on e_2. Again, using $q = \partial_3 v_1$, we have $\int_{\partial f_2} v_1^2(\mathbf{n}_{f_2}\cdot\widehat{\mathbf{e}}_3)\,\mathrm{d}f = 0$, which implies $v_1 = 0$ on e_3. Finally, testing with $q = \partial_1^2 v_1$ and $q = \partial_2^2 v_1$ on f_3, and with $q = \partial_1^2 v_1$ and $q = \partial_3^2 v_1$ on f_2, we deduce easily that $v_1 = 0$ on f_2 and f_3.

Let now λ_i, $i = 1,2,3,4$ be the barycentric coordinates associated to $\widehat{K}$. Then, since $v_1 = 0$ on f_2 and f_3, there exists a $\psi \in \mathbb{P}_{k-3}(\widehat{K})$ such that $v_1 = \lambda_3\lambda_2\psi$. Choosing $q = \psi$ in Condition 3. we obtain: $\int_{\widehat{K}} \lambda_3\lambda_2|\psi|^2\mathrm{d}\widehat{\mathbf{x}} = 0$, which implies $\psi = 0$. Therefore $v_1 = 0$ and, of course, we obtain in the same way that $v_i = 0$, $i = 2,3$.

(ii) We set $\widehat{\Pi}_\pi(v_i\widehat{\mathbf{e}}_i) = \boldsymbol{\varphi}_i$ with $\boldsymbol{\varphi}_i \in \mathbb{P}^3_{k-1}(\widehat{K})$. By (10) $\boldsymbol{\varphi}_i = (\varphi_{i1}, \varphi_{i2}, \varphi_{i3})$ verifies:

$$\text{for every } j\,,\ j \neq i: \quad \left\{ \begin{array}{ll} \int_{e_j} \varphi_{ij}\, q \,\mathrm{d}e = 0 & \forall\, q \in \mathbb{P}_{k-1}(e_j) \\ \int_{f_\ell} \varphi_{ij}\, q \,\mathrm{d}f = 0 & \forall\, q \in \mathbb{P}_{k-2}(f_\ell) \quad j \neq \ell\,, \\ \int_{\widehat{K}} \varphi_{ij}\, q \,\mathrm{d}\widehat{\mathbf{x}} = 0 & \forall\, q \in \mathbb{P}_{k-3}(\widehat{K}) \end{array} \right.$$

This implies that φ_{ij} for $j \neq i$ verifies $k + k(k-1) + \frac{1}{6}k(k-1)(k-2)$ constraints which are independent since (10) are unisolvent degrees of freedom. Since $\varphi_{ij} \in \mathbb{P}_{k-1}(\widehat{K})$ and the dimension of $\mathbb{P}_{k-1}(\widehat{K})$ is equal to $\frac{1}{6}k(k+1)(k+2)$, i.e. to the number of independent constraints, $\varphi_{ij} = 0$ for all $j \neq i$. We have proved that $\widehat{\Pi}_\pi$ is diagonal. □

Proposition 2.4 *Let $\widehat{K}$ be the reference tetrahedron. The interpolation operator $\widehat{\Pi}$ coincides with $\widehat{\Pi}_\pi$ on $\mathbf{W}^{1,p} \cap \ker\{\mathbf{curl}\}$.*

Proof: Let $\mathbf{v} \in \mathbf{W}^{1,p} \cap \ker\{\mathbf{curl}\}$. There exists $q \in \mathbf{W}^{2,p}(\widehat{K})$ such that $\nabla q = \mathbf{v}$. Then the commuting diagram property yields that $\widehat{\Pi}\,\mathbf{v} = \nabla \widehat{\pi} q$. Since $\widehat{\pi} q \in \mathbb{P}_k(\widehat{K})$, we deduce that $\widehat{\Pi}\mathbf{v} \in \mathbb{P}^3_{k-1}(\widehat{K})$. Thus, $\widehat{\Pi}\,\mathbf{v} = \widehat{\Pi}_\pi \widehat{\Pi}\,\mathbf{v}$. By virtue of the following lemma, $\widehat{\Pi}_\pi \widehat{\Pi} = \widehat{\Pi}_\pi$. Therefore $\widehat{\Pi}\,\mathbf{v} = \widehat{\Pi}_\pi \mathbf{v}$. □

Lemma 2.5 *Let Π_A and Π_B be two projection operators $H \to A$ and $H \to B$, with $A \subset B$, defined by two spaces of degrees of freedom $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$. Then $\Pi_A \Pi_B = \Pi_A$.*

Proof: Let $u \in H$. The degrees of freedom in $\mathcal{B}$ are zero on $\Pi_B u - u$. Since they contain those of $\mathcal{A}$, $\Pi_A(\Pi_B u - u) = 0$. □

The operator $\widehat{\Pi}$ is then split as $\widehat{\Pi} = \widehat{\Pi}_\pi + (\widehat{\Pi} - \widehat{\Pi}_\pi)$. This allows to obtain a non-standard Bramble Hilbert estimate for low order edge elements $\mathcal{N}_1$:

Lemma 2.6 *Let $k = 1$ and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{W}^{1,p}(\mathbf{curl}, \widehat{K})$, $p > 2$, such that $\mathbf{curl}\,\mathbf{v} \in \mathbb{R}^3$. The following estimate holds:*

$$\|v_i - (\widehat{\Pi}\mathbf{v})_i\|_{\mathbf{L}^p(\widehat{K})} \lesssim \|\nabla v_i\|_{\mathbf{L}^p(\widehat{K})} + \|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^p(\widehat{K})}. \tag{12}$$

Proof: Let $\mathbf{v}$ be a vector field verifying the assumption of the Lemma. We have:

$$v_i - (\widehat{\Pi}\mathbf{v})_i = v_i - (\widehat{\Pi}_\pi \mathbf{v})_i - \big((\widehat{\Pi}\mathbf{v})_i - (\widehat{\Pi}_\pi \mathbf{v})_i\big). \tag{13}$$

Now, by construction $\widehat{\Pi}\mathbf{v} - \widehat{\Pi}_\pi \mathbf{v} = \mathbf{b} \times \mathbf{x}$, for some $\mathbf{b} \in \mathbb{R}^3$, and $\mathbf{curl}(\widehat{\Pi}\mathbf{v} - \widehat{\Pi}_\pi \mathbf{v}) = \mathbf{curl}\,\widehat{\Pi}\mathbf{v} = \mathbf{curl}\,\mathbf{v} = 5\mathbf{b}$ since $\widehat{\Pi}_\pi \mathbf{v}$ is a constant vector of $\mathbb{R}^3$. This gives the estimate:

$$\|(\widehat{\Pi}\mathbf{v})_i - (\widehat{\Pi}_\pi \mathbf{v})_i\|_{L^p(\widehat{K})} \leq \|\widehat{\Pi}\mathbf{v} - \widehat{\Pi}_\pi \mathbf{v}\|_{\mathbf{L}^p(\widehat{K})} \lesssim \|\,\mathbf{curl}\,\mathbf{v}\|_{\mathbf{L}^p(\widehat{K})}.$$

On the other hand, since $\widehat{\Pi}_\pi$ is diagonal and reproduces constants, the standard Bramble-Hilbert estimate, see [19], gives:

$$\|v_i - (\widehat{\Pi}_\pi \mathbf{v})_i\|_{\mathbf{L}^p(\widehat{K})} \lesssim \|\nabla v_i\|_{\mathbf{L}^p(\widehat{K})}.$$

Inserting these last two estimates into (13), we obtain (12). □

2.3 Properties of the projection operator for Raviart-Thomas elements

We now concentrate on the properties of the Raviart-Thomas interpolation operator.

Lemma 2.7 *Let* $\widehat{K}$ *be the reference tetrahedron. Let* $k \geq 1$.
(i) The degrees of freedom

$$\begin{aligned} &1.\ \int_{f_i} (\boldsymbol{\xi} \cdot \widehat{\mathbf{e}}_i)\, q \,\mathrm{d}f && \forall\, q \in \mathbb{P}_{k-1}(f_i) \quad \forall\, i = 1,2,3 \\ &2.\ \int_{\widehat{K}} \boldsymbol{\xi} \cdot \mathbf{q} \,\mathrm{d}\widehat{\mathbf{x}} && \forall\, \mathbf{q} \in \mathbb{P}^3_{k-2}(\widehat{K}); \end{aligned} \tag{14}$$

are unisolvent on $\mathbb{P}^3_{k-1}(\widehat{K})$ *and define a projector* $\widehat{R}_\pi$ *on* $\mathbf{W}^{1,p}(\widehat{K})$, $p \geq 2$.
(ii) The operator $\widehat{R}_\pi$ *is diagonal in the basis* $(\widehat{\mathbf{e}}_1, \widehat{\mathbf{e}}_2, \widehat{\mathbf{e}}_3)$, *i.e.*

$$\forall \boldsymbol{\xi} = \sum_{j=1}^{3} \xi_j \widehat{\mathbf{e}}_j, \qquad \widehat{R}_\pi(\xi_i \widehat{\mathbf{e}}_i) = (\widehat{R}_\pi \boldsymbol{\xi})_i\, \widehat{\mathbf{e}}_i, \quad i = 1,2,3.$$

Proof: (i) The number of degrees of freedom in (14) is $3\big(\frac{1}{2}k(k+1) + \frac{1}{6}(k-1)k(k+1)\big)$ which is equal to $3\big(\frac{1}{6}k(k+1)(k+2)\big)$, the dimension of $\mathbb{P}^3_{k-1}(\widehat{K})$. To prove that these degrees of freedom are unisolvent, it suffices to prove that if $\boldsymbol{\xi}$ belongs to $\mathbb{P}^3_{k-1}(\widehat{K})$ and has all its degrees (14) zero, then $\boldsymbol{\xi} = 0$. Let us fix i and let ξ_i be the i-th component $\boldsymbol{\xi} \cdot \widehat{\mathbf{e}}_i$ of $\boldsymbol{\xi}$. We have

$$\begin{aligned} &1.\ \int_{f_i} \xi_i\, q \,\mathrm{d}f = 0 && \forall\, q \in \mathbb{P}_{k-1}(f_i) \quad \forall\, i = 1,2,3 \\ &2.\ \int_{\widehat{K}} \xi_i\, q \,\mathrm{d}\widehat{\mathbf{x}} = 0 && \forall\, q \in \mathbb{P}_{k-2}(\widehat{K}). \end{aligned}$$

Condition 1. with $q = \xi_i|_{f_i}$ gives that $\xi_i|_{f_i} = 0$. Condition 2. with $q = \partial_i \xi_i$ gives after integration by parts

$$\int_{\partial \widehat{K}} v_i^2 (\mathbf{n} \cdot \widehat{\mathbf{e}}_i) \,\mathrm{d}f = 0.$$

The contributions on the faces f_j with $j \neq i$ are zero because $\mathbf{n} \cdot \widehat{\mathbf{e}}_i = 0$. The contribution on f_i is also zero as just proved. Therefore, since $\mathbf{n} \cdot \widehat{\mathbf{e}}_i$ is a non-zero constant on the last face f_4 of $\widehat{K}$, we find that $\xi_i|_{f_4} = 0$. We deduce that $\boldsymbol{\xi}$ cancels all Raviart-Thomas degrees of freedom (8). Therefore $\boldsymbol{\xi} = 0$.

(ii) We set $\widehat{R}_\pi \boldsymbol{\xi}_i = \boldsymbol{\varphi}_i$ with $\boldsymbol{\varphi}_i \in \mathbb{P}^3_{k-1}$. By (14) $\boldsymbol{\varphi}_i = (\varphi_{i1}, \varphi_{i2}, \varphi_{i3})$ verifies:

$$\int_{f_j} \varphi_{ij}\, q = 0 \quad \forall\, q \in \mathbb{P}_{k-1}(f_j) \quad \text{and} \quad \int_{\widehat{K}} \varphi_{ij}\, q = 0 \quad \forall\, q \in \mathbb{P}_{k-2}(\widehat{K}), \quad \forall\, j \neq i.$$

This implies that φ_{ij} for $j \neq i$ verifies $\frac{1}{6}k(k+1)(k+2)$ constraints which are independent since (14) are unisolvent degrees of freedom. Since $\varphi_{ij} \in \mathbb{P}_{k-1}(\widehat{K})$ and the dimension of $\mathbb{P}_{k-1}(\widehat{K})$ is equal to the number of independent constraints, $\varphi_{ij} = 0$ for all $j \neq i$. We have proved that $\widehat{R}_\pi$ is diagonal. $\Box$

Proposition 2.8 *Let* $\widehat{K}$ *be the reference tetrahedron. The interpolation operator* $\widehat{R}$ *coincides with* $\widehat{R}_\pi$ *on* $\mathbf{W}^{1,p}(\widehat{K}) \cap \mathrm{Ker}\{\mathrm{div}\}$.

Proof: Let $\boldsymbol{\xi}$ be such that $\mathrm{div}\, \boldsymbol{\xi} = 0$. Therefore $\boldsymbol{\xi}$ is a curl and by the commutative diagram property (33), $\mathrm{div}\, \widehat{R}\boldsymbol{\xi} = 0$. We have $\widehat{R}\boldsymbol{\xi} = \boldsymbol{\varphi} + \psi \widehat{\mathbf{x}}$ with $\boldsymbol{\varphi} \in \mathbb{P}^3_{k-1}(\widehat{K})$ and $\psi \in \overline{\mathbb{P}}_{k-1}(\widehat{K})$. Thus

$$\mathrm{div}\, \widehat{R}\boldsymbol{\xi} = \mathrm{div}\, \boldsymbol{\varphi} + 3\psi + \widehat{\mathbf{x}} \cdot \widehat{\nabla}\psi = \mathrm{div}\, \boldsymbol{\varphi} + (k+2)\psi.$$

As the degree of $\mathrm{div}\, \boldsymbol{\varphi}$ is $k-2$, we find that $\psi = 0$. Therefore $\widehat{R}\boldsymbol{\xi} \in \mathbb{P}^3_{k-1}(\widehat{K})$ and $\widehat{R}\boldsymbol{\xi} = \widehat{R}_\pi \widehat{R}\boldsymbol{\xi}$. By Lemma 2.5 we deduce that $\widehat{R}\boldsymbol{\xi} = \widehat{R}_\pi \boldsymbol{\xi}$. $\Box$

The analogue Lemma to 2.6 for Raviart-Thomas elements is the following:

Lemma 2.9 *Let* $k=1$, *and* $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)\in \mathbf{W}^{1,p}(\mathrm{div},\widehat{K})$, $p\geq 2$. *The following estimate holds:*

$$\|\xi_i-(\widehat{R}\boldsymbol{\xi})_i\|_{\mathbf{L}^p(\widehat{K})}\lesssim \|\nabla\xi_i\|_{\mathbf{L}^p(\widehat{K})}+\|\operatorname{div}\boldsymbol{\xi}\|_{\mathbf{L}^2(\widehat{K})}. \tag{15}$$

Proof: Let $\boldsymbol{\xi}$ be a vector field verifying the assumption of the Lemma. We have:

$$\xi_i-(\widehat{R}\boldsymbol{\xi})_i=\xi_i-(\widehat{R}_\pi\boldsymbol{\xi})_i-\big((\widehat{R}\boldsymbol{\xi})_i-(\widehat{R}_\pi\boldsymbol{\xi})_i\big). \tag{16}$$

Now, by construction $\widehat{R}\boldsymbol{\xi}-\widehat{R}_\pi\boldsymbol{\xi}=b\,\mathbf{x}$, for some $b\in\mathbb{R}$, and $\operatorname{div}(\widehat{R}\boldsymbol{\xi}-\widehat{R}_\pi\boldsymbol{\xi})=\operatorname{div}\widehat{R}\boldsymbol{\xi}=3\,b$ since $\widehat{R}_\pi\boldsymbol{\xi}$ is a constant vector of $\mathbb{R}^3$. Finally, by Theorem 2.1, $\operatorname{div}\widehat{R}\boldsymbol{\xi}=\widehat{\pi}^\star\operatorname{div}\boldsymbol{\xi}$, thus $\|\operatorname{div}\widehat{R}\boldsymbol{\xi}\|_{L^p(\widehat{K})}\lesssim\|\operatorname{div}\widehat{R}\boldsymbol{\xi}\|_{L^2(\widehat{K})}\lesssim\|\operatorname{div}\boldsymbol{\xi}\|_{L^p(\widehat{K})}$. This gives the estimate:

$$\|(\widehat{R}\boldsymbol{\xi})_i-(\widehat{R}_\pi\boldsymbol{\xi})_i\|_{L^p(\widehat{K})}\leq\|\widehat{R}\boldsymbol{\xi}-\widehat{R}_\pi\boldsymbol{\xi}\|_{\mathbf{L}^p(\widehat{K})}\lesssim\|\operatorname{div}\boldsymbol{\xi}\|_{\mathbf{L}^p(\widehat{K})}.$$

On the other hand, since $\widehat{R}_\pi$ is diagonal and reproduces constants, the Bramble-Hilbert estimate gives:

$$\|\xi_i-(\widehat{R}_\pi\boldsymbol{\xi})_i\|_{\mathbf{L}^p(\widehat{K})}\lesssim\|\nabla\xi_i\|_{\mathbf{L}^p(\widehat{K})}.$$

Inserting these last two estimates into (16), we obtain (15). □

Remark 2.10 *Lemma* 2.9 *seems more powerful than Lemma* 2.6 *since it is valid for any vector field* $\boldsymbol{\xi}\in\mathbf{W}^{1,p}(\mathrm{div},\widehat{K})$, *while Lemma* 2.6 *holds only for vectors with constant curl. This is a consequence of the fact that, along the vertical arrows of the commuting diagram in Theorem* 2.1, *the only projection operator which is stable in* L^2 *is* $\widehat{\pi}^\star$. *On the other hand, the same techniques can not be applied for* $k\geq 2$ *since in this case* $\widehat{\Pi}_\pi\mathbf{v}$ *and* $\widehat{R}_\pi\boldsymbol{\xi}$ *are not curl and divergence free respectively.*

3 Anisotropic elementwise estimates

In this section we concentrate on the local approximation properties of edge elements on a physical element K obtained by an affine transformation from a reference cube or tetrahedron $\widehat{K}$.

More precisely, in Section 3.1 we fix our requirements on the mapping Φ_K from $\widehat{K}$ to the physical elements. We authorize K to be stretched in one or two directions, but we need to guarantee some "non-degeneracy" conditions which are discussed. Next, in Sections 3.2 and 3.3 we prove local $\mathrm{L}^p(K)$ error estimates for the interpolation operator in the physical element for the electric field and its curl.

3.1 Mapping to the physical element

Let K be the generic element of a mesh $\mathscr{T}_h$ in a family $(\mathscr{T}_h)_{h\in\mathfrak{h}}$, and $\widehat{K}$ be the reference element. We assume that every K is obtained from $\widehat{K}$ by means of an **affine** and invertible mapping $\Phi_K:\widehat{K}\to K$. Let us recall the induced mappings for the spaces in the deRham commuting diagram, *cf* Theorem 2.1:

- The scalar functions $\widehat{q}$ in $H^1(\widehat{K})$ and in the reference space $\mathcal{P}_k$ are mapped simply by

$$q\circ\Phi_K=\widehat{q}.$$

- The vector fields $\widehat{\mathbf{v}}$ in $\mathbf{H}(\mathbf{curl}, \widehat{K})$ and $\mathcal{N}_k$ are mapped as 1-forms:

$$\widehat{\mathbf{v}} = D\Phi_K^\top(\mathbf{v} \circ \Phi_K), \quad \text{i.e.} \quad \mathbf{v} \circ \Phi_K = D\Phi_K^{-1,\top} \widehat{\mathbf{v}}. \tag{17}$$

- The vector fields $\widehat{\boldsymbol{\xi}}$ in $\mathbf{H}(\mathrm{div}, \widehat{K})$ and $\mathcal{R}_k$ are mapped as 2-forms, i.e., by means of the Piola mapping [12], [31]:

$$\boldsymbol{\xi} \circ \Phi_K = (\det D\Phi_K)^{-1} \, D\Phi_K \, \widehat{\boldsymbol{\xi}}. \tag{18}$$

These choices ensure that the commuting diagram property formulated in Theorem 2.1 still holds for the corresponding space on the physical domains (see [34] and references therein).

Related to its situation in Ω (namely the proximity of an edge), the element K is associated with a local system of Cartesian coordinates $\mathbf{x}^K = (x_1^K, x_2^K, x_3^K)$ and $\Phi_K : \widehat{\mathbf{x}} \mapsto \mathbf{x}^K$. The template for our anisotropic elements is obtained by the mapping $\Phi_K(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = (x_1^K, x_2^K, x_3^K) = (d_1\widehat{x}_1, d_2\widehat{x}_2, d_3\widehat{x}_3)$, where d_1, d_2 and d_3 are the characteristic dimensions of K.

In general, the mapping Φ_K can be represented as $\Phi_K\widehat{\mathbf{x}} = D\Phi_K\widehat{\mathbf{x}} + \mathbf{c}_K$ and we require that the matrix $D\Phi_K$ satisfies the assumption:

Assumption 1 *For any $h \in \mathfrak{h}$ and any $K \in \mathscr{T}_h$, there exists a diagonal scaling matrix*

$$H_K = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

and two matrices B_K and $\breve{B}_K$ which, together with their inverses, are bounded independently of h and K, such that:

$$D\Phi_K = H_K \, B_K = \breve{B}_K \, H_K. \tag{19}$$

Remark 3.1 (i) *In Assumption* 1*, the uniform boundedness condition on $\breve{B}_K$ and $\breve{B}_K^{-1}$ implies the maximum angle condition investigated in* [36], [29] *and also the regular vertex condition used in* [1] *to obtain anisotropic estimates for Raviart-Thomas elements.*

(ii) *Assumption* 1 *implies conditions* (3.2-3.3) *in* [7].

(iii) *As we have $B_K = H_K^{-1}\breve{B}_K H_K$, the requirement $\|B_K\| \lesssim C$ combined with the boundedness of $\breve{B}_K$ can be interpreted as follows: if the element is "long" in one direction, then, this direction has to be "fairly orthogonal" to the other two. The same condition has been indirectly used in* [40] *and* [30].

In the following two paragraphs, we give estimates for the L^p norm of the local interpolation errors for the Nédélec interpolation operator Π_K and for the Raviart-Thomas interpolation operator R_K when acting on curls. This applies to the electric and magnetic fields in a natural way. The upper bounds are weighted semi-norms.

3.2 Estimates for the local Nédélec interpolation operator Π_K

We denote by $\widehat{\Pi}$ the corresponding Nédélec interpolation operator on the reference element $\widehat{K}$. The operator $\widehat{\Pi}$ matches the degrees of freedom (5) or (6) between the interpolated field and its interpolant. Let $\mathbf{v} \in \mathcal{C}^\infty(\mathbb{R}^3)^3$, and denote by Π_K the physical interpolant on the generic physical element K. The transport relation is similar to (17):

$$(\Pi_K \mathbf{v}) \circ \Phi_K = D\Phi_K^{-1,\top} \widehat{\Pi} \widehat{\mathbf{v}}. \tag{20}$$

We start with estimates by first order derivatives.

Proposition 3.2 *Let $k \geq 1$ and $p > 2$. Under Assumption* 1, *there hold the estimates:*

(i) *For any $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{W}^{1,p}(K)$: $\|\mathbf{v} - \Pi_K \mathbf{v}\|_{\mathbf{L}^p(K)} \lesssim \sum_{j=1}^{3} \sum_{\ell=1}^{3} \frac{d_j d_\ell}{\min_i d_i} \|\partial_\ell v_j\|_{\mathrm{L}^p(K)}$.*

(ii) *If the interpolation operator $\widehat{\Pi}$ is diagonal: $\|\mathbf{v} - \Pi_K \mathbf{v}\|_{\mathbf{L}^p(K)} \lesssim \sum_{\ell=1}^{3} d_\ell \|\partial_\ell \mathbf{v}\|_{\mathbf{L}^p(K)}$.*

Remark 3.3 *The condition $p > 2$ is necessary because the interpolation operator $\widehat{\Pi}$ is continuous from $\mathbf{W}^{1,p}(\widehat{K}) \to \mathcal{N}_k$ only for $p > 2$, see* [3].

Proof: *(i)* According to Assumption 1, we have $|D\Phi_K^{-1}| \lesssim (\min_i d_i)^{-1}$. With (20) this allows to estimate:

$$\begin{aligned} \int_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \, \mathrm{d}\mathbf{x} &= |\det D\Phi_K| \int_{\widehat{K}} |\mathbf{v} \circ \Phi_K - (\Pi_K \mathbf{v}) \circ \Phi_K|^p \, \mathrm{d}\widehat{\mathbf{x}} \\ &= |\det D\Phi_K| \int_{\widehat{K}} |D\Phi_K^{-1,\top} \widehat{\mathbf{v}}(\widehat{\mathbf{x}}) - D\Phi_K^{-1,\top} \widehat{\Pi} \widehat{\mathbf{v}}(\widehat{\mathbf{x}})|^p \, \mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim |\det H_K| \int_{\widehat{K}} \frac{1}{(\min_i d_i)^p} \, |\widehat{\mathbf{v}} - \widehat{\Pi} \widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}}. \end{aligned} \tag{21}$$

Using now the Bramble-Hilbert estimate and the fact that $\widehat{\Pi}$ leaves constants unchanged, we have:

$$\int_{\widehat{K}} |\widehat{\mathbf{v}} - \widehat{\Pi} \widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}} \lesssim \int_{\widehat{K}} |\widehat{\nabla} \widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}}, \quad p > 2. \tag{22}$$

Since $\widehat{\mathbf{v}} = D\Phi_K^\top (\mathbf{v} \circ \Phi_K)$, the derivation rule yields the relation on $\widehat{K}$:

$$\widehat{\nabla} \widehat{\mathbf{v}} = D\Phi_K^\top \big((\nabla \mathbf{v}) \circ \Phi_K \big) D\Phi_K.$$

Here we have used the matrix notation (with $\widehat{\partial}_1 = \partial_{\widehat{x}}$, etc...)

$$\widehat{\nabla} \widehat{\mathbf{v}} = \begin{pmatrix} \widehat{\partial}_1 \widehat{v}_1 & \widehat{\partial}_2 \widehat{v}_1 & \widehat{\partial}_3 \widehat{v}_1 \\ \widehat{\partial}_1 \widehat{v}_2 & \widehat{\partial}_2 \widehat{v}_2 & \widehat{\partial}_3 \widehat{v}_2 \\ \widehat{\partial}_1 \widehat{v}_3 & \widehat{\partial}_2 \widehat{v}_3 & \widehat{\partial}_3 \widehat{v}_3 \end{pmatrix} \quad \text{and} \quad \nabla \mathbf{v} = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix}$$

Using now the decomposition $D\Phi_K = H_K B_K$, we rewrite

$$\widehat{\nabla} \widehat{\mathbf{v}} = B_K^\top H_K \big((\nabla \mathbf{v}) \circ \Phi_K \big) H_K B_K. \tag{23}$$

We use this formula and (22) in (21) to obtain:

$$
\begin{aligned}
\int\limits_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \,\mathrm{d}\mathbf{x} &\lesssim \frac{|\det H_K|}{(\min\limits_i d_i)^p} \int\limits_{\widehat{K}} |\widehat{\nabla}\widehat{\mathbf{v}}|^p \,\mathrm{d}\widehat{\mathbf{x}} \\
&\lesssim \frac{1}{(\min\limits_i d_i)^p} \int\limits_K |B_K^\top H_K \,(\nabla \mathbf{v})\, H_K B_K|^p \,\mathrm{d}\mathbf{x} \\
&\lesssim \frac{1}{(\min\limits_i d_i)^p} \int\limits_K |H_K \,(\nabla \mathbf{v})\, H_K|^p \,\mathrm{d}\mathbf{x} \\
&\lesssim \sum_{j=1}^3 \sum_{\ell=1}^3 \frac{d_j^p}{(\min\limits_i d_i)^p} \int\limits_K |d_\ell \partial_\ell v_j|^p \,\mathrm{d}\mathbf{x},
\end{aligned}
\tag{24}
$$

which ends the proof of *(i)*.

(ii) Let us suppose that the interpolation operator $\widehat{\Pi}$ is diagonal. In this case, for $\widehat{\mathbf{v}} = (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)^\top$ and $\widehat{\mathbf{v}}_i = \widehat{v}_i \widehat{\mathbf{e}}_i$, we have $(\widehat{\Pi}\widehat{\mathbf{v}})_i \widehat{\mathbf{e}}_i = \widehat{\Pi}\widehat{\mathbf{v}}_i$. Therefore

$$
\widehat{\mathbf{v}}_i - \widehat{\Pi}\widehat{\mathbf{v}}_i = \big(\widehat{v}_i - (\widehat{\Pi}\widehat{\mathbf{v}}_i)_i\big)\widehat{\mathbf{e}}_i \,.
$$

This fact can be used to improve (21) in the following way: Combining (20) with the factorization $D\Phi_K = \breve{B}_K H_K$ we obtain the relation between $\widehat{\Pi}$ and Π_K:

$$
(\Pi_K \mathbf{v}) \circ \Phi_K = \breve{B}_K^{-1,\top} H_K^{-1} \widehat{\Pi}\widehat{\mathbf{v}}. \tag{25}
$$

This gives

$$
\begin{aligned}
\int\limits_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \,\mathrm{d}\mathbf{x} \; &\lesssim |\det H_K| \int\limits_{\widehat{K}} |H_K^{-1} \widehat{\mathbf{v}}(\widehat{\mathbf{x}}) - H_K^{-1} \widehat{\Pi}\widehat{\mathbf{v}}(\widehat{\mathbf{x}})|^p \,\mathrm{d}\widehat{\mathbf{x}} \\
&\lesssim |\det H_K| \sum_{i=1}^3 \int\limits_{\widehat{K}} |H_K^{-1} \big(\widehat{\mathbf{v}}_i(\widehat{\mathbf{x}}) - \widehat{\Pi}\widehat{\mathbf{v}}_i(\widehat{\mathbf{x}})\big)|^p \,\mathrm{d}\widehat{\mathbf{x}} \\
&\lesssim |\det H_K| \sum_{i=1}^3 \int\limits_{\widehat{K}} \frac{1}{(d_i)^p} \; |\widehat{\mathbf{v}}_i - \widehat{\Pi}\widehat{\mathbf{v}}_i|^p \,\mathrm{d}\widehat{\mathbf{x}}.
\end{aligned}
\tag{26}
$$

Then we can use the Bramble-Hilbert estimate on each $\widehat{\mathbf{v}}_i$. Finally we have to come back to K. Let $\mathbf{v}_i$ be the field transported from $\widehat{\mathbf{v}}_i$, i.e. $\widehat{\mathbf{v}}_i = D\Phi_K^\top(\mathbf{v}_i \circ \Phi_K)$. Note that $\mathbf{v}_i$, in general, is not parallel to $\widehat{\mathbf{e}}_i$. However, the vector

$$
\breve{\mathbf{v}}_i := \breve{B}_K^\top \mathbf{v}_i = H_K^{-1} \widehat{\mathbf{v}}_i \circ \Phi_K^{-1}
$$

has, like $\widehat{\mathbf{v}}_i$, only its i-th component non-zero. Therefore, writing

$$
\begin{aligned}
\widehat{\nabla}\widehat{\mathbf{v}}_i &= H_K \breve{B}_K^\top \big((\nabla \mathbf{v}_i) \circ \Phi_K\big) H_K B_K \\
&= H_K \big((\nabla \breve{\mathbf{v}}_i) \circ \Phi_K\big) H_K B_K
\end{aligned}
$$

instead of (23), we obtain

$$\begin{aligned}\int_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \, \mathrm{d}\mathbf{x} &\lesssim \sum_{i=1}^{3} \frac{|\det H_K|}{(d_i)^p} \int_{\widehat{K}} |\widehat{\nabla} \widehat{\mathbf{v}}_i|^p \, \mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim \sum_{i=1}^{3} \frac{1}{(d_i)^p} \int_K |H_K \, (\nabla \check{\mathbf{v}}_i) \, H_K B_K|^p \, \mathrm{d}\mathbf{x} \\ &\lesssim \sum_{i=1}^{3} \frac{1}{(d_i)^p} \int_K |d_i \, (\nabla \check{\mathbf{v}}_i) \, H_K \,|^p \, \mathrm{d}\mathbf{x} \\ &\lesssim \sum_{i=1}^{3} \int_K |d_\ell \partial_\ell \check{B}_K^\top \mathbf{v}\,|^p \, \mathrm{d}\mathbf{x} \lesssim \sum_{\ell=1}^{3} \int_K |d_\ell \partial_\ell \mathbf{v}|^p \, \mathrm{d}\mathbf{x},\end{aligned} \tag{27}$$

which ends the proof of *(ii)*. □

Finally, when the assumptions of Lemma 2.6 are fulfilled, (22) can be replaced by (12). We obtain the following Proposition:

Proposition 3.4 *Let* $k = 1$ *and that the triple* $(d_1,\, d_2,\, d_3)$ *verifies*

$$\frac{d_\ell \, d_j}{d_i} \lesssim \max_j d_j \quad \text{for all distinct} \quad i\,, j\,, \ell.$$

Let $\mathbf{v} \in \mathbf{W}^{1,p}(\mathbf{curl}, K)$, $p > 2$, *be a vector field with constant curl. Then:*

$$\|\mathbf{v} - \Pi_K \mathbf{v}\|_{\mathbf{L}^p(K)} \lesssim \sum_{\ell=1}^{3} d_\ell \, \|\partial_\ell \mathbf{v}\|_{\mathbf{L}^p(K)} + (\max_j d_j) \, \| \, \mathbf{curl}\, \mathbf{v}\|_{\mathbf{L}^p(K)}. \tag{28}$$

Proof: Similarly as in (26), we compute:

$$\begin{aligned}\int_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \, \mathrm{d}\mathbf{x} \; &\lesssim |\det H_K| \int_{\widehat{K}} |H_K^{-1} \widehat{\mathbf{v}}(\widehat{\mathbf{x}}) - H_K^{-1} \widehat{\Pi} \widehat{\mathbf{v}}(\widehat{\mathbf{x}})|^p \, \mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim |\det H_K| \int_{\widehat{K}} \sum_{i=1}^{3} \frac{1}{d_i^p} (|\widehat{\nabla} \widehat{\mathbf{v}}_i(\widehat{\mathbf{x}})|^p + |\widehat{\mathbf{curl}} \widehat{\mathbf{v}}(\widehat{\mathbf{x}})|^p) \, \mathrm{d}\widehat{\mathbf{x}}.\end{aligned} \tag{29}$$

This implies:

$$\begin{aligned}\int_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \, \mathrm{d}\mathbf{x} \; &\lesssim \sum_{\ell=1}^{3} \int_K |d_\ell \partial_\ell \mathbf{v}|^p \, \mathrm{d}\mathbf{x} + \sum_{\ell,j=1}^{3} \frac{1}{(\min_i \; d_i)^p} \int_K |d_j d_\ell (\partial_j v_\ell - \partial_\ell v_j)|^p \, \mathrm{d}\mathbf{x} \\ &\lesssim \sum_{\ell=1}^{3} \int_K |d_\ell \partial_\ell \mathbf{v}|^p \, \mathrm{d}\mathbf{x} + (\max_j d_j)^p \| \, \mathbf{curl}\, \mathbf{v}\|_{\mathbf{L}^p(K)}^p\end{aligned} \tag{30}$$

where in the last line we have used the assumption $\dfrac{d_j \, d_\ell}{d_i} \leq \max_j d_j$ for all distinct $i\,, j\,, \ell\,.$ □

Let us denote by $\mathbf{d}$ the three dimensions (d_1, d_2, d_3) and by $\mathbf{d}^{\boldsymbol{\alpha}} = d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3}$. The general estimates involving semi-norms of higher degree are:

Proposition 3.5 *Let $\ell \geq 2$, $k \geq \ell$ and $p \geq 2$. Under Assumption 1, there hold the estimates:*

(i) *For any $\mathbf{v} \in \mathbf{W}^{\ell,p}(K)$:* $\|\mathbf{v} - \Pi_K \mathbf{v}\|_{\mathbf{L}^p(K)} \lesssim \sum_{j=1}^{3} \frac{d_j}{\min_i d_i} \sum_{|\boldsymbol{\alpha}|=\ell} \mathbf{d}^{\boldsymbol{\alpha}} \, \|\partial^{\boldsymbol{\alpha}} v_j\|_{\mathrm{L}^p(K)}$.

(ii) *If the interpolation operator $\widehat{\Pi}$ is diagonal:* $\|\mathbf{v} - \Pi_K \mathbf{v}\|_{\mathbf{L}^p(K)} \lesssim \sum_{|\boldsymbol{\alpha}|=\ell} \mathbf{d}^{\boldsymbol{\alpha}} \, \|\partial^{\boldsymbol{\alpha}} \mathbf{v}\|_{\mathrm{L}^p(K)}$.

Proof: By the Bramble-Hilbert estimate, we know that (22) can be replaced by

$$\int_{\widehat{K}} |\widehat{\mathbf{v}} - \widehat{\Pi}\widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}} \lesssim \int_{\widehat{K}} \sum_{|\boldsymbol{\alpha}|=\ell} |\widehat{\partial}^{\boldsymbol{\alpha}} \widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}}. \tag{31}$$

Following exactly the same argument as before, we then have instead of (24) :

$$\begin{aligned}\int_K |\mathbf{v} - \Pi_K \mathbf{v}|^p \, \mathrm{d}\mathbf{x} &\lesssim \frac{|\det H_K|}{(\min\limits_i d_i)^p} \sum_{|\boldsymbol{\alpha}|=k} \int_{\widehat{K}} |\widehat{\partial}^{\boldsymbol{\alpha}} \widehat{\mathbf{v}}|^p \, \mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim \frac{|\det H_K|}{(\min\limits_i d_i)^p} \sum_{|\boldsymbol{\alpha}|=k} \int_{\widehat{K}} \sum_j |d_j \mathbf{d}^{\boldsymbol{\alpha}} (\partial^{\boldsymbol{\alpha}} v_j) \circ \Phi_K)|^p \, \mathrm{d}\widehat{\mathbf{x}}.\end{aligned}$$

Whence estimate *(i)*. The proof of *(ii)* is similar. □

3.3 Local Raviart-Thomas interpolation estimate for a curl

In order to estimate the term $\int_K |\,\mathbf{curl}(\mathbf{v} - \Pi_K \mathbf{v})|^p d\mathbf{x}$ of the interpolation error, we make use of the commuting diagram property Theorem 2.1. Let R_K be the local Raviart-Thomas interpolant obtained by means of (18):

$$R_K \boldsymbol{\xi} \circ \Phi_K = (\det D\Phi_K)^{-1} D\Phi_K \, \widehat{R} \, \widehat{\boldsymbol{\xi}} \tag{32}$$

Theorem 2.1 says that

$$\mathbf{curl}\, \Pi_K \mathbf{v} = R_K \, \mathbf{curl}\, \mathbf{v}. \tag{33}$$

We then have to provide bounds in $\mathbf{L}^p(K)$ for the quantity $\boldsymbol{\xi} - R_K \boldsymbol{\xi}$, with $\boldsymbol{\xi} = \mathbf{curl}\, \mathbf{v}$.

Proposition 3.6 *Let $\ell \geq 1$, $k \geq \ell$ and $p \geq 2$. Under Assumption 1, for any $\boldsymbol{\xi} \in \mathbf{W}^{\ell,p}(K)$ with* $\operatorname{div} \boldsymbol{\xi} = 0$ *there hold the estimates:*

$$\|\boldsymbol{\xi} - R_K \boldsymbol{\xi}\|_{\mathbf{L}^p(K)} \lesssim \sum_{|\boldsymbol{\alpha}|=\ell} \mathbf{d}^{\boldsymbol{\alpha}} \, \|\partial^{\boldsymbol{\alpha}} \boldsymbol{\xi}\|_{\mathbf{L}^p(K)}. \tag{34}$$

Proof: Let us assume that $\ell = 1$. The relations (18) and (32) together with the factorization $\breve{B}_K H_K$ of $D\Phi_K$ yields

$$\int_K |\boldsymbol{\xi} - R_K \boldsymbol{\xi}|^p \, \mathrm{d}\mathbf{x} \lesssim |\det H_K|^{1-p} \int_{\widehat{K}} |H_K \widehat{\boldsymbol{\xi}} - H_K \widehat{R} \widehat{\boldsymbol{\xi}}|^p \, \mathrm{d}\widehat{\mathbf{x}}.$$

Using Lemma 2.7 and Proposition 2.8, and since $\widehat{R}_\pi$ keeps constants unchanged, we obtain

$$\begin{aligned}\int_K |\boldsymbol{\xi} - R_K\boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x} &\lesssim |\det H_K|^{1-p} \int_{\widehat{K}} \sum_{j=1}^{3} d_j^p \,|\widehat{\boldsymbol{\xi}}_j - (\widehat{R}_\pi \widehat{\boldsymbol{\xi}})_j|^p \,\mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim |\det H_K|^{1-p} \int_{\widehat{K}} \sum_{j=1}^{3} d_j^p \,|\widehat{\nabla}\widehat{\xi}_j|^p \,\mathrm{d}\widehat{\mathbf{x}}.\end{aligned} \tag{35}$$

We proceed as in the proof of Proposition 3.2, *(ii)*. Using both factorizations of $D\Phi_K$, we obtain

$$\widehat{\nabla}\widehat{\boldsymbol{\xi}} = (\det D\Phi_K)\, D\Phi_K^{-1} \,\nabla\boldsymbol{\xi} \circ \Phi_K \; D\Phi_K = (\det D\Phi_K)\, H_K^{-1} \breve{B}_K^{-1} \,\nabla\boldsymbol{\xi} \circ \Phi_K \; H_K B_K.$$

Thus:

$$\begin{aligned}\int_K |\boldsymbol{\xi} - R_K\boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x} &\lesssim |\det H_K|^{1-p} \int_{\widehat{K}} \sum_{j=1}^{3} d_j^p \sum_{i=1}^{3} |\, d_j^{-1} d_i (\partial_i \xi_j) \circ \Phi_K|^p \,|\det D\Phi_K|^p \,\mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim \int_K |\sum_{i=1}^{3} d_i \partial_i \boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x}\,,\end{aligned} \tag{36}$$

which ends the proof when $\ell = 1$. The general case is similar. □

For non divergence free fields, instead of (34) estimates similar to those of Proposition 3.5 *(i)* would be obtained, i.e. with additional factors of the type $\frac{d_j}{\min_i d_i}$ on the right hand side.

When $k = 1$, a better estimate can be obtained starting from Lemma 2.9. We state here the following Proposition with the aim of showing that results in [1] are recovered by our technique.

Proposition 3.7 *Let* $k = 1$ *and* $\boldsymbol{\xi} \in \mathbf{W}^{1,p}(\mathrm{div}, K)$, $p \geq 2$. *We have:*

$$\|\boldsymbol{\xi} - R_K\boldsymbol{\xi}\|_{\mathbf{L}^p(K)} \lesssim \sum_{\ell=1}^{3} d_\ell \,\|\partial_\ell \boldsymbol{\xi}\|_{\mathbf{L}^p(K)} + (\max_j d_j)\, \|\operatorname{div}\boldsymbol{\xi}\|_{L^p(K)}. \tag{37}$$

Proof: We estimate:

$$\begin{aligned}\int_K |\boldsymbol{\xi} - R_K\boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x} &\lesssim |\det H_K|^{1-p} \int_{\widehat{K}} \sum_{i=1}^{3} d_i^p |\widehat{\boldsymbol{\xi}}_i - (\widehat{R}\widehat{\boldsymbol{\xi}})_i|^p \,\mathrm{d}\widehat{\mathbf{x}} \\ &\lesssim |\det H_K|^{1-p} \int_{\widehat{K}} \sum_{i=1}^{3} d_i^p (|\widehat{\nabla}\widehat{\boldsymbol{\xi}}_i|^p + |\widehat{\mathrm{div}}\widehat{\boldsymbol{\xi}}|^p) \,\mathrm{d}\widehat{\mathbf{x}}.\end{aligned} \tag{38}$$

Using that $\widehat{\mathrm{div}}\widehat{\boldsymbol{\xi}} = |\det D\Phi_K| \operatorname{div}\boldsymbol{\xi} \circ \Phi_K$, and the same reasoning as in (36), we obtain:

$$\int_K |\boldsymbol{\xi} - R_K\boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x} \lesssim \int_K \sum_{i=1}^{3} d_i \,|\partial_i \boldsymbol{\xi}|^p \,\mathrm{d}\mathbf{x} \; + (\max_j d_j)^p \|\operatorname{div}\boldsymbol{\xi}\|_{L^p(K)}^p. \tag{39}$$

□

4 Regularity results for the electric field in a polyhedral domain

In order to provide regularity results for the solution of problem (1) and also to define our requirements on the mesh, we are going to identify subregions of the domain Ω governed by a corner $\mathbf{c}$ or an edge $\mathbf{e}$ or both. We refer to [32] and [22] for the introduction of similar subregions and coordinates.

We denote by $\mathscr{E}$ and $\mathscr{C}$ the set of edges and corners of the polyhedron Ω. Moreover for every $\mathbf{c} \in \mathscr{C}$, we denote by $\mathscr{E}_{\mathbf{c}}$ the set of edges $\mathbf{e}$ such that $\mathbf{c} \subset \overline{\mathbf{e}}$, and for every $\mathbf{c}$, by $\mathscr{C}_{\mathbf{e}}$ the set of two corners which are endpoints of $\mathbf{e}$. For any $\mathbf{c} \in \mathscr{C}$ and $\mathbf{e} \in \mathscr{E}$, define:

$$r_{\mathbf{e}}(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{e}), \quad r_{\mathbf{c}}(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{c}) \qquad \mathbf{x} \in \Omega. \tag{40}$$

Let now $\mathbf{c} \in \mathscr{C}$ and $B_r(\mathbf{c})$ be a ball centered in $\mathbf{c}$ with radius r which is not intersecting any other corner of Ω. We denote by $G_{\mathbf{c}} \subset \mathbb{S}^2$ the spherical polygonal domain corresponding to $\partial B_r(\mathbf{c}) \cap \Omega$. There is a bijection between the vertices $\widehat{\mathbf{y}}_{\mathbf{e}}$ of $G_{\mathbf{c}}$ and the edges $\mathbf{e}$ in $\mathscr{E}_{\mathbf{c}}$. For every $\mathbf{e} \in \mathscr{E}_{\mathbf{c}}$, let $\mathscr{V}(\mathbf{x}_{\mathbf{e}})$ be a neighborhood of $\widehat{\mathbf{y}}_{\mathbf{e}}$ in $G_{\mathbf{c}}$ such that $\mathscr{V}(\widehat{\mathbf{y}}_{\mathbf{e}})$ does not contain any other vertex of $G_{\mathbf{c}}$.

We introduce spherical coordinates $(r_{\mathbf{c}}, \vartheta_{\mathbf{c}})$, $\vartheta \in \mathbb{S}^2$ associated to the corner $\mathbf{c}$. This allows to define:

$$\begin{aligned} \mathscr{V}_{\mathbf{e}}^{\mathbf{c}} &= \{(r_{\mathbf{c}}, \vartheta_{\mathbf{c}}),\ r_{\mathbf{c}} < \varepsilon, \quad \vartheta_{\mathbf{c}} \in \mathscr{V}(\widehat{\mathbf{y}}_{\mathbf{e}})\}, \\ \mathscr{V}_{\mathbf{c}}^{0} &= \{(r_{\mathbf{c}}, \vartheta_{\mathbf{c}}),\ r_{\mathbf{c}} < \varepsilon, \quad \vartheta_{\mathbf{c}} \in G_{\mathbf{c}} \setminus (\cup_{\mathbf{e} \in \mathscr{E}_{\mathbf{c}}} \overline{\mathscr{V}}(\widehat{\mathbf{y}}_{\mathbf{e}}))\}, \end{aligned} \tag{41}$$

where $\varepsilon \in \mathbb{R}_+$ is small enough to ensure that no other corner except $\mathbf{c}$ belongs to $\mathscr{V}_{\mathbf{c}}^0$ and $\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}$.

Besides the neighborhoods $\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}$ and $\mathscr{V}_{\mathbf{c}}^0$, we introduce $\mathscr{V}_{\mathbf{e}}^0$ such that $\overline{\mathscr{V}_{\mathbf{e}}^0}$ does not contain any other edge than $\mathbf{e}$, nor any corner and such that $\overline{\mathbf{e}}$ is contained in $\overline{\mathscr{V}_{\mathbf{e}}^0} \cup \big(\bigcup_{\mathbf{c} \in \mathscr{C}_{\mathbf{e}}} \overline{\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}}\big)$. And finally choose $\mathscr{V}^0$ such that $\overline{\mathscr{V}^0}$ contains no edge and no corner and such that

$$\Omega = \mathscr{V}^0 \cup \bigcup_{\mathbf{e} \in \mathscr{E}} \mathscr{V}_{\mathbf{e}}^0 \cup \bigcup_{\mathbf{c} \in \mathscr{C}} \Big(\mathscr{V}_{\mathbf{c}}^0 \cup \bigcup_{\mathbf{e} \in \mathscr{E}_{\mathbf{c}}} \mathscr{V}_{\mathbf{e}}^{\mathbf{c}} \Big).$$

In the regions $\mathscr{V}_{\mathbf{e}}^0$ and $\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}$ associated with the edge $\mathbf{e}$ we choose a local system of Cartesian coordinates $\mathbf{x}^{\mathbf{e}} = (x_1^{\mathbf{e}}, x_2^{\mathbf{e}}, x_3^{\mathbf{e}})$ in which the direction of the edge is $x_3^{\mathbf{e}}$. The subscript $\perp$ will always denote the directions transverse to the edge: For example, if $\boldsymbol{\alpha}$ is a derivation multi-index, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\perp, \alpha_3)$ means $\boldsymbol{\alpha}_\perp$ derivatives in $(x_1^{\mathbf{e}}, x_2^{\mathbf{e}})$ and α_3 in $x_3^{\mathbf{e}}$. Then we define the space:

$$\begin{aligned} \mathrm{M}_\gamma^{m,p}(\Omega) := \Big\{ & u \in \mathrm{L}^p(\Omega) \ : \ \forall\, |\boldsymbol{\alpha}| \le m \quad \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}^0), \\ & \forall\, \mathbf{c} \in \mathscr{C} \quad r_{\mathbf{c}}^{\gamma + |\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}_{\mathbf{c}}^0) \\ & \forall\, \mathbf{e} \in \mathscr{E} \quad r_{\mathbf{e}}^{\gamma + |\boldsymbol{\alpha}_\perp|} \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}_{\mathbf{e}}^0) \ \text{ and } \ r_{\mathbf{c}}^{\alpha_3} r_{\mathbf{e}}^{\gamma + |\boldsymbol{\alpha}_\perp|} \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}) \Big\}. \end{aligned} \tag{42}$$

We denote by $\mathbf{M}_\gamma^{m,p}(\Omega)$ the corresponding space of vector-valued functions, i.e. $\mathrm{M}_\gamma^{m,p}(\Omega)^3$.

Theorem 4.1 *There exists $\beta_\Omega > 0$ and $\delta_\Omega > 0$ so that the following regularity results holds for the solutions $\mathbf{u}$ of (3) when λ is not an eigenvalue of this problem. If $\mathbf{f}$ is $C^\infty(\overline{\Omega})$ and divergence-free, there exists a potential $q \in \mathrm{H}_0^1(\Omega)$ such that for all $p \in [2, 2 + \delta_\Omega)$, for all $\beta \in (0, \beta_\Omega)$ and all $m > 0$:*

$$\mathbf{u} = \mathbf{u}_0 + \nabla q \quad \textit{with} \quad \mathbf{u}_0 \in \mathbf{M}_{-1-\beta}^{m,p}(\Omega) \ \textit{and} \ q \in \mathrm{M}_{-1-\beta}^{m+1,p}(\Omega). \tag{43}$$

As a consequence of this splitting, $\mathbf{u}$ belongs to a anisotropic space in which an improved regularity for the tangential component along the edges holds: We define

$$\begin{aligned}\overline{\mathsf{M}}^{m,p}_\gamma(\Omega) := \Big\{ \mathbf{u}\in \mathsf{M}^{m,p}_\gamma(\Omega)\ :\ \forall\,\mathbf{e}\in\mathscr{E},\ \ \text{and with } u_{\mathbf{e},3} \text{ the component of } \mathbf{u} \text{ along } \mathbf{e},\\ \forall\,|\boldsymbol{\alpha}|\le m\quad r_{\mathbf{e}}^{-1+\gamma+|\boldsymbol{\alpha}_\perp|}\partial^{\boldsymbol{\alpha}}u_{\mathbf{e},3}\in \mathrm{L}^p(\mathscr{V}^0_{\mathbf{e}})\\ r_{\mathbf{c}}^{1+\alpha_3}r_{\mathbf{e}}^{-1+\gamma+|\boldsymbol{\alpha}_\perp|}\partial^{\boldsymbol{\alpha}}u_{\mathbf{e},3}\in \mathrm{L}^p(\mathscr{V}^{\mathbf{c}}_{\mathbf{e}})\Big\}.\end{aligned} \tag{44}$$

Note that we have the inclusions $\mathsf{M}^{m,p}_{\gamma-1}(\Omega)\subset\overline{\mathsf{M}}^{m,p}_\gamma(\Omega)\subset\mathsf{M}^{m,p}_\gamma(\Omega)$ and that the operator ∇ is continuous from $\mathrm{M}^{m+1,p}_{\gamma-1}(\Omega)$ into $\overline{\mathsf{M}}^{m,p}_\gamma(\Omega)$. If we combine this with the continuity of the operator $\mathbf{curl}:\mathsf{M}^{m,p}_{-1-\beta}(\Omega)\to\mathsf{M}^{m-1,p}_{-\beta}(\Omega)$, we obtain

Corollary 4.2 *Under the conditions of Theorem* 4.1*,* $\mathbf{curl}\,\mathbf{u}=\mathbf{curl}\,\mathbf{u}_0$ *belongs to* $\mathsf{M}^{m,p}_{-\beta}(\Omega)$*. Moreover* $\mathbf{u}\in\overline{\mathsf{M}}^{m,p}_{-\beta}(\Omega)$*.*

The proof of the theorem is beyond the scope of this paper. Actually it is not an easy consequence of the known theory about Maxwell singularities [21] and it requires the use of sophisticated techniques from the theory of singularities for elliptic problems [26]. Further details can be found in [14] which is announcing the general theory for both the Laplace and Maxwell operators. Moreover, the statement could be improved regarding the requirements on the right hand side $\mathbf{f}$; but these results are not needed here and the related theory is still under investigation.

5 Convergence Analysis for the Galerkin method

In this section, we introduce the discretization of the problem (3) by anisotropically refined edge elements, and provide *a-priori* estimates for the associated Galerkin error.

This analysis requires some preparation: in Section 5.1 we set notations and assumptions for the definition of anisotropic refined meshes, in Section 5.2 we use the local estimates obtained in Section 3 in order to prove algebraic convergence for the best approximation error. These results allow to conclude algebraic optimality for the Galerkin problem when the underlying continuous problem is strongly coercive (cf (69)). When this is not the case, some additional results are needed to conclude convergence of the Galerkin scheme: We need to prove the so-called discrete compactness property [35]. This is investigated in Section 7. In Section 6 we present some extensions of the present theory.

5.1 Anisotropically refined meshes

Let $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ be a family of meshes (hexahedral or tetrahedral) which verifies Assumption 1, §3.1. For each $h\in\mathfrak{h}$ fixed, we define subsets of $\mathscr{T}_h$:

- $\mathcal{L}_0$ denotes the set of elements $K\in\mathscr{T}_h$ such that $\overline{K}\cap\mathscr{C}\neq\emptyset$, i.e., one corner of K coincides with one corner of Ω. Note that $\#\mathcal{L}_0\simeq\#\mathscr{C}$;
- $\mathcal{L}_1$ denotes the set of elements $K\in\mathscr{T}_h$ such that $\overline{K}\cap\{\overline{\mathbf{e}},\ \mathbf{e}\in\mathscr{E}\}\neq\emptyset$, i.e., K has one edge or one corner sitting on an edge or a corner of Ω. Note that $\mathcal{L}_0\subset\mathcal{L}_1$.
- $\mathcal{L}_2$ is the set of elements $K\in\mathscr{T}_h\setminus\mathcal{L}_1$ which share an edge or a corner with a $K\in\mathcal{L}_1$;

$r_{\mathbf{e},K} = \min_{\mathbf{x}\in K} r_{\mathbf{e}}(\mathbf{x})$ and $r_{\mathbf{c},K} = \min_{\mathbf{x}\in K} r_{\mathbf{c}}(\mathbf{x}) \quad \forall\, K \not\subset \mathcal{L}_1$

A discrete version of the family of neighborhoods defined in Section 4 is defined as:

$$\begin{aligned}
\mathscr{V}^0(h) &= \{K \in \mathscr{T}_h \;:\; K \cap \mathscr{V}^0 \neq \emptyset\},\\
\mathscr{V}^0_{\mathbf{e}}(h) &= \{K \in \mathscr{T}_h \;:\; K \cap \mathscr{V}^0_{\mathbf{e}} \neq \emptyset\},\\
\mathscr{V}^0_{\mathbf{c}}(h) &= \{K \in \mathscr{T}_h \;:\; K \cap \mathscr{V}^0_{\mathbf{c}} \neq \emptyset\},\\
\mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h) &= \{K \in \mathscr{T}_h \;:\; K \cap \mathscr{V}^{\mathbf{c}}_{\mathbf{e}} \neq \emptyset\}.
\end{aligned}$$

In order to provide an algebraically optimal (in the sense of Definition 1.2) edge element method, we now have to give precise bounds for the possible choices of $\mathbf{d} = (d_1, d_2, d_3)$ for elements in different region of the mesh, i.e., provide a set of assumptions that $\mathscr{T}_h$ has to verify once k and the set of degrees of freedom $\mathcal{N}_k$ are fixed. In the edge regions $\mathscr{V}^0_{\mathbf{e}}(h)$, $\mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h)$ the local system of Cartesian coordinates $\mathbf{x}^K$ is chosen equal to the system $\mathbf{x}^{\mathbf{e}}$ introduced in Section 4. In the other regions, where the elements will be supposed isotropic, any system of Cartesian coordinates can be chosen.

In order to prove algebraic convergence, we need the following assumption on the mesh:

Assumption 2$_{k,\beta}$ *Refinement for fixed* $k \in \mathbb{N}$ *and* $\beta \in (0,1)$*:*

$K \in \mathscr{V}^0_{\mathbf{c}}(h) \setminus \mathcal{L}_1$	$d_1 \simeq d_2 \simeq d_3 \simeq h\, r_{\mathbf{c},K}^{1-\beta/k}$	
$K \in \mathscr{V}^0_{\mathbf{e}}(h) \setminus \mathcal{L}_1$	$d_1 \simeq d_2 \simeq h\, r_{\mathbf{e},K}^{1-\beta/k}$	$d_3 \simeq h$
$K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h) \setminus \mathcal{L}_1$	$d_1 \simeq d_2 \simeq h\, r_{\mathbf{e},K}^{1-\beta/k}$	$d_3 \simeq h\, r_{\mathbf{c},K}^{1-\beta/k}$
$K \in \mathcal{L}_0$	$d_1 \simeq d_2 \simeq d_3 \simeq h^{k/\beta}$	
$K \in \mathscr{V}^0_{\mathbf{e}}(h) \cap \mathcal{L}_1$	$d_1 \simeq d_2 \simeq h^{k/\beta}$	$d_3 \simeq h$
$K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h) \cap \mathcal{L}_1 \setminus \mathcal{L}_0$	$d_1 \simeq d_2 \simeq h^{k/\beta}$	$d_3 \simeq h\, r_{\mathbf{c},K}^{1-\beta/k}$
$K \in \mathscr{V}^0(h)$	$d_1 \simeq d_2 \simeq d_3 \lesssim h.$	

For elements belonging to intersections of $\mathscr{V}^0(h)$, $\mathscr{V}^0_{\mathbf{c}}(h)$, $\mathscr{V}^0_{\mathbf{e}}(h)$, $\mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h)$ *any of the admissible strategies can be adopted since in such regions, they are equivalent to each other. The constants hidden in the symbols* $\lesssim$ *and* $\simeq$ *are uniform on the whole family* $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$.

Proposition 5.1 *The number of elements of the mesh* $\mathscr{T}_h$ *in a family satisfying Assumption* $2_{k,\beta}$ *is* $O(h^{-3})$.

This proposition is a consequence of its 1D version, that we state in the following way (and where τ corresponds to k/β):

Lemma 5.2 *Let* $\tau \geq 1$. *Let* $\big(\gamma_n\big)_n$ *be a sequence of numbers such that* $c^{-1} \leq \gamma_n < c$ *for a* $c > 0$ *and all* $n \in \mathbb{N}$*,. Let us fix* $N \in \mathbb{N}$ *and define the two sequences* $\big(\delta_n\big)_n$ *and* $\big(t_n\big)_n$ *by*

$$\delta_1 = t_1 = \gamma_1 N^{-\tau} \quad \textit{and} \quad \forall n \geq 1 : \delta_{n+1} = \gamma_{n+1} N^{-1} t_n^{1-1/\tau} \textit{ and } t_{n+1} = t_n + \delta_{n+1}. \tag{45}$$

Then $t_N = O(1)$ *and* $\delta_N = O(N^{-1})$.

Proof: 1. Let us start with the usual refinement strategy which consists of taking $t_n = (n/N)^\tau$ and $\delta_1 = t_1$, $\delta_{n+1} = t_{n+1} - t_n$. The conclusion of the lemma is satisfied for this couple of sequences. Let us prove that (45) is satisfied. We compute that, for $n \ge 1$

$$\delta_{n+1}\Big(N^{-1}t_n^{1-1/\tau}\Big)^{-1} = n\big((1+\tfrac{1}{n})^\tau - 1\big) =: \gamma^0_{n+1}.$$

It is easy to see that $\tau \le \gamma^0_{n+1} \le \tau\, 2^{\tau-1}$ for all $n \ge 1$. Therefore these sequences satisfy the assumptions of the lemma for $\gamma_n = \gamma^0_n$.
2. Let $\gamma > 0$. Let us now consider $t_n = \gamma(n/N)^\tau$ and $\delta_1 = t_1$, $\delta_{n+1} = t_{n+1} - t_n$. There holds

$$\delta_{n+1}\Big(N^{-1}t_n^{1-1/\tau}\Big)^{-1} = \gamma^{1/\tau}\gamma^0_{n+1}.$$

These sequences still satisfy the assumptions and the conclusion of the lemma.
3. If $\gamma_n \le \gamma'_n$, the sequences δ_n, t_n, δ'_n, t'_n constructed by formula (45) are such that $\delta_n \le \delta'_n$ and $t_n \le t'_n$. Let $\big(\delta_n\big)_n$ and $\big(t_n\big)_n$ satisfy (45). It is easy to construct $\gamma_{\min}$ and $\gamma_{\max}$ such that

$$\forall n \ge 1, \quad \gamma_{\min}^{1/\tau}\gamma^0_{n+1} \le \gamma_{n+1} \le \gamma_{\max}^{1/\tau}\gamma^0_{n+1}.$$

Since the couples of sequences of type 2. satisfy the conclusion of the lemma, the same holds for the sequences $\big(\delta_n\big)_n$ and $\big(t_n\big)_n$. □

5.2 Best approximation error

This section is devoted to the proof of the following theorem:

Theorem 5.3 *Let the degree $k \ge 1$ be fixed. Let $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ be a family of meshes verifying Assumptions* 1 *and* $2_{k,\beta}$ *for a positive index β such that $\beta \le \beta_\Omega$ where β_Ω is the parameter of Theorem* 4.1. *Then, for every $\mathbf{u}$ verifying* (43)*, there holds:*

$$\inf_{\mathbf{v}_h\in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X \le Ch^k \le C'N^{-k/3}, \tag{46}$$

where C and C' are constants depending on k, but not on h or N.

The proof of this theorem needs several steps which are presented in form of lemmas.
First about the relative sizes of the elements in the regions $\mathscr{V}^0_{\mathbf{c}}(h)$, $\mathscr{V}^0_{\mathbf{e}}(h)$, $\mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h)$.

Lemma 5.4 (Relative sizes) *For all $\mathbf{x} \in K$, $K \in \mathscr{T}_h \setminus \mathcal{L}_1$, there holds*

$$\begin{aligned} &K \in \mathscr{V}^0_{\mathbf{c}}(h) \setminus \mathcal{L}_1, \quad r_{\mathbf{c}}(\mathbf{x}) \simeq r_{\mathbf{c},K} \gtrsim d_1 (\simeq d_2 \simeq d_3)\\ &K \in \mathscr{V}^0_{\mathbf{e}}(h) \setminus \mathcal{L}_1, \quad r_{\mathbf{e}}(\mathbf{x}) \simeq r_{\mathbf{e},K} \gtrsim d_1 (\simeq d_2)\\ &K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h) \setminus \mathcal{L}_1, \quad r_{\mathbf{e}}(\mathbf{x}) \simeq r_{\mathbf{e},K} \gtrsim d_1 (\simeq d_2) \quad \textit{and} \quad r_{\mathbf{c}}(\mathbf{x}) \simeq r_{\mathbf{c},K} \gtrsim d_3. \end{aligned} \tag{47}$$

If, moreover $K \in \mathcal{L}_2$:

$$\begin{aligned} &K \in \mathscr{V}^0_{\mathbf{c}}(h) \cap \mathcal{L}_2, \quad r_{\mathbf{c}}(\mathbf{x}) \simeq r_{\mathbf{c},K} \simeq d_1 \simeq h^{k/\beta}\\ &K \in \mathscr{V}^0_{\mathbf{e}}(h) \cap \mathcal{L}_2, \quad r_{\mathbf{e}}(\mathbf{x}) \simeq r_{\mathbf{e},K} \simeq d_1 \simeq h^{k/\beta}\\ &K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h) \cap \mathcal{L}_2, \quad r_{\mathbf{e}}(\mathbf{x}) \simeq r_{\mathbf{e},K} \simeq d_1 \simeq h^{k/\beta} \ \ \textit{and} \ \ r_{\mathbf{c}}(\mathbf{x}) \simeq r_{\mathbf{c},K} \simeq d_3 \simeq h^{k/\beta}. \end{aligned} \tag{48}$$

Proof: Let $K \in \mathscr{V}_{\mathbf{c}}^0(h) \setminus \mathcal{L}_1$. By definition $r_{\mathbf{c},K} \leq r_{\mathbf{c}}(\mathbf{x})$. We also know that $r_{\mathbf{c}}(\mathbf{x}) \leq r_{\mathbf{c},K} + ch\, r_{\mathbf{c},K}^{1-\beta/k}$ (c denotes a constant). We need then to prove: $r_{\mathbf{c},K} + ch\, r_{\mathbf{c},K}^{1-\beta/k} \lesssim r_{\mathbf{c},K}$. Which is equivalent to $h \lesssim r_{\mathbf{c},K}^{\beta/k}$, i.e. to $h^{k/\beta} \lesssim r_{\mathbf{c},K}$, which is always the case by Assumption $2_{k,\beta}$ for $K \in \mathscr{V}_{\mathbf{c}}^0(h) \setminus \mathcal{L}_1$.

Since in the same region the characteristic dimensions d_i are all equivalent to $h\, r_{\mathbf{c},K}^{1-\beta/k}$, the same proof yields that $d_i \lesssim r_{\mathbf{c},K}$. If, moreover, K belongs to the layer $\mathcal{L}_2$, $r_{\mathbf{c},K}$ coincides with the value of $r_{\mathbf{c}}(\mathbf{x})$ for $\mathbf{x}$ in an element $K' \in \mathcal{L}_1$, then $r_{\mathbf{c},K} \lesssim h^{k/\beta}$, which gives that $d_i \gtrsim r_{\mathbf{c},K}$.

For the regions $\mathscr{V}_{\mathbf{e}}^0(h)$ and $\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h)$, the proof is similar. □

Lemma 5.5 *There exists a smooth cut-off function χ_h which is 0 on $\mathcal{L}_1$ and 1 outside $\mathcal{L}_1 \cup \mathcal{L}_2$, and which satisfies the estimates*

$$\begin{aligned} &K \in \mathscr{V}_{\mathbf{c}}^0(h) \cap \mathcal{L}_2, \quad \|\partial^{\boldsymbol{\alpha}} \chi_h\|_{\infty,K} \lesssim r_{\mathbf{c},K}^{-|\boldsymbol{\alpha}|} \\ &K \in \mathscr{V}_{\mathbf{e}}^0(h) \cap \mathcal{L}_2, \quad \|\partial^{\boldsymbol{\alpha}} \chi_h\|_{\infty,K} \lesssim r_{\mathbf{e},K}^{-|\boldsymbol{\alpha}_\perp|} \\ &K \in \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h) \cap \mathcal{L}_2, \quad \|\partial^{\boldsymbol{\alpha}} \chi_h\|_{\infty,K} \lesssim r_{\mathbf{c},K}^{-\alpha_3}\, r_{\mathbf{e},K}^{-|\boldsymbol{\alpha}_\perp|}. \end{aligned} \tag{49}$$

The existence of such a function is a consequence of the estimates (48).

Lemma 5.6 *The following continuity estimates hold for $\varphi \in \mathrm{M}_\gamma^{m,p}(\Omega)$ and $\boldsymbol{\varphi} \in \overline{\mathbf{M}}_\gamma^{m,p}(\Omega)$:*

$$\|\chi_h \varphi\|_{\mathrm{M}_\gamma^{m,p}(\Omega)} \lesssim \|\varphi\|_{\mathrm{M}_\gamma^{m,p}(\Omega)} \tag{50}$$

$$\forall \boldsymbol{\alpha},\ |\boldsymbol{\alpha}| = 1, \quad \|(\partial^{\boldsymbol{\alpha}} \chi_h)\varphi\|_{\mathrm{M}_{\gamma+1}^{m,p}(\Omega)} \lesssim \|\varphi\|_{\mathrm{M}_\gamma^{m,p}(\Omega)} \tag{51}$$

$$\|\chi_h \boldsymbol{\varphi}\|_{\overline{\mathbf{M}}_\gamma^{m,p}(\Omega)} \lesssim \|\boldsymbol{\varphi}\|_{\overline{\mathbf{M}}_\gamma^{m,p}(\Omega)}. \tag{52}$$

Proof: In order to prove (50), we prove the estimate: $\|\chi_h \varphi\|_{\mathrm{L}_\gamma^{m,p}(\mathscr{V}_{\mathbf{e}}^{\mathbf{c}})} \lesssim \|\varphi\|_{\mathrm{L}_\gamma^{m,p}(\mathscr{V}_{\mathbf{e}}^{\mathbf{c}})}$. The estimates in $\mathscr{V}_{\mathbf{e}}^0$ and $\mathscr{V}_{\mathbf{c}}^0$ are similar.

$$\begin{aligned} \|\chi_h \varphi\|_{\mathrm{L}_\gamma^{m,p}(\mathscr{V}_{\mathbf{e}}^{\mathbf{c}})}^p &= \sum_{|\boldsymbol{\alpha}| \leq m} \int_{\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}} \big(r_{\mathbf{c}}^{\alpha_3} r_{\mathbf{e}}^{\boldsymbol{\alpha}_\perp + \gamma}\big)^p \sum_{|\mathbf{k}| \leq \boldsymbol{\alpha}} |\,\partial^{\boldsymbol{\alpha}-\mathbf{k}} \chi_h \partial^{\mathbf{k}} \varphi|^p \,\mathrm{d}\mathbf{x} \\ &\lesssim \sum_{|\boldsymbol{\alpha}| \leq m} \sum_{|\mathbf{k}| \leq \boldsymbol{\alpha}} \int_{\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}} \big(r_{\mathbf{c}}^{\alpha_3} r_{\mathbf{e}}^{\boldsymbol{\alpha}_\perp + \gamma}\big)^p \big(r_{\mathbf{c}}^{k_3 - \alpha_3} r_{\mathbf{e}}^{\mathbf{k}_\perp - \boldsymbol{\alpha}_\perp}\big)^p |\partial^{\mathbf{k}} \varphi|^p \,\mathrm{d}\mathbf{x} \\ &\lesssim \sum_{|\mathbf{k}| \leq m} \int_{\mathscr{V}_{\mathbf{e}}^{\mathbf{c}}} \big(r_{\mathbf{c}}^{k_3} r_{\mathbf{e}}^{\mathbf{k}_\perp + \gamma}\big)^p |\partial^{\mathbf{k}} \varphi|^p \,\mathrm{d}\mathbf{x} \lesssim C \|\varphi\|_{\mathrm{L}_\gamma^{m,p}(\mathscr{V}_{\mathbf{e}}^{\mathbf{c}})}^p \end{aligned} \tag{53}$$

The proof of the (51) and (52) follows the same line. □

From now on we adopt the notation $\|\cdot\|_{p,K}$ to denote the norm in $\mathrm{L}^p(K)$ or $\mathbf{L}^p(K)$.

Lemma 5.7 (Local Estimates) *Let Assumptions 1 and $2_{k,\beta}$ hold true. For all $K \in \mathscr{T}_h \setminus \mathcal{L}_1$ and $\boldsymbol{\varphi} \in \mathbf{W}^{k,p}(K)$ such that $\mathbf{curl}\,\boldsymbol{\varphi} \in \mathbf{W}^{k,p}(K)$, it holds:*

$$\|\boldsymbol{\varphi} - \Pi_K \boldsymbol{\varphi}\|_{p,K} \lesssim h^k \|\boldsymbol{\varphi}\|_{\overline{\mathbf{M}}_{-\beta}^{k,p}(K)} \tag{54}$$

$$\|\,\mathbf{curl}\,\boldsymbol{\varphi} - \mathbf{curl}\,\Pi_K \boldsymbol{\varphi}\|_{p,K} \lesssim h^k \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathbf{M}_{-\beta}^{k,p}(K)}. \tag{55}$$

Proof: We make use of the anisotropic local interpolation estimates obtained in Section 3.

The two dimensions d_1 and d_2 are always equivalent: Let $d_\perp := \min\{d_1, d_2\}$. We have in any case $d_\perp \lesssim d_3$. Using Proposition 3.5 *(i)* for $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, we obtain

$$\begin{aligned} &\|\boldsymbol{\varphi} - \Pi_K \boldsymbol{\varphi}\|_{p,K} \lesssim D_1 + D_2 + D_3, \qquad \text{with:} \\ &D_i = \sum_{|\boldsymbol{\alpha}|=k} \|\mathbf{d}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \varphi_i\|_{p,K},\ i = 1,2, \quad D_3 = \sum_{|\boldsymbol{\alpha}|=k} \frac{d_3}{d_\perp} \|\mathbf{d}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \varphi_3\|_{p,K}. \end{aligned} \tag{56}$$

For $i = 1,2$ we have

$$\begin{aligned} K \in \mathscr{V}_{\mathbf{e}}^0(h) \setminus \mathcal{L}_1 \ : \quad & D_i \lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|(1-\beta/k)} \partial^{\boldsymbol{\alpha}} \varphi_i\|_{p,K} \\ K \in \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h) \setminus \mathcal{L}_1 \ : \quad & D_i \lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{c}}^{\alpha_3(1-\beta/k)} r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|(1-\beta/k)} \partial^{\boldsymbol{\alpha}} \varphi_i\|_{p,K} \end{aligned} \tag{57}$$

and for $i = 3$

$$\begin{aligned} K \in \mathscr{V}_{\mathbf{e}}^0(h) \setminus \mathcal{L}_1 \ : \quad & D_3 \lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{e}}^{(|\boldsymbol{\alpha}_\perp|-1)(1-\beta/k)} \partial^{\boldsymbol{\alpha}} \varphi_3\|_{p,K} \\ K \in \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h) \setminus \mathcal{L}_1 \ : \quad & D_3 \lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{c}}^{(1+\alpha_3)(1-\beta/k)} r_{\mathbf{e}}^{(|\boldsymbol{\alpha}_\perp|-1)(1-\beta/k)} \partial^{\boldsymbol{\alpha}} \varphi_3\|_{p,K} \end{aligned} \tag{58}$$

Now, since $|\boldsymbol{\alpha}_\perp| \le k$, for $K \in \mathscr{V}_{\mathbf{e}}^0(h)$ we have $r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|(1-\beta/k)} \le r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|-\beta}$ and $r_{\mathbf{e}}^{(|\boldsymbol{\alpha}_\perp|-1)(1-\beta/k)} \le r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|-1-\beta}$, whereas for $K \in \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h)$, since it holds $r_{\mathbf{e}} \le r_{\mathbf{c}}$, we have

$$r_{\mathbf{c}}^{\alpha_3(1-\beta/k)} r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|(1-\beta/k)} \le r_{\mathbf{c}}^{\alpha_3} r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|-\beta}, \quad r_{\mathbf{c}}^{(1+\alpha_3)(1-\beta/k)} r_{\mathbf{e}}^{(|\boldsymbol{\alpha}_\perp|-1)(1-\beta/k)} \le r_{\mathbf{c}}^{1+\alpha_3} r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|-\beta-1}.$$

Moreover, any element $K \in \mathscr{V}_{\mathbf{c}}^0(h) \setminus \mathcal{L}_1$ is regular and the estimate goes smoothly:

$$\|\boldsymbol{\varphi} - \Pi_K \boldsymbol{\varphi}\|_{p,K} \lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{c}}^{|\boldsymbol{\alpha}|-\beta} \mathbf{u}\|_{p,K} \lesssim h^k \|\boldsymbol{\varphi}\|_{\mathsf{M}_{-\beta}^{k,p}(K)} \tag{59}$$

Finally, for $K \in \mathscr{V}^0(h)$ the estimate is standard since the element K is regular. Thus (56)-(59) imply the local estimate (54).

The estimate (55) on the curl part is easier since the interpolation operator is diagonal – cf. Proposition 2.8 and (34). We show it for a $K \in \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}(h)$:

$$\begin{aligned} \|\,\mathbf{curl}\,\boldsymbol{\varphi} - \mathbf{curl}\,\Pi_K \boldsymbol{\varphi}\|_{p,K} &\lesssim \sum_{|\boldsymbol{\alpha}|=k} \|\mathbf{d}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{p,K} \\ &\lesssim h^k \sum_{|\boldsymbol{\alpha}|=k} \|r_{\mathbf{c}}^{\alpha_3(1-\beta/k)} r_{\mathbf{e}}^{|\boldsymbol{\alpha}_\perp|(1-\beta/k)} \partial^{\boldsymbol{\alpha}}\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{p,K} \\ &\lesssim h^k \|\,\mathbf{curl}\,\boldsymbol{\varphi}\|_{\mathsf{M}_{-\beta}^{k,p}(K)}. \end{aligned} \tag{60}$$

The other cases go similarly, and this ends the proof. □

Proof of Theorem 5.3. We make use of the decomposition (43) introduced in Theorem 4.1 and apply the cut-off function in the following way:

$$\mathbf{u} = \{\chi_h \mathbf{u}_0 + \nabla\chi_h q\} + (1-\chi_h)\mathbf{u}_0 + \nabla(1-\chi_h)q \tag{61}$$

and the best approximation error is estimated taking as an approximation $\Pi^k(\chi_h\mathbf{u}_0 + \nabla\chi_h q)$, where Π^k denotes the global interpolation operator for edge elements of degree k. This immediately implies:

$$\begin{aligned}\inf_{\mathbf{v}_h\in X_h}\|\mathbf{u}-\mathbf{v}_h\|_X \leq \quad &\textit{(i)} \qquad \|\chi_h\mathbf{u}_0 - \Pi^k(\chi_h\mathbf{u}_0)\|_X \\ &\textit{(ii)} \quad + \|\nabla\chi_h q - \Pi^k(\nabla\chi_h q)\|_X \\ &\textit{(iii)} \quad + \|(1-\chi_h)\mathbf{u}_0\|_X + \|\nabla(1-\chi_h)q\|_X\,.\end{aligned} \tag{62}$$

We need now to estimate each term *(i)*, *(ii)* and *(iii)* on the right hand side.

(i) Since $\mathbf{u}_0$ belongs to $\mathsf{M}^{k,p}_{-1-\beta}(\Omega)$, it also belongs to $\overline{\mathsf{M}}^{k,p}_{-\beta}(\Omega)$, therefore, by Lemma 5.6, $\chi_h\mathbf{u}_0 \in \overline{\mathsf{M}}^{k,p}_{-\beta}(\Omega)$. Since $\mathbf{u}_0$ also belongs to $\mathsf{M}^{k+1,p}_{-1-\beta}(\Omega)$, its curl is in $\mathsf{M}^{k,p}_{-\beta}(\Omega)$. Then we use the identity

$$\mathbf{curl}\,\chi_h\mathbf{u}_0 = \nabla\chi_h\times\mathbf{u}_0 + \chi_h\ \mathbf{curl}\,\mathbf{u}_0\,,$$

and Lemma 5.6 to deduce that $\mathbf{curl}\,\chi_h\mathbf{u}_0 \in \mathsf{M}^{k,p}_{-\beta}(\Omega)$. Therefore we can apply Lemma 5.7 to $\boldsymbol{\varphi} = \chi_h\mathbf{u}_0$. Since the support of $\chi_h\mathbf{u}_0$ is contained in $\Omega\setminus\mathcal{L}_1$, we can sum over all $K\in\mathscr{T}_h$ and obtain:

$$\begin{aligned}\|\chi_h\mathbf{u}_0 - \Pi^k(\chi_h\mathbf{u}_0)\|_X \ &\lesssim\ h^k\big\{\|\mathbf{u}_0\|_{\overline{\mathsf{M}}^{k,p}_{-\beta}(\Omega)} + \|\,\mathbf{curl}\,\mathbf{u}_0\|_{\mathsf{M}^{k,p}_{-\beta}(K)}\big\} \\ &\lesssim\ h^k\,\|\mathbf{u}_0\|_{\mathsf{M}^{k+1,p}_{-1-\beta}(\Omega)}.\end{aligned} \tag{63}$$

(ii) Since q belongs to $\mathrm{M}^{k+1,p}_{-1-\beta}(\Omega)$, by Lemma 5.6, $\chi_h q$ also belongs to $\mathrm{M}^{k+1,p}_{-1-\beta}(\Omega)$, therefore $\nabla\chi_h q$ belongs to $\overline{\mathsf{M}}^{k,p}_{-\beta}(\Omega)$. Since its curl is zero, we can directly apply Lemma 5.7 to $\boldsymbol{\varphi} = \nabla\chi_h q$ and obtain

$$\|\nabla\chi_h q - \Pi^k(\nabla\chi_h q)\|_X \ \lesssim\ h^k\|q\|_{\mathrm{M}^{k+1,p}_{-\beta}(\Omega)}. \tag{64}$$

(iii) As a consequence of $\mathbf{u}_0 \in \mathsf{M}^{1,p}_{-1-\beta}(\Omega)$ and $q\in\mathrm{M}^{1,p}_{-\beta}(\Omega)$, the three fields $(1-\chi_h)\mathbf{u}_0$, $\mathbf{curl}((1-\chi_h)\mathbf{u}_0)$ and $\nabla(1-\chi_h)q$ all belong to $\mathsf{M}^{0,p}_{-\beta}(\Omega)$. Moreover their supports are contained in $\Omega_h := \bigcup_{K\in\mathcal{L}_1\cup\mathcal{L}_2}\overline{K}$. As a consequence of Lemma 5.8 below, we obtain immediately

$$\|(1-\chi_h)\mathbf{u}_0\|_X + \|\nabla(1-\chi_h)q\|_X \ \lesssim\ h^k\big\{\|\mathbf{u}_0\|_{\mathsf{M}^{1,p}_{-1-\beta}(\Omega)} + \|q\|_{\mathrm{M}^{1,p}_{-\beta}(\Omega)}\big\}. \tag{65}$$

The combination of the four estimates (62)-(65) gives that there exists $\mathbf{v}_h\in X_h$ such that

$$\|\mathbf{u}-\mathbf{v}_h\| \lesssim h^k\big\{\|\mathbf{u}_0\|_{\mathsf{M}^{k+1,p}_{-1-\beta}(\Omega)} + \|q\|_{\mathrm{M}^{k+1,p}_{-\beta}(\Omega)}\big\}. \tag{66}$$

The conclusion of Theorem 5.3 is now a consequence of the Céa estimate (70). ☐

Lemma 5.8 *Under Assumption* $2_{k,\beta}$, *let* $\Omega_h := \bigcup_{K\in\mathcal{L}_1\cup\mathcal{L}_2}\overline{K}$ *and* $\psi\in\mathrm{M}^{0,p}_{-\beta}(\Omega)$. *Then*

$$\|\psi\|_{p,\Omega_h} \lesssim h^k\|\psi\|_{\mathrm{M}^{0,p}_{-\beta}(\Omega)}.$$

Proof: The layer domain Ω_h is contained in $\mathscr{V}_{\mathbf{c}}^0 \cup \mathscr{V}_{\mathbf{e}}^0 \cup \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}$. We note that, as a consequence of Lemma 5.4 and particularly (48), in $\Omega_h \cap \mathscr{V}_{\mathbf{c}}^0$ and $\Omega_h \cap (\mathscr{V}_{\mathbf{e}}^0 \cup \mathscr{V}_{\mathbf{e}}^{\mathbf{c}})$ there holds $r_{\mathbf{c}} \lesssim h^{k/\beta}$ and $r_{\mathbf{e}} \lesssim h^{k/\beta}$, respectively. Therefore for $K \in \mathcal{L}_1 \cup \mathcal{L}_2$,

$$\begin{aligned}
\text{if } K \subset (\mathscr{V}_{\mathbf{c}}^0): \qquad \|\psi\|_{p,K} &\lesssim \big(\sup_K r_{\mathbf{c}}^{\beta}\big)\, \|r_{\mathbf{c}}^{-\beta}\psi\|_{p,K} \lesssim h^k \|\psi\|_{\mathrm{M}^{0,p}_{-\beta}(K)} \\
\text{if } K \subset (\mathscr{V}_{\mathbf{e}}^0 \cup \mathscr{V}_{\mathbf{e}}^{\mathbf{c}}): \qquad \|\psi\|_{p,K} &\lesssim \big(\sup_K r_{\mathbf{e}}^{\beta}\big)\, \|r_{\mathbf{e}}^{-\beta}\psi\|_{p,K} \lesssim h^k \|\psi\|_{\mathrm{M}^{0,p}_{-\beta}(K)}.
\end{aligned}$$

Adding these estimates ends the proof. □

5.3 Algebraic optimality of the Galerkin approximation

Let now $X_h \subset \mathbf{H}_0(\mathbf{curl},\Omega)$ be the edge element space generated by $\mathcal{N}_k$, i.e.:

$$X_h = \{\mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl},\Omega) \ : \ \forall K \in \mathscr{T}_h \quad D\Phi_K^{\top}(\mathbf{v}_h\big|_K \circ \Phi_k) \in \mathcal{N}_k\}. \tag{67}$$

The Galerkin problem associated with (3) reads: Find $\mathbf{u}_h \in X_h$ such that:

$$\int_{\Omega} (\mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h - \lambda \mathbf{u}_h \cdot \mathbf{v}_h) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in X_h \tag{68}$$

Theorem 5.9 *Let $\mathbf{u}$ be the solution of problem* (3) *with $\lambda \notin [0,+\infty[$ and a regular right hand side $\mathbf{f}$. Let the degree $k \geq 1$ be fixed. Let $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ be a family of meshes verifying Assumptions* 1 *and* $2_{k,\beta}$ *for a positive index β such that $\beta \leq \beta_\Omega$ where β_Ω is the parameter of Theorem* 4.1. *Then, the variational problem* (68) *is well-posed and moreover:*

$$\|\mathbf{u} - \mathbf{u}_h\|_X \leq Ch^k \leq C' N^{-k/3}$$

where C and C' depend on k, but not on h or N.

Proof: It is enough to prove that, if $\lambda \notin [0,\infty)$, the bilinear form $a_\lambda : (\mathbf{u},\mathbf{v}) \mapsto \int_\Omega (\mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} - \lambda \mathbf{u}\cdot\mathbf{v})$ is strongly coercive on X, i.e., there exists $\alpha \in \mathbb{C}$ such that

$$\mathrm{Re}\,\big(\alpha\, a_\lambda(\mathbf{u},\overline{\mathbf{u}})\big) \geq \|\mathbf{u}\|_X^2. \tag{69}$$

Since λ is not real ≥ 0, we can write $-\lambda$ in the form $-\lambda = \rho \mathrm{e}^{i\theta}$ with $\theta \in (-\pi,\pi)$. Then, if we choose $\alpha = \mathrm{e}^{-i\theta/2}$, we obtain the strong coercivity (69) since then $\mathrm{Re}\,\alpha$ and $-\mathrm{Re}\,\alpha\lambda$ are both positive. The variational problem (68) admits then a unique solution. Moreover, the Céa-type estimate:

$$\|\mathbf{u} - \mathbf{u}_h\|_X \lesssim \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X. \tag{70}$$

holds. Theorem 5.3 allows to conclude. □

6 Extensions

In this section we consider three different extensions for the theory presented in Section 5.2 to the case of over-refinement meshes (so not verifying Assumption $2_{k,\beta}$) and for piecewise homogeneous materials.

It is worth noting that in Section 6.2 and 6.3 the over-refinement we consider is different from the tensor product hexahedral meshes of Section 6.1. This is an edge-corner refinement in the sense of Assumption $2_{k,\beta}$ around fictitious internal edges and corners for the convenience of mesh design. In the case of piecewise-homogeneous materials on polyhedral partition of Ω, there appear actual internal edges and corners. In both situations, we have to revisit the decomposition (43) in a gradient and a less singular part to conclude.

6.1 Tensor product hexahedral meshes

If hexahedral elements are used in the family $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$, the projection operators $\widehat{\Pi}$ and $\widehat{R}$ are diagonal. Thus, thanks to Proposition 3.2 *(ii)*, in estimates (56), the bound D_3 has a similar structure as D_1 and D_2:

$$D_3 = \sum_{|\boldsymbol{\alpha}|=k} \|\mathbf{d}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\alpha}}\varphi_3\|_{p,K}. \tag{71}$$

This allows for proving the local estimates (54)-(55) under the relaxed assumption on the meshes:

Assumption $3_{k,\beta}$ *The number of elements K of $\mathscr{T}_h$ is $O(h^{-3})$ and the elements satisfy:*

$$\begin{array}{lll}
K \in \mathscr{V}^0_{\mathbf{c}}(h)\setminus\mathcal{L}_1 & h^{k/\beta}\lesssim d_1,\ d_2,\ d_3 \lesssim h\, r_{\mathbf{c},K}^{1-\beta/k} & \\
K \in \mathscr{V}^0_{\mathbf{e}}(h)\setminus\mathcal{L}_1 & h^{k/\beta}\lesssim d_1,\ d_2 \lesssim h\, r_{\mathbf{e},K}^{1-\beta/k} & h^{k/\beta}\lesssim d_3\lesssim h \\
K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h)\setminus\mathcal{L}_1 & h^{k/\beta}\lesssim d_1,\ d_2 \lesssim h\, r_{\mathbf{e},K}^{1-\beta/k} & h^{k/\beta}\lesssim d_3\lesssim h\, r_{\mathbf{c},K}^{1-\beta/k} \\
K \in \mathcal{L}_0 & d_1\simeq d_2\simeq d_3\simeq h^{k/\beta} & \\
K \in \mathscr{V}^0_{\mathbf{e}}(h)\cap\mathcal{L}_1 & d_1\simeq d_2\simeq h^{k/\beta} & h^{k/\beta}\lesssim d_3\lesssim h \\
K \in \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}(h)\cap\mathcal{L}_1\setminus\mathcal{L}_0 & d_1\simeq d_2\simeq h^{k/\beta} & h^{k/\beta}\lesssim d_3\lesssim h\, r_{\mathbf{c},K}^{1-\beta/k} \\
K \in \mathscr{V}^0(h) & h^{k/\beta}\lesssim d_1,\ d_2,\ d_3\lesssim h. &
\end{array}$$

The lower bound $h^{k/\beta}$ for all dimensions of all elements in the mesh allows to keep the size estimates (47)-(48) (Lemma 5.4). We can check that all arguments in the proof of Theorem 5.3 are still valid and thus we obtain

Theorem 6.1 *Let the degree of the elements $k\geq 1$ be fixed. Let $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ be a family of* hexahedral *meshes verifying Assumptions* 1 *and* $3_{k,\beta}$*, and let the right hand side* $\mathbf{f}$ *be regular enough. Then, if the Céa-type estimate* (70) *holds, the Galerkin method* (68) *is algebraically optimal of degree* k.

As a corollary of this theorem, we obtain that graded tensor product meshes provide algebraically optimal Galerkin methods.

Corollary 6.2 *With a fixed $k\geq 1$ let $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ be a family of hexahedral meshes constructed by a macro decomposition of Ω in hexahedra, each of them being meshed by a tensor mesh, product of 1D graded mesh of the form* (45) *with $\tau = k/\beta$. Then, if the Céa-type estimate* (70) *holds, the Galerkin method* (68) *is algebraically optimal of degree* k.

6.2 Over-refinement

It may happen that a decomposition of the domain in hexahedral macro-elements Ω_j helps for the mesh design: Without significant loss of degrees of freedom, a refined mesh on Ω can be constructed by the combination of refined meshes on the Ω_j where the edges and corners are now those of the Ω_j: For example, a refined mesh on the Fichera corner $\Omega := (-1,1)^3 \setminus (-1,0]^3$ can be obtained by refined meshes on the seven cubes of which Ω consists. The outcome is an over-refined mesh on Ω where the sets $\mathscr{E}$ and $\mathscr{C}$ of edges and corners are now augmented by those of the Ω_j: let us denote by $\mathscr{E}^*$ and $\mathscr{C}^*$ those augmented sets. With these new sets we associate regions $\mathscr{V}^0_{\mathbf{c}}$, $\mathscr{V}^0_{\mathbf{e}}$, $\mathscr{V}^{\mathbf{c}}_{\mathbf{e}}$ and distance functions $r_{\mathbf{e}}$ and $r_{\mathbf{c}}$ like before.

Such an over-refined mesh satisfies Assumption $3_{k,\beta}$, and if it is hexahedral, Theorem 6.1 yields the algebraic optimality. If the mesh is tetrahedral, we have to revisit decomposition (43).

Of course, there are no singularities inside the internal edges or at the internal corners. We still have a corner type singular behavior near the external corners along the internal edges. It is possible to prove that the decomposition (43) can be replaced by the following more precise version

$$
\begin{aligned}
\mathbf{u} = \mathbf{u}_0 + \nabla q \quad & \text{where} \quad q = q_0 + q_1 + q_{\text{reg}} \quad \text{and:} \\
& \mathbf{u}_0 \in \mathbf{M}^{*,m,p}_{-1-\beta}(\Omega) \\
& q_0 \in \mathrm{M}^{*,m+1,p}_{-1-\beta}(\Omega), \quad q_1 \in \mathrm{W}^{m+1,p}_{-1-\beta}(\Omega), \quad q_{\text{reg}} \in \mathrm{W}^{m+1,p}(\Omega).
\end{aligned} \tag{72}
$$

Here we have made use of new spaces $\mathrm{M}^{*,m,p}_{\gamma}(\Omega)$ and $\mathbf{M}^{*,m,p}_{\gamma}(\Omega)$: They are defined as $\mathrm{M}^{m,p}_{\gamma}(\Omega)$ and $\mathbf{M}^{m,p}_{\gamma}(\Omega)$, simply replacing the sets $\mathscr{E}$ and $\mathscr{C}$ by their augmented versions $\mathscr{E}^*$ and $\mathscr{C}^*$. As for $\mathrm{W}^{m,p}_{\gamma}(\Omega)$, it bears weights only at the corners $\mathbf{c} \in \mathscr{C}$ (we denote by $\mathscr{V}_{\mathbf{c}}$ the global corner neighborhood $\mathscr{V}_{\mathbf{c}} := \mathscr{V}^0_{\mathbf{c}} \cup \bigcup_{\mathbf{e}\in\mathscr{E}_{\mathbf{c}}} \mathscr{V}^{\mathbf{c}}_{\mathbf{e}}$):

$$
\begin{aligned}
\mathrm{W}^{m,p}_{\gamma}(\Omega) := \Big\{ u \in \mathrm{L}^p(\Omega) \ : \ \forall\, |\boldsymbol{\alpha}| \le m \quad & \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}^0), \\
\forall\, \mathbf{c} \in \mathscr{C} \quad & r_{\mathbf{c}}^{\gamma+|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^p(\mathscr{V}_{\mathbf{c}}) \Big\}.
\end{aligned} \tag{73}
$$

The decomposition (72) allows for proving the algebraic optimality of degree k for any tetrahedral over-refined mesh in the above sense: There we use the fact that the projection operator $\widehat{\Pi}$ is diagonal on gradients and the estimate in Proposition 3.2 *(ii)*.

6.3 Piecewise-homogeneous materials

We assume now that Ω admits a decomposition into a finite number of polyhedral subdomains Ω_j, $j = 1, \ldots, J$, so that the magnetic permeability μ and the electric permittivity ε are constant (and equal to some positive numbers μ_j and ε_j) on Ω_j. The electric problem has now to interpreted as a *transmission problem* between the subdomains Ω_j with the interface conditions $[\mathbf{u} \times \mathbf{n}] = 0$ and $[\varepsilon \mathbf{u} \cdot \mathbf{n}] = 0$. The variational form of the electric problem is, instead of(3):

Find $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ *such that:*

$$
\int_\Omega \mu^{-1}\, \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} - \lambda \varepsilon \mathbf{u} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega). \tag{74}
$$

The singularities of $\mathbf{u}$ are now also present along internal edges and corners (i.e. the edges and corners of the Ω_j which are not contained in $\partial\Omega$). They are described in [24]. There are two main differences from the fully homogeneous case:

1. The singularities may be stronger: in general $\mathbf{u}$ is not piece-wise $\mathbf{H}^{1/2}$;
2. The expansion along the internal edges $\mathbf{e}$ contain regular terms of the form $(r_\mathbf{e},\theta_\mathbf{e},z_\mathbf{e})\mapsto d_\mathbf{e}(z_\mathbf{e})$ in local cylindrical coordinates, with $d_\mathbf{e}$ smooth inside $\mathbf{e}$. Such terms do not fit well with the weighted spaces M.

We define the sets $\mathscr{E}^*$ and $\mathscr{C}^*$ as in Section 6.2. Then, if $\mathbf{f}$ is regular enough, we have the following decomposition for all $m>0$, similar to (72)

$$\begin{aligned}\mathbf{u}=\mathbf{u}_0+\nabla q \quad & \text{where} \quad q=q_0+q_1+q_{\text{reg}} \quad \text{and:}\\ & \mathbf{u}_0\in \mathbf{M}^{\bullet,m,p}_{-1-\beta}(\Omega)\\ & q_0\in \mathrm{M}^{\bullet,m+1,p}_{-1-\beta}(\Omega) \quad \text{with} \quad \nabla q_0\in \mathrm{M}^{\bullet,m,p}_{-\beta}(\Omega),\\ & q_1\in \mathrm{W}^{*,m+1,p}_{-1-\beta}(\Omega), \quad q_{\text{reg}}\in \mathrm{W}^{m+1,p}(\Omega).\end{aligned} \tag{75}$$

where $\beta=\beta(\Omega)>0$ and $p=p(\Omega)>2$. Here $\mathrm{M}^{\bullet,m,p}_{\gamma}(\Omega)$ and $\mathbf{M}^{\bullet,m,p}_{\gamma}(\Omega)$ are defined as $\mathrm{M}^{*,m,p}_{\gamma}(\Omega)$ and $\mathbf{M}^{*,m,p}_{\gamma}(\Omega)$ above with the distinction that the derivatives are now considered in *each* Ω_j *separately*, which means that the regularity in $\mathrm{M}^{\bullet,m,p}_{\gamma}(\Omega)$ and $\mathbf{M}^{\bullet,m,p}_{\gamma}(\Omega)$ is a piecewise regularity inside each Ω_j.

The space $\mathrm{W}^{*,m,p}_{\gamma}(\Omega)$ bears weights at all corners $\mathbf{c}\in\mathscr{C}^*$ and is defined similarly as $\mathrm{W}^{m,p}_{\gamma}(\Omega)$ in (73):

$$\begin{aligned}\mathrm{W}^{*,m,p}_{\gamma}(\Omega):=\Big\{u\in \mathrm{L}^p(\Omega)\ :\ \forall\,|\boldsymbol{\alpha}|\le m \quad & \partial^{\boldsymbol{\alpha}}u\in \mathrm{L}^p(\mathscr{V}^0),\\ \forall\,\mathbf{c}\in\mathscr{C}^* \quad & r_\mathbf{c}^{\gamma+|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}}u\in \mathrm{L}^p(\mathscr{V}_\mathbf{c})\Big\}.\end{aligned} \tag{76}$$

Note that if $\beta<\frac{3}{p}-1$, the part q_{reg} is no more of any use since then $\mathrm{W}^{m+1,p}(\Omega)\subset \mathrm{W}^{*,m+1,p}_{-1-\beta}(\Omega)$.

The Galerkin problem associated with (74) reads: Find $\mathbf{u}_h\in X_h$ such that:

$$\int_\Omega (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h\cdot\mathbf{curl}\,\mathbf{v}_h-\lambda\varepsilon\mathbf{u}_h\cdot\mathbf{v}_h)=\int_\Omega \mathbf{f}\cdot\mathbf{v}_h \qquad \forall\,\mathbf{v}_h\in X_h \tag{77}$$

The results in the previous sections can be extended as

Theorem 6.3 *Let the degree* $k\ge 1$ *be fixed. We assume* (i) *or* (ii)*:*

(i) *The family of meshes* $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ *satisfies Assumptions* 1 *and* $2_{k,\beta}$ *inside each* Ω_j.

(ii) *The family of meshes* $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ *satisfies Assumptions* 1 *and* $3_{k,\beta}$ *inside each* Ω_j *and are formed with* hexahedral *elements.*

Then, if the Céa-type estimate (70) *holds, the Galerkin method* (77) *is algebraically optimal of degree* k.

7 Discrete compactness and the eigenvalue problem

In Section 5, relying on the coercivity of problem (3) when $\lambda\in\mathbb{C}\setminus\mathbb{R}_+$, we have proved algebraic optimality for our Galerkin approximation. The extension to positive λ is not straightforward. A necessary condition for that is the "spectral correctness" of the Galerkin approximation, which means that all discrete eigenvalues converge towards eigenvalues of problem (3) and, conversely, all eigenvalues of problem (3) are approximated, all respecting multiplicities.

Due to the infinite dimensional kernel $\mathcal{K}$ of problem (3), the spectral correctness is not an easy consequence of the coercivity of the form $(\mathbf{u},\mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{u},\mathbf{v})$ on $\mathbf{H}_0(\mathbf{curl},\Omega)$, because the embedding of $\mathbf{H}_0(\mathbf{curl},\Omega)$ into $L^2(\Omega)$ is not compact. Related to the fact that the condition $\operatorname{div}\mathbf{v} = 0$ eliminates $\mathcal{K}$, the subspace V of divergence-free fields:

$$V = \big\{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl},\Omega)\ :\ \operatorname{div}\mathbf{v} = 0\big\},$$

is compactly embedded in $L^2(\Omega)$. The condition $\operatorname{div}\mathbf{v} = 0$ cannot be forced into the discrete spaces, but only imitated in the form of the *discrete divergence free condition*: The discrete counterpart of V is V_h, the subspace of discrete divergence-free fields defined as

$$V_h = \big\{\mathbf{v}_h \in X_h\ :\ \textstyle\int_\Omega \mathbf{v}_h \cdot \nabla p_h = 0,\ \forall\, p_h \in P_h\big\},$$

where P_h is the space of piecewise polynomial continuous functions generated by $\mathcal{P}_k$ (see Section 2.1).

The *discrete compactness property* [35] is a necessary and sufficient condition for the spectral correctness of the Galerkin projection [10] and [16]. This property is a consequence of the following condition: There exists a sequence δ_h, $\delta_h \to 0$ when $h \to 0$ such that

$$\forall\, \mathbf{v}_h \in V_h \quad \exists\, \mathbf{v} \in V\ :\ \|\mathbf{v} - \mathbf{v}_h\|_X \lesssim \delta_h \|\mathbf{v}_h\|_X \tag{78}$$

see [10, 11], [17] and also [13].

7.1 The source problem

It is well known that the problem (68) is well-posed and quasi-optimal if and only if the discrete inf-sup condition:

$$\exists\, \alpha \in \mathbb{R}\ :\ \inf_{\mathbf{v}_h \in X_h} \sup_{\mathbf{u}_h \in X_h} \frac{a_\lambda(\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{u}_h\|_X \|\mathbf{v}_h\|_X} \geq \alpha > 0 \tag{79}$$

holds uniformly in $h \in \mathfrak{h}$. Using [15, Theorem 4.1] (see also [13]), we know that (79) holds for h sufficiently small if condition (78) holds.

For the proof of (78), we fix $\mathbf{v}_h \in V_h$ and choose for $\mathbf{v}$ the solution of the problem:

$$\mathbf{v} \in \mathbf{H}_0(\mathbf{curl},\Omega) \quad : \quad \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{v}_h\,, \quad \operatorname{div}\mathbf{v} = 0. \tag{80}$$

We first prove a regularity result for $\mathbf{v}$ in the form a splitting in a “regular” field and a gradient:

Lemma 7.1 *For all $p \geq 2$, $\mathbf{v}$ can be split into*

$$\mathbf{v} = \mathbf{w} + \nabla q \quad \textit{with} \quad \mathbf{w} \in \mathbf{W}^{1,p}_0(\Omega) \quad \textit{and} \quad q \in \mathrm{H}^1_0(\Omega):\ \Delta q \in \mathrm{L}^p(\Omega). \tag{81}$$

with estimates

$$\|\mathbf{w}\|_{\mathbf{W}^{1,p}(\Omega)} + \|\Delta q\|_{\mathrm{L}^p(\Omega)} \lesssim \|\mathbf{v}_h\|_X + \|\,\mathbf{curl}\,\mathbf{v}_h\|_{\mathbf{L}^p(\Omega)} \tag{82}$$

Proof: We note that $\mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,\mathbf{v}_h$ belongs to $\mathbf{L}^p(\Omega)$ for all $p \geq 2$. The proof consists in proving (81) for any $\mathbf{v} \in \mathbf{H}_0(\mathbf{curl},\Omega)$ such that $\mathbf{curl}\,\mathbf{v} \in \mathbf{L}^p(\Omega)$ and $\operatorname{div}\mathbf{v} = 0$.

Let $\mathcal{O}$ be an open ball containing $\overline{\Omega}$. Let $\widetilde{\mathbf{v}}$ be the extension by zero of $\mathbf{v}$ to $\mathcal{O}$. Then $\mathbf{curl}\,\widetilde{\mathbf{v}}$ is the extension by zero of $\mathbf{curl}\,\mathbf{v}$ to $\mathcal{O}$. The straightforward generalization to the spaces L^p of [3, Lemma 3.5] gives the existence of $\mathbf{w}_0 \in \mathbf{W}^{1,p}(\Omega)$ such that $\operatorname{div}\mathbf{w}_0 = 0$ and $\mathbf{curl}\,\mathbf{w}_0 = \mathbf{curl}\,\widetilde{\mathbf{v}}$.

Since $\mathbf{curl}\,\mathbf{w}_0 = 0$ in $\mathcal{O}\setminus\Omega$ there exists $\psi \in \mathrm{W}^{2,p}(\mathcal{O}\setminus\Omega)$ such that $\mathbf{w}_0 = \nabla\psi$ in $\mathcal{O}\setminus\Omega$. There exists an extension $\widetilde{\psi} \in \mathrm{W}^{2,p}(\mathcal{O})$ of ψ to Ω: $\widetilde{\psi}\big|_{\mathcal{O}\setminus\Omega} = \psi$.

We set $\mathbf{w} := \mathbf{w}_0 - \nabla\widetilde{\psi}$ in $\mathcal{O}$. Thus

$$\mathbf{w} \in \mathrm{W}^{1,p}(\mathcal{O}) \quad \text{and} \quad \mathbf{w}\big|_{\mathcal{O}\setminus\Omega} = 0.$$

Therefore $\mathbf{w}$ belongs to $\mathbf{W}^{1,p}_0(\Omega)$.

Let $\mathbf{v}_1$ be the difference $\mathbf{v}-\mathbf{w}$. Since $\mathbf{curl}\,\mathbf{v}_1 = \mathbf{curl}\,\mathbf{v} - \mathbf{curl}\,\mathbf{w}_0 = 0$ in Ω, there exists $q \in \mathrm{H}^1(\Omega)$ such that $\mathbf{v}_1 = \nabla q$. Since

$$\operatorname{div}\mathbf{v}_1 = \operatorname{div}\mathbf{v} - \operatorname{div}\mathbf{w} = \Delta\widetilde{\psi} \in \mathrm{L}^p(\Omega)$$

we obtain that Δq belongs to $\mathrm{L}^p(\Omega)$, which ends the proof, since the estimates can be proved at each stage. $\square$

Corollary 7.2 *The Nédélec interpolation operator* Π^k *is well defined for the solution* $\mathbf{v}$ *of problem* (80).

Proof: In the splitting (81) of $\mathbf{v}$, the potential q belongs to $\mathrm{H}^{\delta+3/2}(\Omega)$ for a $\delta > 0$, hence is continuous on $\overline{\Omega}$. Therefore the Lagrange interpolation operator π^k is well defined on q and we have

$$\Pi^k\mathbf{w} + \nabla(\pi^k q) = \Pi^k(\mathbf{w}+\nabla q), \tag{83}$$

which gives sense to $\Pi^k\mathbf{v}$. $\square$

Lemma 7.3 *For the solution* $\mathbf{v}$ *of problem* (80) *the following estimate holds*

$$\|\mathbf{v}-\mathbf{v}_h\|_X \lesssim \|\mathbf{v}-\Pi^k\mathbf{v}\|_{\mathbf{L}^2(\Omega)}. \tag{84}$$

Proof: The proof is known as NÉDÉLEC's trick, see [34]. For completeness we recall its simple arguments. Since $\mathbf{curl}(\mathbf{v}-\mathbf{v}_h) = 0$, the norms $\|\mathbf{v}-\mathbf{v}_h\|_X$ and $\|\mathbf{v}-\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}$ coincide. In order to estimate the $\mathbf{L}^2$ norm we evaluate the scalar product:

$$\begin{aligned}\|\mathbf{v}-\mathbf{v}_h\|^2_{\mathbf{L}^2(\Omega)} &= (\mathbf{v}-\mathbf{v}_h, \mathbf{v}-\mathbf{v}_h)\\ &= (\mathbf{v}-\mathbf{v}_h, \mathbf{v}-\Pi^k\mathbf{v}+\Pi^k\mathbf{v}-\mathbf{v}_h).\end{aligned} \tag{85}$$

If we prove that

$$(\mathbf{v}-\mathbf{v}_h, \Pi^k\mathbf{v}-\mathbf{v}_h) = 0 \tag{86}$$

estimate (84) is clearly a consequence of (85). So, let us prove (86).

Since $\mathbf{v}_h$ belongs to X_h, $\Pi^k\mathbf{v}_h = \mathbf{v}_h$, therefore $\Pi^k\mathbf{v}-\mathbf{v}_h = \Pi^k(\mathbf{v}-\mathbf{v}_h)$.

Since $\mathbf{curl}(\mathbf{v}-\mathbf{v}_h) = 0$, there exists $q \in \mathrm{H}^1(\Omega)$ such that $\mathbf{v}-\mathbf{v}_h = \nabla q$. Since $\Pi^k(\mathbf{v}-\mathbf{v}_h)$ makes sense, $\pi^k q$ also makes sense and we have

$$\Pi^k\mathbf{v}-\mathbf{v}_h = \Pi^k(\mathbf{v}-\mathbf{v}_h) = \nabla(\pi^k q).$$

Now, $(\mathbf{v}, \Pi^k\mathbf{v}-\mathbf{v}_h) = (\mathbf{v}, \nabla\pi^k q)$ is zero since $\operatorname{div}\mathbf{v} = 0$, and $(\mathbf{v}_k, \Pi^k\mathbf{v}-\mathbf{v}_h) = (\mathbf{v}_h, \nabla\pi^k q)$ is zero since $\mathbf{v}_h \in V_h$ and $\pi^k q \in P_h$. We have obtained (86), which ends the proof. $\square$

Now we are able to prove that condition (78) holds for low order edge elements on anisotropic tetrahedral meshes and for edge elements of any order on hexahedral anisotropic refined meshes, provided an approximation result holds for the solutions of the scalar Dirichlet problem:

$$q \in \mathrm{H}^1_0(\Omega), \quad \text{and} \quad \Delta q = g. \tag{87}$$

Assumption 4 (i) *or* (ii) *holds:*
(i) *The meshes* $\mathscr{T}_h$ *are tetrahedral,* $k = 1$.
(ii) *The meshes* $\mathscr{T}_h$ *are hexahedral,* $k \geq 1$*, and there exists* $\sigma^* > 0$ *and* $p^* > 2$ *such that for all* $p \in (2, p^*)$ *and for all* $g \in \mathrm{L}^p(\Omega)$ *the following estimate holds between the solutions* q *of* (87) *and its interpolate* $\pi^k q$*:*

$$\|q - \pi^k q\|_{\mathrm{H}^1(\Omega)} \lesssim h^{\sigma^*} \|g\|_{\mathrm{L}^p(\Omega)}. \tag{88}$$

We present (88) as an assumption here, because no proof seems to exist in published form. From [5] we conclude, however, that a proof, using techniques similar to [6], is possible.

Theorem 7.4 *Let the family of meshes* $\left\{\mathscr{T}_h\right\}_{h\in\mathfrak{h}}$ *satisfy Assumptions* 1 *and* $2_{k,\beta}$. *We assume moreover that Assumption* 4 *holds. Then there exists* $\sigma > 0$ *such that the solution* $\mathbf{v}$ *of problem* (80) *satisfy the following approximation estimate*

$$\|\mathbf{v} - \Pi^k \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \lesssim h^{\sigma} \|\mathbf{v}_h\|_X. \tag{89}$$

Therefore (78) *holds.*

Proof: Using (81) and (83), we obtain

$$\|\mathbf{v} - \Pi^k \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{w} - \Pi^k \mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|q - \pi^k q\|_{\mathrm{H}^1(\Omega)}. \tag{90}$$

To bound the first piece $\|\mathbf{w} - \Pi^k \mathbf{w}\|_{\mathbf{L}^2(\Omega)}$, we use Proposition 3.2 which gives us

$$\|\mathbf{w} - \Pi^k \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \lesssim \ h\|\mathbf{w}\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for hexahedral meshes}$$

For tetrahedral meshes and when $k = 1$, we are exactly in the situation of Proposition 3.4 since $\mathbf{curl}\,\mathbf{w}$ is a constant vector. We then have:

$$\|\mathbf{w} - \Pi^k \mathbf{w}\|_{\mathbf{L}^2(\Omega)} \lesssim h\|\mathbf{w}\|_{\mathrm{W}^{1,p}(\Omega)} \quad \text{for tetrahedral meshes and } k = 1.$$

Concerning the second piece $\|q - \pi^k q\|_{\mathrm{H}^1(\Omega)}$, we use (88) in any situation: For tetrahedral meshes and $k = 1$, it is proved in [7] and we assume in *(ii)* that it holds for hexahedral meshes.

Therefore we have obtained that there exists $\sigma' > 0$ and $p^* > 2$ such that for all $p \in (2, p^*)$

$$\|\mathbf{v} - \Pi^k \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \lesssim h^{\sigma'}(\|\mathbf{w}\|_{\mathrm{W}^{1,p}(\Omega)} + \|\Delta q\|_{L^p(\Omega)}).$$

Combining with (82), that implies

$$\|\mathbf{v} - \Pi^k \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \lesssim h^{\sigma'}(\|\mathbf{v}_h\|_X + \|\,\mathbf{curl}\,\mathbf{v}_h\|_{\mathbf{L}^p(\Omega)}).$$

We end the proof by virtue of the inverse inequality

$$\|\,\mathbf{curl}\,\mathbf{v}_h\|_{\mathbf{L}^p(\Omega)} \lesssim h^{3(\frac{1}{p}-\frac{1}{2})\frac{k}{\beta}}\|\,\mathbf{curl}\,\mathbf{v}_h\|_{\mathbf{L}^2(\Omega)}.$$

Choosing p close enough to 2 so that $\sigma' + 3(\frac{1}{p} - \frac{1}{2})\frac{k}{\beta}$ remains positive, we have obtained (89).
□

Finally, combining Céa's estimate which is a consequence of the inf-sup condition (79) with the interpolation estimates of Theorem 5.3 we have obtained:

Theorem 7.5 *Let* $\lambda > 0$ *be not an eigenvalue of* (3) *and the family of meshes* $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ *verify the assumptions in Theorem* 7.4. *There exists a* h_0 *such that, for all* $h < h_0$ *the variational problem* (68) *admits a unique solution. If* $\mathbf{u}$ *is the solution of* (3)*, there holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_X \lesssim \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X.$$

Moreover, for sufficiently regular right hand side $\mathbf{f}$*:* $\|\mathbf{u} - \mathbf{u}_h\|_X \lesssim h^k$.

As far as discontinuous coefficients are concerned, as it is shown in [13], the condition (78) implies wellposedness also for the transmission problem (74). We have:

Corollary 7.6 *Let* $\lambda > 0$ *be not an eigenvalue of* (74) *and the family of meshes* $\{\mathscr{T}_h\}_{h\in\mathfrak{h}}$ *verify the assumptions in Theorem* 7.4. *There exists a* h_0 *such that, for all* $h < h_0$ *the variational problem* (77) *admits a unique solution. If* $\mathbf{u}$ *is the solution of* (74)*, there holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_X \lesssim \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X.$$

Moreover, for sufficiently regular right hand side $\mathbf{f}$*:* $\|\mathbf{u} - \mathbf{u}_h\|_X \lesssim h^k$.

7.2 The eigenvalue problem

The eigenvalue problem associated to (3) reads: *Find* $\lambda > 0$, $\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega)$ *such that:*

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} = \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega). \tag{91}$$

The eigenvalues λ are a positive increasing sequence $\{\lambda_j\}_j$ and have a finite multiplicity (Theorem 1.1). Let us denote by E_λ the associated eigenspace and by m its dimension.

The corresponding Galerkin eigenvalue problem is then: *Find* $\lambda_h > 0$, $\mathbf{u}_h \in X_h$ *such that:*

$$\int_\Omega \mathbf{curl}\,\mathbf{u}_h \cdot \mathbf{curl}\,\mathbf{v}_h = \lambda_h \int_\Omega \mathbf{u}_h \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in X_h. \tag{92}$$

As a consequence of the discrete compactness property, and therefore of (78), problems (92) are a spurious-free spectrally correct approximation of (91) in the sense of [16]. As a consequence of [28, Theorem 1 and 3], this implies the following estimates for any eigenvalue λ of (91):
(i) There exists exactly m eigenvalues $\lambda_{h,i}$, $i = 1, \ldots, m$ (counted with their multiplicity) of (92) such that $\lim_{h\to 0} \lambda_{h,i} = \lambda$, and moreover the following estimate holds:

$$\max_{i=1,\ldots,m} |\lambda - \lambda_{h,i}| \;\lesssim\; \Big(\sup_{\substack{\mathbf{u}\in E_\lambda \\ \|\mathbf{u}\|_X = 1}} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X \Big)^2.$$

(ii) Let $E_{h,\lambda}$ be the union of the discrete eigenspaces associated to $\{\lambda_{h,i}\}_{1\le i\le m}$, then

$$\sup_{\substack{\mathbf{u}_h\in E_{h,\lambda} \\ \|\mathbf{u}_h\|_X = 1}} \inf_{\mathbf{u} \in E_\lambda} \|\mathbf{u} - \mathbf{u}_h\|_X + \sup_{\substack{\mathbf{u}\in E_\lambda \\ \|\mathbf{u}\|_X = 1}} \inf_{\mathbf{u}_h \in E_{h,\lambda}} \|\mathbf{u} - \mathbf{u}_h\|_X \;\lesssim\; \sup_{\substack{\mathbf{u}\in E_\lambda \\ \|\mathbf{u}\|_X = 1}} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_X.$$

Since the best approximation result (46) also holds for eigenvectors, we obtain the following:

Theorem 7.7 *Let the assumptions of Theorem* 7.4 *hold. With the notation above, the following holds for* h *small enough:*

$$\max_{i=1,\dots,m} |\lambda - \lambda_{h,i}| \lesssim h^{2k}$$

$$\sup_{\substack{\mathbf{u}_h \in E_{h,\lambda} \\ \|\mathbf{u}_h\|_X = 1}} \inf_{\mathbf{u}\in E_\lambda} \|\mathbf{u} - \mathbf{u}_h\|_X + \sup_{\substack{\mathbf{u} \in E_{\lambda} \\ \|\mathbf{u}\|_X = 1}} \inf_{\mathbf{u}_h\in E_{h,\lambda}} \|\mathbf{u} - \mathbf{u}_h\|_X \lesssim h^{k}.$$

As far as the transmission problem is concerned, the same kind of result hold thanks to the validity of the discrete compactness property [16] and the regularity and approximation results stated in Section 6.3.

8 Conclusive remarks

We have proved algebraic optimality of order k for a class of refined meshes using edge elements of degree k in the following two cases

1. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ and any $k \geq 1$, using hexahedral elements or tetrahedral elements;

2. For any $\lambda > 0$ which is not an eigenvalue and any k using hexahedral elements; if tetrahedral elements are used our proof is limited to $k = 1$.

Let us comment on our assumptions for the mesh, in connection with the assumptions used in the papers [7] (Laplace equation on polyhedra) and [40] (Maxwell equations on tensor product domains $\Omega = G \times Z$, with a polygon G and an interval Z).

Taking the special form of the domain $G \times Z$ into account, the assumptions on the refinement parameters are all similar.

In [40], pentahedral and tetrahedral meshes are used. The pentahedral elements are tensor product of uniformly refined triangles inside G by subintervals in Z. Thus they exactly satisfy our Assumption 1. Note that our results would extend naturally to a pentahedral mesh, or to a mixed pentahedral-hexahedral mesh, provided Assumption 1 holds. The tetrahedral elements used in [40] are obtained by dividing each pentahedron in three tetrahedra K_1, K_2 and K_3. Both K_1 and K_2 have one face inside a plane G_1 transverse to the edge. Thus they satisfy Assumption 1. In contrast, K_3 has two of its vertices in such a plane G_1 and the two others in another parallel plane G_2. It does not satisfy Assumption 1. We would not be surprised if the result of [40] obtained for $k = 1$ cannot be extended to $k \geq 2$, due to the fact that Propositions 3.4 and 3.7 have no clear generalization to $k \geq 2$.

Assumptions (3.2-3.3) of [7] are weaker than Assumption 1 because, in particular, they do not impose any limitation on the aspect ratio of the elements. From our proofs, we see that such a limitation is connected to the vector nature of interpolant, and appears to be necessary as soon as there is no commutation property with the projection on the axes of coordinates.

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